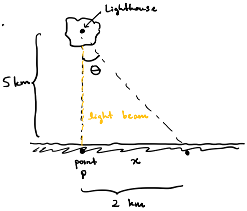
Monday, May 10, 2021 3:01 PM

1.



x = distance from point P to point that light is shining on.

Want $\frac{dx}{dt}$ when $x=2$

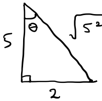
Given: light makes 8 rev. per minute.

$$\Rightarrow \frac{d\theta}{dt} = 8\cdot 2\pi \ \frac{\text{rad}}{\text{min}}$$

$$= 16\pi \ \frac{\text{rad.}}{\text{min}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{x}{5}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{dx}{dt}$$



$\sqrt{5^2+2^2} = \sqrt{29}$

So $\sec^2\theta = \left(\frac{\text{hyp}}{\text{adj}}\right)^2 = \left(\frac{\sqrt{29}}{5}\right)^2 = \frac{29}{25}$

So $\frac{dx}{dt} = 5\cdot\sec^2\theta\cdot\frac{d\theta}{dt}$

$$= 5\cdot\frac{29}{25}\cdot 16\pi = \frac{29\cdot 16\pi}{5} \ \frac{\text{km}}{\text{min}}$$

2. $f(x) = \frac{e^x}{x} = \frac{t}{b}$

(a) increasing / decreasing?

$$f'(x) = \frac{b\cdot t' - t\cdot b'}{b^2} = \frac{x\cdot e^x - e^x\cdot 1}{x^2} = \frac{e^x\cdot(x-1)}{x^2}$$

Critical points: where $f'(x)=0$ or DNE.

In our case, $x=0$ and $x=1$ are critical points

x		0	$\frac{1}{2}$	1	
$f'(x)$	−	DNE	−	0	+

$$\frac{e^{1/2}\cdot(\frac{1}{2}-1)}{(\frac{1}{2})^2} = \frac{+\cdot(-)}{+}$$

f is decreasing

f is increasing

$x=-100$

$$\frac{e^{-100}\cdot(-100-1)}{(-100)^2} = \frac{(+)\cdot(-)}{+} = -$$

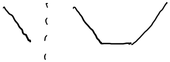
$x=100$: $\frac{e^{100}\cdot(100-1)}{100^2} = \frac{(+)\cdot(+)}{+} = +$

Decreasing: $(-\infty,0)\cup(0,1)$

Increasing: $(1,\infty)$

(b) Critical points: $x=0$, $x=1$

x		0		1	
$f'(x)$	$-$	DNE	$-$	0	$+$



← "picture" of graph

$x=0$: neither

$x=1$: local min.

(c) Intervals of concavity:

$$f'(x) = \frac{e^x\cdot(x-1)}{x^2} = \frac{t}{b}$$

$$f''(x) = \frac{b\cdot t' - t\cdot b'}{b^2}$$

$b = x^2 \rightarrow b' = 2x$

$t' = e^x\cdot(x-\cancel{1}) + \cancel{e^x\cdot 1}$

$= e^x\cdot x$

$$= \frac{x^2\cdot(e^x\cdot x) - [e^x\cdot(x-1)\cdot 2x]}{(x^2)^2}$$

$$= \frac{x^3\cdot e^x - e^x\cdot 2x^2 + e^x\cdot 2x}{x^4}$$

$$= \frac{e^x\cdot(x^3-2x^2+2x)}{x^4}$$

factor out x

Set $0 = f''(x) = \dfrac{e^x\cdot(x^3-2x^2+2x)}{x^4} = \dfrac{e^x\cdot(x^2-2x+2)}{x^3}$

$0 = x^2-2x+2 = ax^2+bx+c$

$\Rightarrow x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a} = \dfrac{2\pm\boxed{\sqrt{2^2-4\cdot1\cdot2}}}{2}$ $= \sqrt{-4}$ imaginary!

never $= 0$. x^2-2x+2 is always positive

$f''(x)$ = DNE when $x=0$.

x	-5	0	5
$f''(x)$	$-$	DNE	$+$

$x=-5$: $\dfrac{e^{-5}\cdot(+)}{(-)} = -$

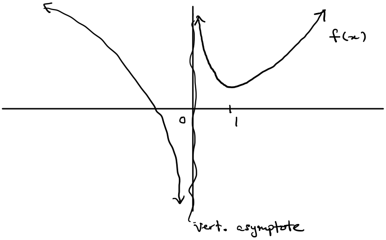
$x=5$: $\dfrac{e^{5}\cdot(+)}{(+)} = +$

plug into $f''(x)$

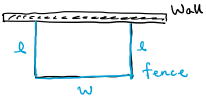
Concave down: $(-\infty, 0)$

Concave up: $(0, \infty)$

No inflection points.



