MAT 211: Linear Algebra

Practice Midterm 1

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Problem 1. Check if $\begin{bmatrix}1\\2\\4\end{bmatrix}$ and $\begin{bmatrix}0\\6\\-3\end{bmatrix}$ are orthogonal vectors.

Answer: Yes.

Solution. The vectors are orthogonal because their dot product is zero:

$$\begin{bmatrix}1\\2\\4\end{bmatrix} \cdot \begin{bmatrix}0\\6\\-3\end{bmatrix} = 1 \cdot 0 + 2 \cdot 6 + 4 \cdot (-3) = 0.$$

□

Problem 2. In the following problems compute $u \cdot v$.

1) $u = \begin{bmatrix}1\\2\\3\end{bmatrix}, \qquad v = \begin{bmatrix}2\\-3\\1\end{bmatrix}$

2) $u = \begin{bmatrix}1\\x\\3\end{bmatrix}, \qquad v = \begin{bmatrix}2\\3\\y\end{bmatrix}$

3) $u = \begin{bmatrix}x\\2\\-3\end{bmatrix}, \qquad v = \begin{bmatrix}3\\4\\x\end{bmatrix}$

Solution: 1) $\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}2\\-3\\1\end{bmatrix} = 1 \cdot 2 + 2 \cdot (-3) + 3 \cdot 1 = -1$

2) $\begin{bmatrix}1\\x\\3\end{bmatrix} \cdot \begin{bmatrix}2\\3\\y\end{bmatrix} = 1 \cdot 2 + x \cdot 3 + 3 \cdot y = 2 + 3x + 3y$

3) $u = \begin{bmatrix}x\\2\\-3\end{bmatrix} \cdot \begin{bmatrix}3\\4\\x\end{bmatrix} = x \cdot 3 + 2 \cdot 4 + (-3) \cdot x = 8.$

□

Problem 3. In the following problems find all k such that u and v are parallel vectors.

1) $u = \begin{bmatrix} k \\ 2 \end{bmatrix}, \qquad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2) $u = \begin{bmatrix} k \\ 1 \\ k \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3) $u = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$

Answer: 1) $k = 4$, 2) no solution, 3) k is any.

Solution. Since u, v are non-zero, u, v are parallel if and only if $u = cv$ for some scalar c, equivalently if the coordinates of u and v are proportional. We have:

1) $k/2 = 2/1$, or $k = 4$

2) $k/1 = 1/2 = k/3$ – no solution.

3) $k/1 = -k/(-1)$; i.e. k is any.

□

Problem 4.

1) Find the general and parametric equations of the line passing through the points $(3, 1)$ and $(1, 0)$.

2) Are the points $(2, 1)$, $(1, 2)$, and $(4, -1)$ on the same line? If yes, find the general and parametric equations of the line passing through these points.

3) Find the parametric equation of the line passing through $(1, 1)$ and $(0, x)$.

Solution. 1) Write $A = (1, 0)$ and $B = (3, 1)$, and $O = (0, 0)$ – the origin. Then $\overrightarrow{AB} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a direction vector of the line. Also $\overrightarrow{OB} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

The point $X = (x, y)$ is on the line if and only if $\overrightarrow{BX}$ is parallel to $\overrightarrow{AB}$; equivalently if $\overrightarrow{BX} = t\overrightarrow{AB}$. Since $\overrightarrow{BX} = \overrightarrow{OX} - \overrightarrow{OB}$, we have

$$\overrightarrow{OX} = \overrightarrow{OB} + t\overrightarrow{AB}$$

or:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This is the vector form of the equation of the line. The parametric equation:

$$x = 3 + 2t$$
$$y = 1 + t.$$

Observe that $n = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is a normal vector, because n is orthogonal to $\overrightarrow{AB}$:

$$n \cdot \overrightarrow{AB} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (-1) \cdot 2 + 2 \cdot 1 = 0.$$

Remark: $\begin{bmatrix} -b \\ a \end{bmatrix}$ is always orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$:

$$\begin{bmatrix} -b \\ a \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = -ba + ab = 0.$$

