MAT 627, Spring 2025, Stony Brook University

Topics in Complex Analysis: Quasiconformal Mappings

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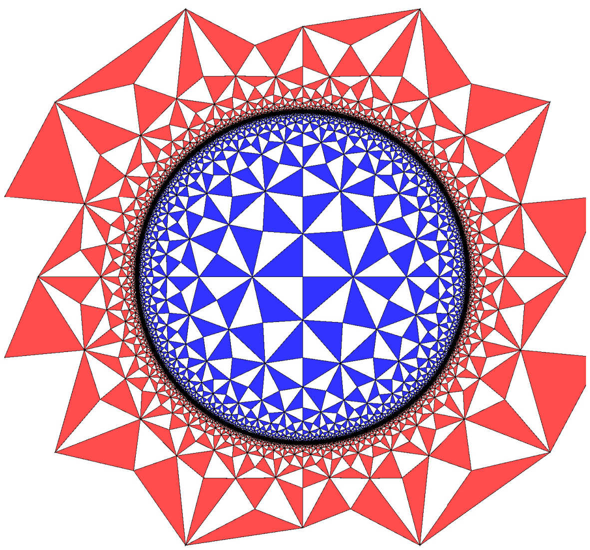
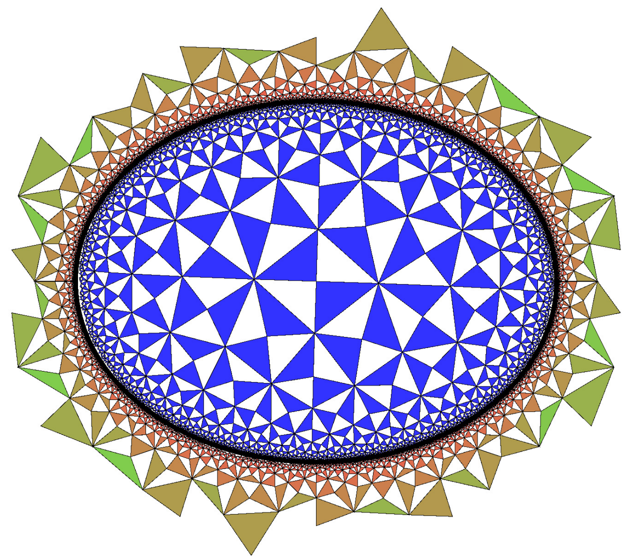

This semester I hope to cover the following topics:

- Review of conformal mappings
- Extremal length and conformal modulus
- Definitions of quasiconformal mappings; geometric and analytic
- Basic properties
- Quasisymmetric maps and boundary extension
- The measurable Riemann mapping theorem
- Removable sets
- Conformal welding
- David maps
- Astala's theorems on area and dimension distortion
- Quasiconformal maps on metric spaces
- Conformal dimension

Extremal Length

Consider a positive function ρ on a domain Ω. We think of ρ as analogous to $|f'|$ where f is a conformal map on Ω.

Just as the image area of a set E can be computed by integrating $\int_E |f'|^2 dxdy$, we can use ρ to define areas by $\int_E \rho^2 dxdy$.

Similarly, we can define $\ell(f(\gamma)) = \int_\gamma |f'(z)|ds$, we can define the ρ-length of a curve γ by $\int_\gamma \rho ds$.

We need γ to be locally rectifiable (so the arclength measure ds is defined) and it is convenient to assume that ρ is Borel (so that its restriction to any curve γ is also Borel and hence measurable for length measure on γ).

Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω.

We say ρ is **admissible** for Γ if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1.$$

In this case we write $\rho \in \mathcal{A}(\Gamma)$.

We define the **modulus** of the path family Γ as

$$\mathrm{Mod}(\Gamma) = \inf_{\rho} \int_M \rho^2 dx dy,$$

where the infimum is over all admissible ρ for Γ.

The **extremal length** of Γ is defined as $\lambda(\Gamma) = 1/M(\Gamma)$.

Note that if the path family Γ is contained in a domain Ω, then we need only consider metrics ρ are zero outside Ω.

Otherwise, we can define a new (smaller) metric by setting $\rho = 0$ outside Ω; the new metric is still admissible, and a smaller integral than before.

Therefore $M(\Gamma)$ can be computed as the infimum over metrics which are only nonzero inside Ω.