3.



Total : 40 ft fencing material.

$2\ell + w = 40 \implies w = 40 - 2\ell.$

Maximize: Area $= \ell \cdot w.$

$A(\ell) = \ell \cdot (40 - 2\ell) = 40 \cdot \ell - 2\ell^2$

Strategy: find local maximum of $A(\ell)$

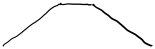
$A'(\ell) = 40 - 4\ell \overset{\text{set}}{=} 0$ to find crit. points

$4\ell = 40 \quad \ell = 10$

ℓ	0	5	10	20
$A'(\ell)$		+	0	−

$\ell = 5$: $A'(5) = 40 - 4 \cdot 5 = +$

$\ell = 15$: $A'(15) = 40 - 4 \cdot 15 = -$



local max $\implies$ global max on this interval $0 \le \ell \le 20$

Rectangle with most area: $\begin{cases} \ell = 10 \\ w = 20 \end{cases}$

4. Find the derivatives

(a) $y = \underbrace{x \cdot \ln x}_{f} \cdot \underbrace{\arccos(x)}_{g}$

$f' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$\frac{dy}{dx} = f' \cdot g + f \cdot g'$

$g' = \frac{-1}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = (\ln x + 1) \cdot \arccos x + x \cdot \ln x \cdot \left(\frac{-1}{\sqrt{1-x^2}}\right)$

(b) $y = \frac{4}{\sqrt{3x^2+2}} = 4 \cdot (3x^2+2)^{-\frac{1}{2}}$

$\frac{dy}{dx} = 4 \cdot \left(-\frac{1}{2}\right) \cdot (3x^2+2)^{-\frac{3}{2}} \cdot 6x$

$= -12x \cdot (3x^2+2)^{-\frac{3}{2}}$

(c) $y = \log_{x^2}(y+3) \iff (x^2)^y = y+3 \iff e^{\ln (x^2)^y} = e^{y+3}$

$\iff e^{y\cdot\ln(x^2)} = e^{y+3}$

Change of base for logs:

$$y = \frac{\ln(y+3)}{\ln x^2} \qquad t' = \frac{1}{y+3}\cdot\frac{dy}{dx}$$

$$= \frac{\ln(y+3)}{2\cdot\ln x} = \frac{t}{b} \qquad b' = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$$

$$\frac{dy}{dx} = \frac{b\cdot t' - t\cdot b'}{b^2} = \frac{(2\cdot\ln x)\cdot\frac{1}{y+3}\cdot\frac{dy}{dx} - \ln(y+3)\cdot\frac{2}{x}}{(2\cdot\ln x)^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{y+3}\cdot\frac{dy}{dx}}{2\cdot\ln x} - \frac{\ln(y+3)\cdot\frac{2}{x}}{(2\cdot\ln x)^2}$$

$$\frac{dy}{dx}\left(1 - \frac{1}{(y+3)\cdot 2\ln x}\right) = -\frac{\ln(y+3)\cdot 2}{x\cdot(2\cdot\ln x)^2}$$

$$\frac{dy}{dx} = \frac{-\left(\frac{\ln(y+3)\cdot 2}{x\cdot(2\ln x)^2}\right)}{\left(1 - \frac{1}{(y+3)\cdot 2\ln x}\right)}$$

(d) $\boxed{2y^2 = \ln\frac{2x}{\sqrt[5]{x^2-5}}}$ solve for y, plug in

$$2y^2 = \ln 2x - \ln(x^2-5)^{1/5}$$

$$2y^2 = \ln 2x - \frac{1}{5}\cdot\ln(x^2-5)$$

$$y^2 = \frac{1}{2}\ln\frac{2x}{\sqrt[5]{x^2-5}}$$

$$y = \sqrt{\frac{1}{2}\cdot\ln\frac{2x}{\sqrt[5]{x^2-5}}}$$

Implicit diff:

$$4\cdot y\cdot\frac{dy}{dx} = \frac{1}{\cancel{2}x}\cdot\cancel{2} - \frac{1}{5}\cdot\frac{1}{(x^2-5)}\cdot 2x$$

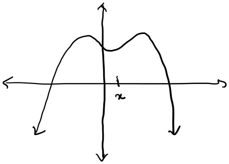
$$= \frac{1}{x} - \frac{2x}{5\cdot(x^2-5)}$$

$$\frac{dy}{dx} = \frac{1}{4y}\cdot\left(\frac{1}{x} - \frac{2x}{5\cdot(x^2-5)}\right)$$

5. $f(x) = 10 + 40x^3 - 3x^5$. When does tangent line have largest slope?