Therefore, $X = (x, y)$ is on the line if and only if $\overrightarrow{BX}$ is orthogonal to n, equivalently:

$$n \cdot (\underbrace{\overrightarrow{OX} - \overrightarrow{OB}}_{\overrightarrow{BX}}) = 0 \quad \text{or} \quad n \cdot \overrightarrow{OX} = n \cdot \overrightarrow{OB},$$

or

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

This is the normal form of the equation of the line. The general form of the equation of the line is obtained by evaluating the dot product:

$$-x + 2y = -1.$$

2) Write $A = (2, 1)$, $B = (1, 2)$, $C = (4, -1)$, then $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel (because $-1/2 = 1/(-2)$), the points A, B, and C are on the same line.
Since $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a direction vector, the vector form of the equation is

$$\overrightarrow{OX} = \overrightarrow{OA} + t\overrightarrow{AB} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The parametric equation:

$$\begin{aligned} x &= 2 - t \\ y &= 1 + t. \end{aligned}$$

Observe that $n = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a normal vector because $n \cdot \overrightarrow{AB} = 0$. The general form of the equation is obtained by evaluating the dot product in $n \cdot \overrightarrow{OX} = n \cdot \overrightarrow{OB}$:

$$x + y = 3.$$

(We can double check that $C = (4, -1)$ is on the line passing through A and B: $4 + (-1) = 3$.)

3) Using x_1, x_2-axes, we have:

$$x_1 = 0 + t$$
$$x_2 = x + t(1 - x).$$

$\square$

Problem 5. Find the general and parametric equations of the plane passing through the point $(1,1,1)$ and orthogonal to the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Solution. Write $A = (1,1,1)$. Since the vector $n = \begin{bmatrix}1\\1\\1\end{bmatrix}$ is normal, the general form of the equation is obtained by evaluating the dot product in $n \cdot \overrightarrow{OX} = n \cdot \overrightarrow{OA}$:

$$x + y + z = 3.$$

To find the parametric equation, it is sufficient to solve the equation $x + y + z = 3$. We set $z = t, y = s$, then $x = 3 - s - t$; i.e.:

$$x = 3 - s - t$$
$$y = s$$
$$z = t.$$

$\square$

Problem 6. Find the general and parametric equations of the plane passing through the points $(0,1,2)$, $(1,0,1)$, and $(2,1,4)$. Does the plane also pass through the origin $(0,0,0)$?

Solution. Write $A = (0,1,2)$, $B = (1,0,1)$, $C = (2,1,4)$, and $O = (0,0,0)$ – the origin. Then $\overrightarrow{AB} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix}2\\0\\2\end{bmatrix}$ are two non-parallel direction vectors of the plane. Also $\overrightarrow{OB} = \begin{bmatrix}1\\0\\1\end{bmatrix}$. The vector form of the equation is

$$\overrightarrow{OX} = \overrightarrow{OB} + s\overrightarrow{AB} + t\overrightarrow{AC},$$

or:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} + s\begin{bmatrix}1\\-1\\-1\end{bmatrix} + t\begin{bmatrix}2\\0\\2\end{bmatrix}.$$

The parametric equation:

$$x = 1 + s + 2t$$
$$y = -s$$
$$z = 1 - s + 2t.$$

Next we need to find a normal vector $n = \begin{bmatrix} n_1 \\ n_2 \\ n_3. \end{bmatrix}$ of the plane. Such vector n is orthogonal to the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Therefore, n satisfies $n \cdot \overrightarrow{AB} = 0$ and $n \cdot \overrightarrow{AC} = 0$, or

$$n_1 - n_2 - n_3 = 0$$

$$2n_1 \qquad + 2n_3 = 0$$

This system has infinitely many solutions, $n_1 = 1, n_2 = 2, n_3 = -1$ is one of non-zero solutions. The normal form of the equation is obtained by evaluating the dot product in $n \cdot \overrightarrow{OX} = n \cdot \overrightarrow{OB}$:

$$x + 2y - z = 0.$$

Since $(0, 0, 0)$ satisfies $x + 2y - z = 0$, the plane passes through the origin $O = (0, 0, 0)$.

