MAT 211: Linear Algebra

Practice problems

Stony Brook University Spring 2019
Dzmitry Dudko

Problem 1.
Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-2\end{bmatrix} \text{ and } T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}.$$

Find the standard matrix of T.

Answer: $\begin{bmatrix}-6/5 & 9/5\\1 & -1\end{bmatrix}$. □

Solution 1. Recall that we write a matrix linear transformation as $T_C(v) = Cv$. We need to find a matrix C such that

$$\begin{bmatrix}-1\\1\end{bmatrix} \overset{T_C}{\longmapsto} \begin{bmatrix}3\\-2\end{bmatrix}$$

$$\begin{bmatrix}3\\2\end{bmatrix} \overset{T_C}{\longmapsto} \begin{bmatrix}0\\1\end{bmatrix}.$$

Let us find matrices A, B such that

$$\begin{bmatrix}-1\\1\end{bmatrix} \overset{T_A}{\longmapsto} \begin{bmatrix}1\\0\end{bmatrix} \overset{T_B}{\longmapsto} \begin{bmatrix}3\\-2\end{bmatrix}$$

$$\begin{bmatrix}3\\2\end{bmatrix} \overset{T_A}{\longmapsto} \begin{bmatrix}0\\1\end{bmatrix} \overset{T_B}{\longmapsto} \begin{bmatrix}0\\1\end{bmatrix}.$$

Then $T_C(v) = T_B(T_A(v))) = BAv$ and the matrix C is equal to BA; more details in Syllabus, April 11 Lecture.

Since

$$\begin{bmatrix}1\\0\end{bmatrix} \overset{T_B}{\longmapsto} \begin{bmatrix}3\\-2\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \overset{T_B}{\longmapsto} \begin{bmatrix}0\\1\end{bmatrix},$$

$B = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}$. Since

$$\begin{bmatrix}-1\\1\end{bmatrix} \overset{T_{A^{-1}}}{\longleftarrow\!\shortmid} \begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}3\\2\end{bmatrix} \overset{T_{A^{-1}}}{\longleftarrow\!\shortmid} \begin{bmatrix}0\\1\end{bmatrix},$$

$A^{-1} = \begin{bmatrix}-1 & 3\\1 & 2\end{bmatrix}$ and $A = \begin{bmatrix}-1 & 3\\1 & 2\end{bmatrix}^{-1}$.

We have $C = BA = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{bmatrix} = \begin{bmatrix} -6/5 & 9/5 \\ 1 & -1 \end{bmatrix}$. □

Solution 2. Let us present $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Write

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solving this system we obtain $c_1 = -2/5$, $c_2 = 1/5$; i.e.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{-2}{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and we have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\frac{-2}{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \frac{-2}{5} T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + \frac{1}{5} T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) =$$

$$-2/5 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + 1/5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6/5 \\ 1 \end{bmatrix}.$$

Similarly, write

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solving this system we obtain $c_1 = 3/5$, $c_2 = 1/5$; i.e.

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and we have

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\frac{3}{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \frac{3}{5} T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + \frac{1}{5} T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) =$$

$$3/5 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + 1/5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9/5 \\ -1 \end{bmatrix}.$$

Since

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -6/5 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 9/5 \\ -1 \end{bmatrix},$$

$\begin{bmatrix} -6/5 & 9/5 \\ 1 & -1 \end{bmatrix}$ is the standard matrix of T. □

Problem 2.
Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & k \end{bmatrix},$$

where k is a real parameter.

1. Find the determinant of A and say for which values of k the matrix A is invertible.
2. Find the dimensions of $\text{null}(A)$ and $\text{col}(A)$ as k varies.
3. For $k = 4$,
   - find the eigenvalues of A;
   - find an eigenvector corresponding to the eigenvalue $\lambda = 5$.

.

Solution. 1. We have

$$\det \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & k \end{bmatrix} = 16 - 4k.$$

The matrix A is invertible if and only if $k \neq 4$.

2. If $k \neq 4$, then A is invertible, thus the dimension of $\text{col}(A)$ is 4 and the dimension of $\text{null}(A)$ is 0.