Modulus and extremal length satisfy several useful properties that we list as a series of lemmas.

Lemma 2.1 (Conformal invariance)**.** *If* Γ *is a family of curves in a domain* Ω *and* f *is a one-to-one holomorphic mapping from* Ω *to* Ω' *then* $M(\Gamma) = M(f(\Gamma))$.

Lemma 2.1 (Conformal invariance). *If* Γ *is a family of curves in a domain* Ω *and* f *is a one-to-one holomorphic mapping from* Ω *to* Ω' *then* $M(\Gamma) = M(f(\Gamma))$.

Proof. This is just the change of variables formulas

$$\int_{\gamma} \rho \circ f |f'| ds = \int_{f(\gamma)} \rho ds,$$
$$\int_{\Omega} (\rho \circ f)^2 |f'|^2 dx dy = \int_{f(\Omega)} \rho dx dy.$$

These imply that if $\rho \in \mathcal{A}(f(\Gamma))$ then $|f'| \cdot \rho \circ f \in \mathcal{A}(f(\Gamma))$, and thus by taking the infimum over such metrics we get $M(f(\Gamma)) \leq M(\Gamma)$

There might be admissible metrics for $f(\Gamma)$ that are not of this form, possibly giving a strictly smaller modulus. However, by switching the roles of Ω and Ω' and replacing f by f^{-1} we see equality does indeed hold. $\square$

Lemma 2.2 (Monotonicity)**.** *If* Γ_0 *and* Γ_1 *are path families such that every* $\gamma \in \Gamma_0$ *contains some curve in* Γ_1 *then* $M(\Gamma_0) \leq M(\Gamma_1)$ *and* $\lambda(\Gamma_0) \geq \lambda(\Gamma_1)$*.*

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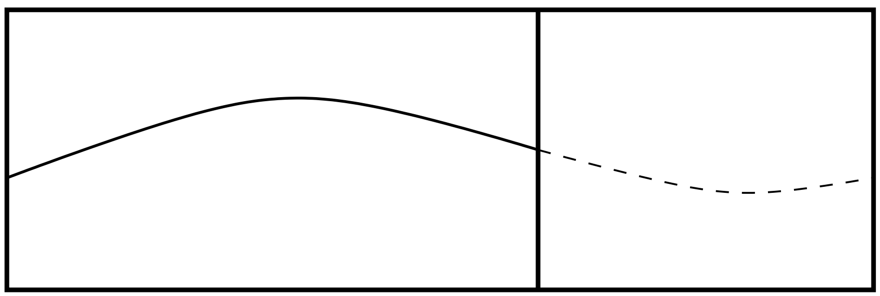
Proof. The proof is immediate since $\mathcal{A}(\Gamma_0) \supset \mathcal{A}(\Gamma_1)$. □



Lemma 2.3 (Grötsch Principle). *If* Γ_0 *and* Γ_1 *are families of curves in disjoint domains then* $M(\Gamma_0 \cup \Gamma_1) = M(\Gamma_0) + M(\Gamma_1)$.

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Proof. Suppose ρ_0 and ρ_1 are admissible for Γ_0 and Γ_1. Take $\rho = \rho_0$ and $\rho = \rho_1$ in their respective domains.

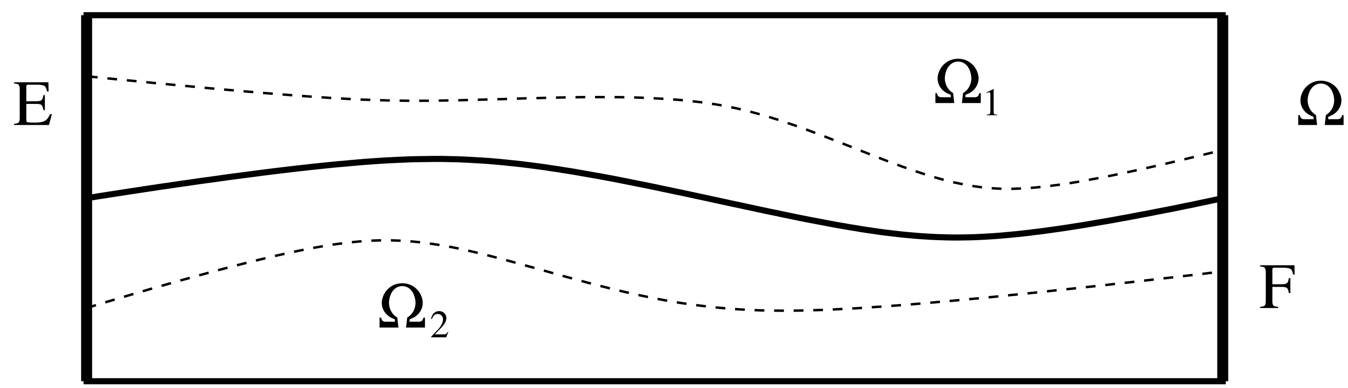
Then it is easy to check that ρ is admissible for $\Gamma_0 \cup \Gamma_1$ and, since the domains are disjoint, $\int \rho^2 = \int \rho_1^2 + \int \rho_2^2$.

