CURVES ON SURFACES BLOWN UP AT THREE POINTS

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It is well known that for any integer $g \geq 0$, there exists a smooth curve $C \subset \mathbb{P}^3$ of genus g. One can ask an analogous question when $\mathbb{P}^3$ is replaced by an arbitrary smooth projective variety X of dimension ≥ 3. However, it is possible that a given variety contains no curves of small genus (i.e., abelian varieties or generic hypersurfaces). A better-posed question to ask is whether all sufficiently large genera are realized by smooth curves, and this is answered in the affirmative by J. A. Chen [1].

For a general surface S, one cannot expect for all sufficiently large genera to be realized as the genus of a smooth curve on S. Nevertheless, it is shown in [1] that the statement holds if one allows integral curves with at worst nodal singularities (and considers their geometric genera).

Let S be any smooth projective surface. In this note, we show that if S is replaced by a birational model $\tilde{S}$, then all sufficiently large genera can be realized by smooth curves on $\tilde{S}$. Moreover, it suffices to choose $\tilde{S}$ to be the blow-up of S at three points. This is equivalent to the statement that on S, all sufficiently large (geometric) genera can be realized by integral curves with at most three singular points.

Theorem 1. *Let S be a smooth projective surface with an ample line bundle H and let $\tilde{S} \to S$ be the blow-up at three points. There exists an integer $g_0 = g_0(S, H)$ such that for any integer $g \geq g_0$, there exists a smooth curve $C \subset \tilde{S}$ of genus g.*

The set-up of the proof follows [1, Theorem 2]. Gauss' Eureka theorem is used to choose solutions to a certain numerical constraint, and smoothness of the curves follows from Reider's theorem. We would like to thank R. Lazarsfeld for suggesting questions along these lines.

1. Proof of Theorem 1

Let us recall the statement of Reider's theorem [2]:

> Let L be an ample line bundle on a smooth projective surface X.
>
> (1) Assume that $L^2 \geq 5$ and $L \cdot C \geq 2$ for all irreducible curves $C \subset X$. Then $K_X + L$ is globally generated.
>
> (2) If $L^2 \geq 10$ and $L \cdot C \geq 3$ for all $C \subset X$, then $K_X + L$ is very ample.

Now let $\pi : \tilde{S} \to S$ be the blow-up of S at three points $x_1, x_2, x_3 \in S$. For simplicity, let us assume that H is very ample on S. Let $E_i \subset \tilde{S}$ denote the exceptional divisor over x_i. Write

$$g_m = p_a(K_{\tilde{S}} + m \cdot \pi^* H);$$

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$$d_m = g_m - g_{m-1}.$$

We will show that for any $m \gg 0$ and for any integer r with $1 \le r \le d_m$, there exist integers $k_i \ge 0$ and a smooth curve

$$C \in \left| K_{\tilde{S}} + m \cdot \pi^* H - \sum_{i=1}^{3} k_i E_i \right|$$

with genus $g(C) = g_m - r$. This would imply the theorem.

Choose integers $k_i \ge 2$ satisfying

$$(*) \qquad \sum_{i=1}^{3} \frac{(k_i - 1)(k_i - 2)}{2} = r + 1.$$

Such a solution always exists by Gauss' Eureka theorem.[1] Let

$$D_{m,\vec{k}} = D(m, k_1, k_2, k_3) := m \cdot \pi^* H - \sum_{i=1}^{3} k_i E_i.$$

A straightforward computation shows

$$\begin{aligned}
p_a(K_{\tilde{S}} + D_{m,\vec{k}}) &= p_a(K_{\tilde{S}} + m \cdot \pi^* H - \sum_{i=1}^{3} k_i E_i) \\
&= 1 + \frac{1}{2}(K_{\tilde{S}} + m \cdot \pi^* H - \sum_{i=1}^{3} k_i E_i)(2K_{\tilde{S}} + m \cdot \pi^* H - \sum_{i=1}^{3} k_i E_i) \\
&= 1 + \frac{1}{2}(K_{\tilde{S}} + m \cdot \pi^* H)(2K_{\tilde{S}} + m \cdot \pi^* H) \\
&+ \frac{1}{2}\left[(K_{\tilde{S}} + m \cdot \pi^* H)(-\sum_{i=1}^{3} k_i E_i) + (2K_{\tilde{S}} + m \cdot \pi^* H)(-\sum_{i=1}^{3} k_i E_i) - \sum_{i=1}^{3} k_i^2 \right] \\
&= p_a(K_{\tilde{S}} + m \cdot \pi^* H) + \frac{1}{2}\left[3\sum_{i=1}^{3} k_i - \sum_{i=1}^{3} k_i^2 \right] \\
&= g_m - r.
\end{aligned}$$

It remains to show that given any $m \gg 0$, we can choose a smooth curve

$$C \in \left| K_{\tilde{S}} + D_{m,\vec{k}} \right|$$

where the k_i satisfy $(*)$.

[1] The original formulation says that any positive integer N can be expressed as a sum

$$N = \sum_{i=1}^{3} \frac{a_i(a_i + 1)}{2}$$

of three triangular numbers, where the a_i are non-negative integers.

We first observe that $\pi^* H - E_i$ is nef for any i since H is very ample. By computation,

$$\sum_{i=1}^{3} \frac{(k_i - 1)(k_i - 2)}{2} - 1 = r \leq d_m = \frac{1}{2}(3K_S \cdot H - H^2) + H^2 \cdot m$$

In particular, only the last term on the right hand side depends on m, so for any $m \gg 0$ and for all i, we have

$$k_i \lesssim C \cdot \sqrt{m}$$

where the constant C only depends on H^2 and $K_S \cdot H$. We may then rewrite

$$D_{m,\vec{k}} = (m - 3 - \sum_{i=1}^{3} k_i) \cdot \pi^* H + \sum_{i=1}^{3} (k_i + 1)\pi^* H - k_i E_i.$$

By our assumption, $k_i \geq 2$ for all i so for $m \gg 0$ the expression on the right is the sum of a globally generated line bundle and 3 line bundles that are ample and globally generated. Therefore, when $m \gg 0$ it follows that $D_{m,\vec{k}}$ satisfies the conditions of Reider's theorem and $K_{\tilde{S}} + D_{m,\vec{k}}$ is globally generated. By Bertini's theorem, the general curve

$$C \in \left| K_{\tilde{S}} + D_{m,\vec{k}} \right|$$

is smooth, which completes the proof.

Remark 2. Note that the minimal value of g_0 as in the theorem can be arbitrarily large. Consider the product of two curves $S = C_1 \times C_2$, and let $\tilde{S}$ be the blow-up at three points. For any smooth curve $C \subset \tilde{S}$ that is not one of the three exceptional divisors, we have

$$g(C) \geq \min\{g(C_1), g(C_2)\}.$$

References

[1] Jungkai A. Chen. On genera of smooth curves in higher dimensional varieties, *Proc. Amer. Math. Soc.* **125** (1997), 2221 – 2225.

[2] Igor Reider. Vector bundles of rank 2 and linear systems on algebraic surfaces, *Annals of Mathematics* **127** (1988), 309 – 316.

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