$S(x) = f'(x) = 120x^2 - 15x^4$

↳ optimize this!



$$S'(x) = 240x - 60x^3 \overset{set}{=} 0$$

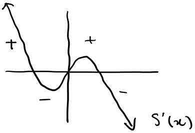
$$(240 - 60x^2)x = 0$$

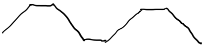
$x = 0$ or $240 - 60x^2 = 0$

$$4 - x^2 = 0$$

$$(2+x)(2-x) = 0 \qquad x = 2, -2.$$

x		-2		0		2	
$S'(x)$	$+$	0	$-$	0	$+$	0	$-$





graph of S looks like

S has 2 local maximum at $x=-2$, $x=2$

$$S(x) = 120x^2 - 15x^4$$

Compare $S(2) = 120(2)^2 - 15\cdot(2)^4$

|| equal

$$S(-2) = 120(-2)^2 - 15(-2)^4$$

$x = 2, -2$ is where f has largest slope.

6. $$\lim_{x\to\frac{\pi}{2}} \frac{\cot x}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{\cancel{\cos x}}{\sin x}}{\cancel{\cos x}}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{\sin x} = \frac{1}{\sin(\frac{\pi}{2})} = \frac{1}{1} = 1$$

$$\lim_{x\to\infty} \frac{2x\cdot\sqrt{x^2+1}}{\sqrt{x^4-1}} = \lim_{x\to\infty} \frac{2x\cdot\cancel{\sqrt{x^2+1}}}{\cancel{\sqrt{x^2+1}}\cdot\sqrt{x^2-1}}$$

$$= \lim_{x\to\infty} \frac{\overset{\to\infty}{2x}}{\underset{\to\infty}{\sqrt{x^2-1}}}$$

like x as $x\to\infty$
use squeeze thm

L'Hopital's Rule

$$= \lim_{x\to\infty} \frac{\cancel{2}}{\frac{1}{2}(x^2-1)^{-\frac{1}{2}}\cdot\cancel{2}x}$$

reciprocals

$$= \lim_{x\to\infty} \frac{(x^2-1)^{\frac{1}{2}}}{\frac{1}{2}\cdot x}$$

$$= \lim_{x\to\infty} \frac{2\cdot\sqrt{x^2-1}}{x}$$

$$\frac{\sqrt{x^2}-1}{x} \le \frac{\sqrt{x^2-1}}{x} \le \frac{\sqrt{x^2}}{x} \quad \text{for } x \ge 1.$$

$$\frac{\sqrt{x^2}-1}{x} = \frac{x-1}{x} \qquad \frac{\sqrt{x^2}}{x} = \frac{x}{x} = 1$$

① $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2}-1}{x} = \lim_{x\to\infty} \frac{x-1}{x} = 1$

② $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2}}{x} = 1$

$\Big\}$ Squeeze $\Longrightarrow$ Thm $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{x} = 1$

(c) $$\lim_{x\to 1} \frac{\overset{\to 0}{x-1}}{\underset{\to \sin(\pi) = 0}{\sin(\pi x)}}$$

Apply L'Hopital's Rule

$$= \lim_{x\to 1} \frac{1}{\cos(\pi x)\cdot \pi} \underset{\text{plug in}}{=} \frac{1}{\cos(\pi)\cdot\pi} = -\frac{1}{\pi} \qquad \cos(\pi) = -1,$$

7. Find antiderivatives:

$$f(x) = \frac{1}{(x+4)^3} + 2\csc x\cdot\cot x$$

$$= (x+4)^{-3} + 2\csc x\cdot\cot x \qquad x^{-3} \underset{\text{anti-der}}{\rightsquigarrow} \frac{x^{-2}}{-2}$$

$$F(x) = \frac{(x+4)^{-2}}{-2} - 2\cdot\csc x + C$$

anti-der.

$$g(x) = 6 + \sqrt{x+3} = 6 + (x+3)^{\frac{1}{2}} \qquad x^{\frac{1}{2}} \rightsquigarrow \frac{x^{3/2}}{3/2}$$

$$G(x) = 6x + \frac{(x+3)^{3/2}}{3/2} + C$$

Anti-der. of $(2x+3)^{\frac{1}{2}} \rightsquigarrow \frac{(2x+3)^{3/2}}{3/2}\cdot\frac{1}{2}$

OH: Tuesday 5-7pm, same Zoom link.