Thus $M(\Gamma_0 \cup \Gamma_1) \leq M(\Gamma_0) + M(\Gamma_1)$. By restricting an admissible metric ρ for $\Gamma_0 \cup \Gamma_1$ to each domain, a similar argument proves the other direction. □

Corollary 2.4 (Parallel Rule)**.** *Suppose* Γ_0 *and* Γ_1 *are path families in disjoint domains* $\Omega_0, \Omega_1 \subset \Omega$ *that connect disjoint sets* E, F *in* $\partial\Omega$*. If* Γ *is the path family connecting* E *and* F *in* Ω*, then*

$$M(\Gamma) \geq M(\Gamma_0) + M(\Gamma_1).$$

Proof. Combine the Grötsch principle and the monotonicity principle. □



Lemma 2.5 (Series Rule)**.** *If* Γ_0 *and* Γ_1 *are families of curves in disjoint domains and every curve of* $\mathcal{F}$ *contains both a curve from both* Γ_0 *and* Γ_1*, then* $\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1)$*.*

Proof. If $\rho_j \in \mathcal{A}(\Gamma_j)$ for $j = 0, 1$, then $\rho_t = (1-t)\rho_0 + t\rho_1$ is admissible for Γ.

Since the domains are disjoint we may assume $\rho_0\rho_1 = 0$.

Integrating ρ^2 then shows

$$M(\Gamma) \leq (1-t)^2 M(\Gamma_0) + t^2 M(\Gamma_1),$$

for each t.

To find the optimal t set $a = M(\Gamma_1)$, $b = M(\Gamma_0)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

$$M(\mathcal{F}) \ \leq \ t^2 a + (1-t^2)b = \frac{b^2 a a^2 b}{(a+b)^2} = \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a}+\frac{1}{b}}$$

or

$$\frac{1}{M(\Gamma)} \geq \frac{1}{M(\Gamma_0)} + \frac{1}{M(\Gamma_1)},$$

which, by definition, is the same as

$$\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1). \quad \square$$

The fundamental example is to compute the modulus of the path family connecting opposite sides of a $a \times b$ rectangle; this serves as the model of almost all modulus estimates.

So suppose $R = [0, b] \times [0, a]$ is a b wide and a high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a.

Lemma 2.6. $\text{Mod}(\Gamma) = a/b$.

Proof. Each curve in Γ has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dx dy = \frac{1}{b^2} ab = \frac{a}{b}.$$

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fg dx)^2 \leq (\int f^2 dx)(\int g^2 dx).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible,

$$\int_0^b \rho(x,y)dx \geq 1,$$

and so by Cauchy-Schwarz

$$1 \leq \int_0^b (1 \cdot \rho(x,y))dx \leq \int_0^b 1^2 dx \cdot \int_0^b \rho^2(x,y)dx.$$

Now integrate with respect to y to get

$$a = \int_0^a 1dy \leq b \int_0^a \int_0^b \rho^2(x,y)dxdy,$$

or

$$\frac{a}{b} \leq \iint_R \rho^2 dxdy,$$

which implies $\mathrm{Mod}(\Gamma) \geq \frac{b}{a}$. Thus $\mathrm{Mod}(\Gamma) = \frac{b}{a}$. $\square$

Lemma 2.7. *If* $A = \{z : r < |z| < R\}$ *then the modulus of the path family connecting the two boundary components is* $2\pi / \log \frac{R}{r}$.