   Consider the case $k = 4$. Then

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}$$

   and we can check that the dimension of $\text{col}(A)$ is 3 and the dimension of $\text{null}(A)$ is 1.

3. For $k = 4$, we calculate the characteristic polynomial of A:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 & 2 \\ 0 & -\lambda & 2 & 0 \\ 0 & 2 & -\lambda & 0 \\ 2 & 0 & 0 & 4-\lambda \end{bmatrix} = (\lambda^2 - 5\lambda)(\lambda^2 - 4).$$

   Therefore, A has eigenvalues $5, 0, -2, 2$.

   For $\lambda = 5$, we have

$$A - \lambda I = \begin{bmatrix} -4 & 0 & 0 & 2 \\ 0 & -5 & 2 & 0 \\ 0 & 2 & -5 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix},$$

and we calculate that $\begin{bmatrix}1\\0\\0\\2\end{bmatrix}$ is an eigenvector corresponding to $\lambda = 5$.

$\square$

Problem 3. Consider the matrix

$$A = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}.$$

1. Find the eigenvalues and corresponding eigenspaces of A. Conclude that A is diagonalizable.

2. Write down a basis $\mathcal{B} = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ consisting of eigenvectors of A. Using this, find an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix.

3. Find the coordinates of $\begin{bmatrix}6\\-1\\-2\end{bmatrix}$ with respect to the basis $\mathcal{B}$; i.e. write $\begin{bmatrix}6\\-1\\-2\end{bmatrix}$ as a linear combination of v_1, v_2, and v_3.

4. Compute $A^{456}\begin{bmatrix}6\\-1\\-2\end{bmatrix}$.

Solution. 1. The eigenvalues of A are 0 and 3;

- span $\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$ is the eigenspace corresponding to 0, and $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}$ are two linearly independent eigenvectors corresponding to 0,
- span $\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$ is the eigenspace corresponding to 3, and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is an eigenvector corresponding to 3.

2. $\mathcal{B} = \left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is a basis consisting of eigenvectors of A. Set

$$S = \begin{bmatrix}1 & 1 & 1\\-1 & 0 & 1\\0 & -1 & 1\end{bmatrix},$$

then $S^{-1}AS$ is a diagonal matrix.

3. Solving the system

$$\begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we obtain $c_1 = 2, c_2 = 3, c_3 = 1$; i.e.:

$$\begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

4. We have:

$$A^{456} \begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix} = A^{456} \left(2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) =$$

$$0^{456} \times 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 0^{456} \times 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3^{456} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3^{456} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

□

Problem 4.

Write the matrix representing the linear transformation T on $\mathbb{R}^2$ that reflects vectors about the line $y = x$. Is it invertible? What about diagonalizability?

Solution. We can compute that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the standard matrix of T.

The transformation T is invertible.

The transformation T has two linearly independent eigenvectors, thus T is diagonalizable. □

Problem 5.

Consider the vector subspace $W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$. Find the projection of $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ onto W and onto $W^\perp$.

Solution. We have:

$$\text{proj}_W \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \frac{\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{4+4+6}{1+4+9} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

and:

$$\text{proj}_{W^\perp}\begin{bmatrix}4\\2\\2\end{bmatrix} = \text{perp}_W\begin{bmatrix}4\\2\\2\end{bmatrix} = \begin{bmatrix}4\\2\\2\end{bmatrix} - \text{proj}_W\begin{bmatrix}4\\2\\2\end{bmatrix} = \begin{bmatrix}4\\2\\2\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}3\\0\\-1\end{bmatrix}.$$

□

Problem 6.
Find all a, b, c such that the following matrices are simultaneously non-invertible

$$\begin{bmatrix}a-4 & -2\\ b & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 1\\ 4-a-c & a+b\end{bmatrix}, \quad \begin{bmatrix}1 & \frac{1}{2}-c\\ 2 & a+b\end{bmatrix}.$$