□

Problem 7. Solve the following system of linear equations:

$$\begin{aligned} x + 2y - 3z &= 9, \\ 2x - y + z &= 0, \\ 4x - y + z &= 4. \end{aligned}$$

Answer $x = 2, y = 5, z = 1$.

Solution 1. We have:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array}\right] \xrightarrow{R_2/(-5)}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -1.4 & 3.6 \\ 0 & -9 & 13 & -32 \end{array}\right] \xrightarrow[R_3 + 9R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -0.2 & 1.8 \\ 0 & 1 & -1.4 & 3.6 \\ 0 & 0 & 0.4 & 0.4 \end{array}\right] \xrightarrow{R_3/0.4} \left[\begin{array}{ccc|c} 1 & 0 & -0.2 & 1.8 \\ 0 & 1 & -1.4 & 3.6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + 1.4R_3]{R_1 + 0.2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

□

Solution 2. To simplify calculations, let us first subtract the second equation from the last equation:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 2 & 0 & 0 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$
\left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & -3 & 9 \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & -3 & 9 \end{array}\right] \xrightarrow{\substack{R_2-2R_1 \\ R_3-R_1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 2 & -3 & 7 \end{array}\right]
$$

$$
\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 2 & -3 & 7 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{-R_3}
$$

$$
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right].
$$

□

Problem 8. Solve the following system of linear equations:

$$
\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right]
$$

Solution. We have:

$$
\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \xrightarrow{\substack{R_2-2R_1 \\ R_3-R_1}}
$$

$$
\left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right] \xrightarrow{\substack{R_2/(-3) \\ R_3/2}} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{\substack{R_1-2R_2 \\ R_3-R_2}}
$$

$$
\left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

The solution: $x_1 = -2 + 3t, x_2 = t, x_3 = 1 + 2s, x_4 = s$.

□

Problem 9. Determine if the following vectors are linearly independent

$$
\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}.
$$

Solution. We need to check if the system

$$
c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

has a non-trivial solution; i.e. when at least one of c_1, c_2, c_3 is not zero. We have:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 1 & -5 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -3 & -9 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right] \xrightarrow{R_2/(-3)}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right] \xrightarrow[R_3 + R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

□

Therefore, the system has a non-trivial solution, for example $c_1 = 4, c_2 = -3, c_3 = 1$:

$$4\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 10. Find all k such that the following vectors are linearly independent

$$\begin{bmatrix} 2k \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Answer: $k \neq 1/2$.

Solution. Two vectors $\begin{bmatrix} 2k \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent if and only if they are not parallel. The vectors $\begin{bmatrix} 2k \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are parallel if and only if $k = 1/2$.

□

Problem 11. Check if the span of the following vectors is $\mathbb{R}^3$.

$$\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix}.$$

Answer: yes.

Solution. The span of three vectors is $\mathbb{R}^3$ if and only if the vectors are linearly independent. Let us check if the vectors are linearly independent.

We need to check if the system

$$c_1\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix} + c_3\begin{bmatrix} 7 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution; i.e. when at least one of c_1, c_2, c_3 is not zero.

We have:

$$\left[\begin{array}{ccc|c} 2 & 1 & 7 & 0 \\ 3 & -4 & 6 & 0 \\ 3 & -2 & 0 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 1 & 7 & 0 \\ 1 & -5 & -1 & 0 \\ 1 & -3 & -7 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -7 & 0 \\ 1 & -5 & -1 & 0 \\ 2 & 1 & 7 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -3 & -7 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 7 & 21 & 0 \end{array}\right] \xrightarrow{R_2/(-2)} \left[\begin{array}{ccc|c} 1 & -3 & -7 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 7 & 21 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - 7R_2]{R_1 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 42 & 0 \end{array}\right] \xrightarrow{R_3/42} \left[\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 + 3R_3]{R_1 + 16R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The last system has only the trivial solution: $c_1 = 0, c_2 = 0, c_3 = 0$. □

Problem 12. Calculate the product

$$\begin{bmatrix} 2k & 1 \\ 1 & k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 2k & 1 \\ 1 & k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2k-1 & -2k+1 \\ 1-k & -1+k \end{bmatrix}.$$

□