More generally, if Γ is the family of paths connecting $r\mathbb{T} = \{|z| = r\}$ to a set $E \subset R\mathbb{T} = \{|z| = R\}$, then $M(\Gamma) \geq |E| / \log \frac{R}{r}$.

Proof. By conformal invariance, we can rescale and assume $r = 1$. Suppose ρ is admissible for Γ. Then for each $z \in E \subset \mathbb{T}$,

$$1 \leq (\int_1^R \rho ds)^2 \leq (\int_1^R \frac{ds}{s})(\int_1^R \rho^2 s ds) = \log R \int_1^R \rho^2 s ds$$

and hence we get

$$\int_0^{2\pi} \int_1^R \rho^2 s ds d\theta \geq \int_E \int_1^R \rho^2 s ds d\theta \geq |E| \int_1^R \rho^2 s ds \geq \frac{|E|}{\log R}.$$

When $E = \mathbb{T}$ we prove the other direction by taking $\rho = (s \log R)^{-1}$. This is an admissible metric and

$$\mathrm{Mod}(\Gamma) \leq \int_0^{2\pi} \int_1^R \rho^2 s ds d\theta = \frac{2\pi}{(\log R)^2} \int_1^R \frac{1}{s} ds = \frac{2\pi}{\log R}. \quad \square$$

Given a Jordan domain Ω and two disjoint closed sets $E, F \subset \partial\Omega$, the **extremal distance** between E and F (in Ω) is the extremal length of the path family in Ω connecting E to F (paths in Ω that have one endpoint in E and one endpoint in F).

The series rule is a sort of "reverse triangle inequality" for extremal distance.

The series rule says that the extremal distance from X to Z in the rectangle is greater than the sum the extremal distance from X to Y in Ω_1 plus the extremal distance from Y to Z in Ω_2.

Extremal distance can be particularly useful when both E and F are connected.

If so, their complement in $\partial\Omega$ also consists of two arcs, and the extremal distance between these is the reciprocal of the extremal distance between E and F.

This holds because of conformal invariance, and the fact that it is true for rectangles.

(We can conformally map Ω to some rectangle, so that E and F go to opposite sides; this follows from the Schwarz-Christoffel formula.)

Obtaining an upper bound for the modulus of a path family usually involves choosing a metric; every metric gives an upper bound.

Giving a lower bound usually involves a Cauchy-Schwarz type argument, which can be harder to do in general cases. However, in the special case of extremal distance between arcs $E, F \subset \partial\Omega$, a lower bound for the modulus can also be computed by giving a upper bound for the reciprocal separating family.

Thus estimates of both types can be given by producing metrics (for different families) and this is often the easiest thing to do.

Lemma 2.8 (Points are removable)**.** *Suppose Q is a quadrilateral with opposite sides E, F and that Γ is the path family in Q connecting E and F. If $z \in \Omega$, let $\Gamma_0 \subset \Gamma$ be the paths that do not contain z. Then* $\mod(\Gamma_0) = \mod(\Gamma)$.

This will be useful later, when we want to prove that quasiconformal map of a punctured disk is actually quasiconformal on the whole disk. The point can be replaced by larger sets.

Proof. Since $\Gamma_0 \subset \Gamma$ we have $\mod(\Gamma_0) \leq \mod(\Gamma)$ by monotonicity.

To prove the other direction we claim that any metric that is admissible for Γ_0 is also admissible for Γ.

Suppose ρ is not admissible for Γ. Then there is a $\gamma \in \Gamma$ so that $\int_\gamma \rho ds < 1 - \epsilon$.

Choose a small $r > 0$ so $D(z, r) \subset \Omega$ and note that by Cauchy-Schwarz

$$(\int_0^r [\int_0^{2\pi} \rho t d\theta] dt)^2 \leq \pi r^2 \int_{D(z,r)} \rho^2 dxdy = o(r^2).$$

Here we have used the fact that since ρ^2 is integrable on Q, we have $\int_{D(z,r)} \rho^2 dxdy \to 0$ as $r \searrow 0$ (see Folland's book).

Hence

$$\int_0^r [\int_{C_t} \rho ds] dt = \int_0^r \ell_\rho(C_t) dt = o(r),$$

where C_t is the circle of radius t around z.