Answer: $a = 2, b = 1, c = -1$. □

Solution. The matrices are non-invertible if and only if their determinants are zero. We need to solve the system:

$$\det\begin{bmatrix}a-4 & -2\\ b & 1\end{bmatrix} = 0$$

$$\det\begin{bmatrix}1 & 1\\ 4-a-c & a+b\end{bmatrix} = 0$$

$$\det\begin{bmatrix}1 & \frac{1}{2}-c\\ 2 & a+b\end{bmatrix} = 0$$

or:

$$a - 4 - (-2)b = 0$$

$$a + b - (4 - a - c) = 0$$

$$a + b - 2\left(\frac{1}{2} - c\right) = 0$$

or:

$$a + 2b = 4$$

$$2a + b + c = 4$$

$$a + b + 2c = 1$$

or:

$$\left[\begin{array}{ccc|c}1 & 2 & 0 & 4\\ 2 & 1 & 1 & 4\\ 1 & 1 & 2 & 1\end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c}1 & 2 & 0 & 4\\ 0 & -3 & 1 & -4\\ 0 & -1 & 2 & -3\end{array}\right] \xrightarrow{R_2 - 3R_3} \left[\begin{array}{ccc|c}1 & 2 & 0 & 4\\ 0 & 0 & -5 & 5\\ 0 & -1 & 2 & -3\end{array}\right]$$

$$\xrightarrow[-R_3]{R_2/(-5)} \left[\begin{array}{ccc|c}1 & 2 & 0 & 4\\ 0 & 0 & 1 & -1\\ 0 & 1 & -2 & 3\end{array}\right] \xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc|c}1 & 2 & 0 & 4\\ 0 & 0 & 1 & -1\\ 0 & 1 & 0 & 1\end{array}\right] \xrightarrow{R_1 - 2R_3} \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 0 & 1 & -1\\ 0 & 1 & 0 & 1\end{array}\right].$$

Therefore, $a = 2, b = 1, c = -1$. □

Problem 7.
Suppose v_1, v_2, v_3 are linearly independent vectors.

1. Find all scalars a and b such that
$$2av_1 - v_2 = v_1 + bv_2.$$
2. Find all scalars k such that the vectors
$$v_1 + 3v_3, \quad 2v_1 + kv_3, \quad 2v_2$$
are linearly dependent.

Solution. 1. We have:
$$(2a-1)v_1 + (-1-b)v_2 = 0.$$
Since v_1, v_2 are linearly independent, we obtain $2a-1=0$ and $-1-b=0$. Answer: $a=\frac{1}{2}, b=-1$.

2. The vectors $v_1+3v_3, \ \ 2v_1+kv_3, \ \ 2v_2$ are linearly independent if and only if there are scalars c_1, c_2, c_3, at least one of which is not zero, such that
$$c_1(v_1+3v_3) + c_2(2v_1+kv_3) + c_3(2v_2) = 0,$$
or:
$$(c_1+2c_2)v_1 + (2c_3)v_2 + (3c_1+kc_2)v_3 = 0.$$
Since v_1, v_2, v_3 are linearly independent, $(c_1+2c_2)v_1+(2c_3)v_2+(3c_1+kc_2)v_3$ is equal to 0 if and only if
$$c_1 + 2c_2 = 0$$
$$2c_3 = 0$$
$$3c_1 + kc_2 = 0.$$
We need to find all k such that the last system has a non-zero solution. Using the determinant test, we have:
$$\det\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 2\\ 3 & k & 0\end{bmatrix} = 0,$$
we obtain that $k=6$.

□

Problem 8. Find an orthogonal basis of span $\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right)$.

Solution. Since $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$ are linearly independent, $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}$ form a basis of $W = \text{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right)$.

The vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix} - t\begin{bmatrix}1\\0\\1\end{bmatrix}$ also form a basis for W, and we need to find t so that $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix} - t\begin{bmatrix}1\\0\\1\end{bmatrix}$ are orthogonal vectors.

We have:

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \left(\begin{bmatrix}1\\0\\2\end{bmatrix} - t\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = 0,$$

or: $3 - 2t = 0$. Thus $t = 3/2$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1/3\\0\\1/3\end{bmatrix}$ is an orthogonal basis for W. □