Thus we can find arbitrarily small circles centered at z whose ρ-length is less than ϵ. Then for the path γ chosen above, replace it by a path that follows γ from E to the first time it hits C_t, then follows an arc of C_t, and then follows γ from the last time it hits C_t to to F.

This path is in Γ_0 but its ρ-length is at most the ρ-length of γ plus the ρ-length of C_t, and this sum is less than 1. Thus ρ is also not admissible for Γ_0. This proves the claim and the lemma. $\square$

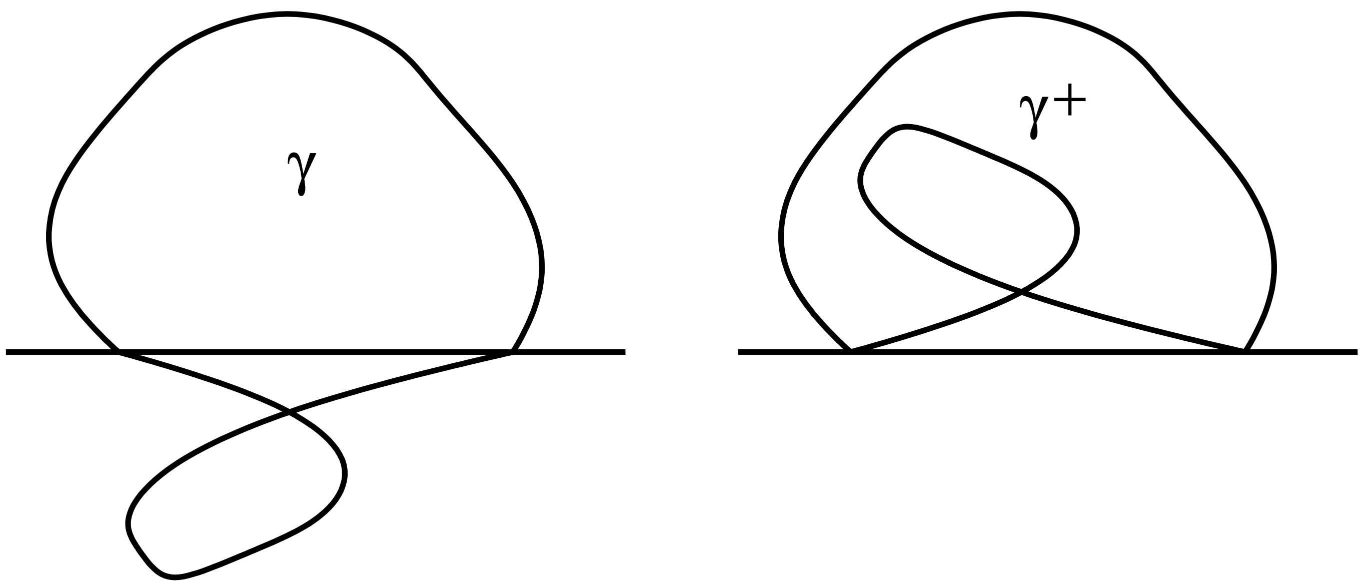
Extremal length, symmetry and Koebe's 1/4-theorem

If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let

$$\gamma_u = \gamma \cap \mathbb{H}, \quad \gamma_\ell = \gamma \cap \mathbb{H}_l, \quad \gamma_+ = \gamma_u \cup \overline{\gamma_\ell},$$

where $\mathbb{H} = \{x + iy : y > 0\}$, $\mathbb{H}_l = \{x + iy : y < 0\}$ denote the upper and lower half-planes.

For a path family Γ, define $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma_+ = \{\gamma_+ : \gamma \in \Gamma\}$.



Lemma 2.9 (Symmetry Rule)**.** *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma_+)$.

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Proof. We start by proving $M(\Gamma) \leq 2M(\Gamma_+)$.

Given a metric ρ admissible for γ_+, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$.

Then for any $\gamma \in \Gamma$,

$$\begin{aligned}
\int_\gamma \sigma ds &= \int_{\gamma_u} \sigma(z) ds + \int_{\gamma_\ell} \sigma(z) ds \\
&\geq \int_{\gamma_u} \rho(z) ds + \int_{\gamma_\ell} \rho(\bar{z}) ds \\
&= \int_{\gamma_u} \rho(z) ds + \int_{\overline{\gamma_\ell}} \rho(z) ds \geq \int_{\gamma_+} \rho ds \geq \inf_{\gamma \in \Gamma} \int_\gamma \rho ds.
\end{aligned}$$

Thus if ρ admissible for Γ_+, then σ is admissible for Γ.

Since $\max(a,b)^2 \leq a^2 + b^2$, integrating gives

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z) dxdy + \int \rho^2(\bar{z}) dxdy \leq 2 \int \rho^2(z) dxdy.$$

Taking the infimum over admissible ρ's for Γ_+ makes the right hand side equal to $2M(\Gamma_+)$, proving $\mathrm{Mod}(\Gamma) \leq 2\mathrm{Mod}(\Gamma_+)$.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}$ and $\sigma = 0$ if $z \in \mathbb{H}_l$. Then

$$\begin{aligned} \int_{\gamma_+} \sigma ds &= \int_{\gamma_+} \rho(z) + \rho(\bar{z}) ds \\ &= \int_{\gamma_u} \rho(z) ds + \int_{\gamma_u} \rho(\bar{z}) ds + \int_{\gamma_e ll} \rho(z) + \int_{\gamma_\ell} \rho(\bar{z}) ds \\ &= \int_{\gamma} \rho(z) ds + \int_{\overline{\gamma}} \rho(\bar{z}) ds \\ &= 2 \inf_{\rho} \int_{\gamma} \rho ds. \end{aligned}$$

Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ_+.

Since $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\begin{aligned} M(\Gamma_+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy \\ &= \frac{1}{4}\int_{\mathbb{H}} (\rho(z)+\rho(\bar{z}))^2 dxdy \\ &\leq \frac{1}{2}\int_{\mathbb{H}} \rho^2(z)dxdy + \int_{\mathbb{H}} \rho^2(\bar{z})dxdy \\ &= \frac{1}{2}\int \rho^2 dxdy. \end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. $\square$

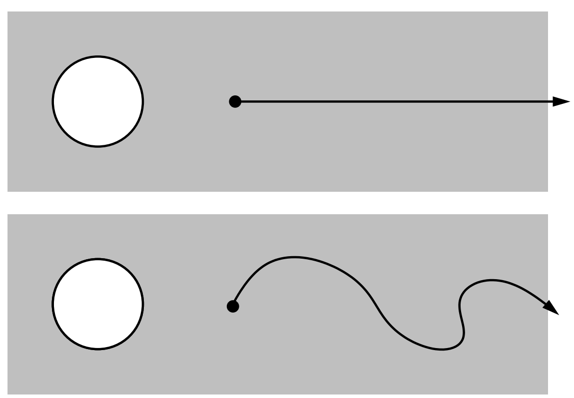
Lemma 2.10. *Let* $\mathbb{D}^* = \{z : |z| > 1\}$ *and* $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ *for some* $R > 1$. *Let* $\Omega = \mathbb{D}^* \setminus K$, *where* K *is a closed, unbounded, connected set in* $\mathbb{D}^*$ *which contains the point* $\{R\}$. *Let* Γ_0, Γ *denote the path families in* Ω, Ω_0 *respectively that separate the two boundary components. Then* $M(\Gamma_0) \leq M(\Gamma)$.

Proof. We use the symmetry principle we just proved. The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma_0^+) = \frac{1}{2}M(\Gamma_0)$.

The family Γ may not be symmetric, but we can replace it by a larger family that is. Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0.

Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so $M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$.

Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$.



Suppose $\gamma \in \Gamma_R$. Since γ has non-zero winding around 0 it must cross both the negative and positive real axes.

If it never crossed $(0, R)$ then the winding around 0 and R would be the same, which false, so γ must cross$(0, R)$ as well.

Choose points $z_- \in \gamma \cap (-\infty, 0)$ and $z_+ \in \gamma \cap (0, R)$. These points divide γ into two subarcs γ_1 and γ_2.

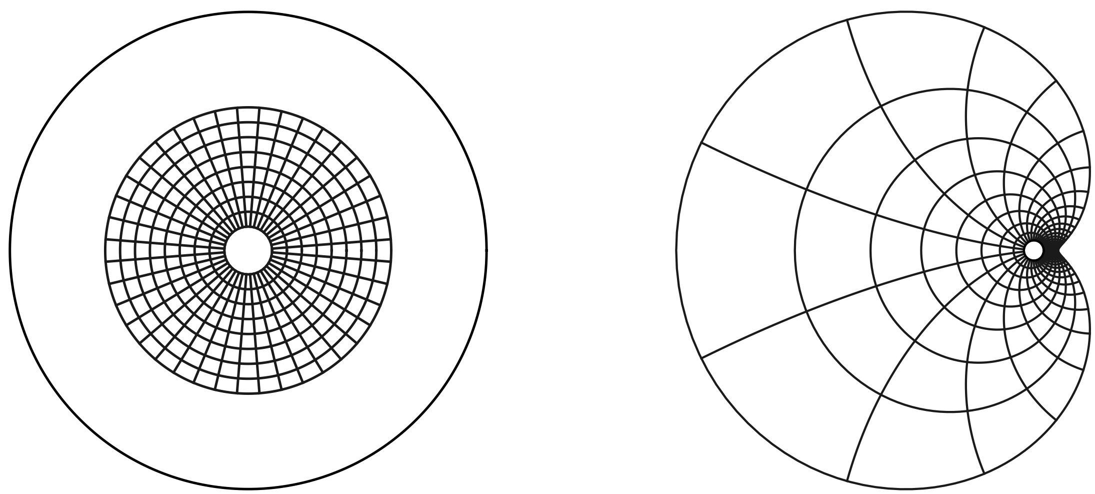
Then $\gamma_+ = (\gamma_1)_+ \cup (\gamma_2)_+$. But if we reflect $(\gamma_2)_+$ into the lower half-plane and join it to $(\gamma_1)_+$ it forms a closed curve γ_0 that is in Γ_0 and $(\gamma_0)_+ = \gamma_+$. Thus $\gamma_+ \in (\Gamma_0)_+$, as desired. $\square$

Let $\Omega_{\epsilon,R} = \{z : |z| > \epsilon\} \setminus [R, \infty)$. Note that $\Omega_{1,R}$ is the domain considered in the previous lemma.

We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$

This conformal maps $\{|z| < 1\}$ to $\mathbb{R}^2 \setminus [\frac{1}{4}, \infty)$ with $k(0) = 0$, $k'(0) = 1$.



Plot of the Koebe function

Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\epsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius $\frac{\epsilon}{4R}(1 \pm O(\frac{\epsilon}{R}))$.

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Thus the modulus of $\Omega_{\epsilon,R}$ is

$$2\pi \log \frac{4R}{\epsilon} + O(\frac{\epsilon}{R}). \tag{2.5}$$

Next we prove the Koebe $\frac{1}{4}$-theorem for conformal maps.

The standard proof of Koebe's $\frac{1}{4}$-theorem uses Green's theorem to estimate the power series coefficients of conformal map (proving the Bieberbach conjecture for the second coefficient).

However here we will present a proof, due to Mateljevic that uses the symmetry property of extremal length.

Theorem 2.11 (The Koebe 1/4-Theorem)**.** *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Theorem 2.11 (The Koebe 1/4-Theorem). *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Proof. Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components).

Let $R = \operatorname{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\epsilon,r} = \{z : \epsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\epsilon} = M(A_{\epsilon,1}) = M(f(A_{\epsilon,1})).$$

Let $\delta = \min_{|z|=\epsilon} |f(z)|$. Since $f'(0) = 1$, we have $\delta = \epsilon + O(\epsilon^2)$.

Note that $f(A_{\epsilon,1}) \subset f(\mathbb{D}) \setminus D(0, \delta)$, so

$$M(f(A_{\epsilon,1})) \leq M(f(\mathbb{D}) \setminus D(0, \delta)).$$

By Lemma 2.10 and Equation (2.5),

$$M(f(\mathbb{D}) \setminus D(0, \delta)) \leq M(\Omega_{\delta,R}) = 2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\epsilon},$$

or

$$\log 4R - \log(\epsilon + O(\epsilon^2)) + O(\frac{\epsilon}{R}) \geq -\log \epsilon,$$

and hence

$$\log 4R \geq -O(\frac{\epsilon}{R}) + \log(1 + O(\epsilon)).$$

Taking $\epsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. $\square$