# MAT 627, Spring 2025, Stony Brook University

# Topics in Complex Analysis: Quasiconformal Mappings

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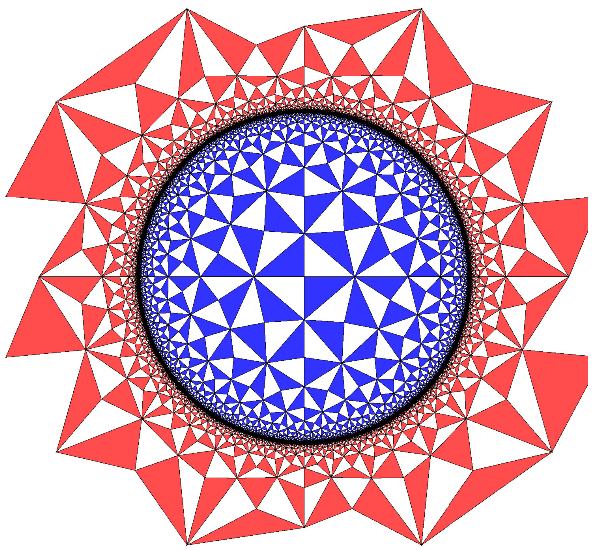
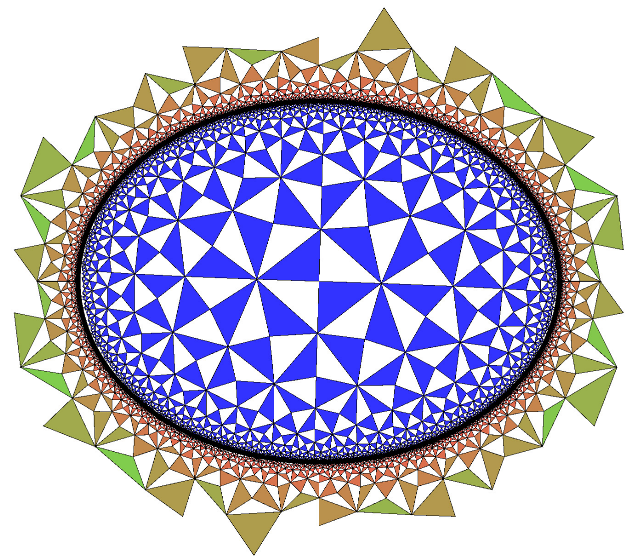




This semester I hope to cover the following topics:

- Review of complex analysis, conformal mappings
- Extremal length, logarithmic capacity, harmonic measures
- Geometric definition, compactness, quasisymmetric maps, quasicircles
- Removable sets
- Analytic definition of QC maps
- The measurable Riemann mapping theorem
- Conformal welding
- Astala's theorems on area and dimension distortion
- Conformal dimension
- David maps
- Quasiconformal maps on metric spaces

## Conformal mappings

Conformal = 1-1 and holomorphic = angle and orientation preserving

**Riemann mapping thm:** for any proper simply connected planar domain $\Omega$, there is a conformal map $f : \mathbb{D} \to \Omega$.

**Uniformization thm:** Any simply connected Riemann surface is conformally equivalent to the disk, the plane or the 2-sphere.

## Conformal invariants

A conformal invariant is a quantity that is invariant under conformal mapping.

Hyperbolic metric

Harmonic measure

Extremal length

First two are very important, but third is easiest to compute or estimate.

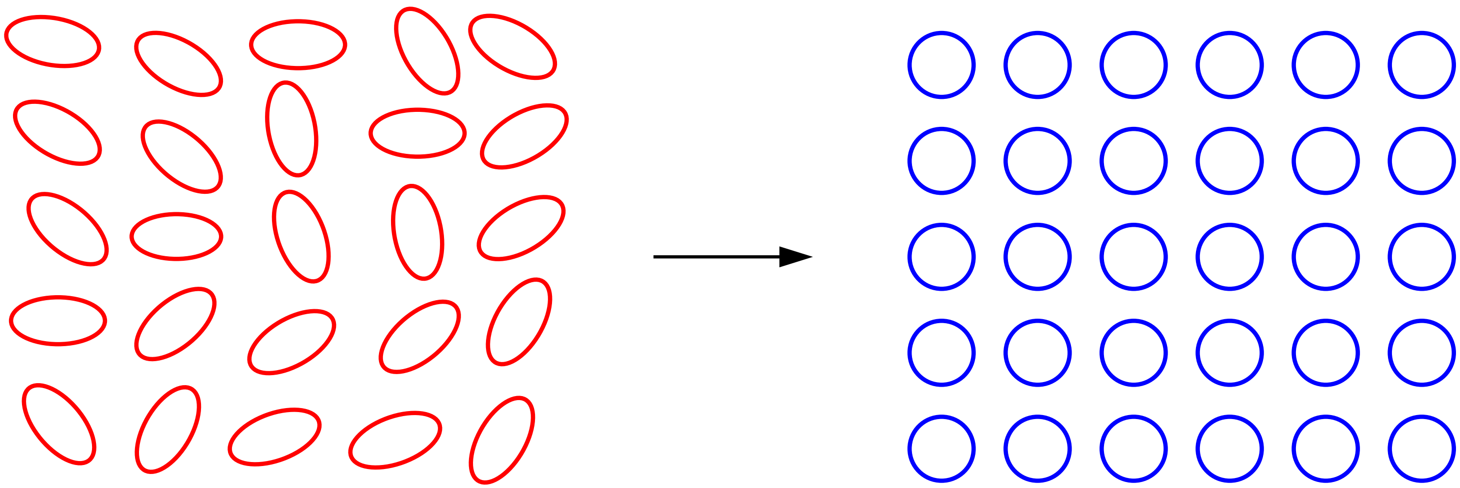
## Quasiconformal maps

Quasiconformal maps are homeomorphisms that preserve extremal length up to a bounded factor.

There are other definitions, including solving the Beltrami equation $f_{\overline{z}} = \mu f_z$.

QC maps can also be defined on metric spaces.

Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

$K$-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

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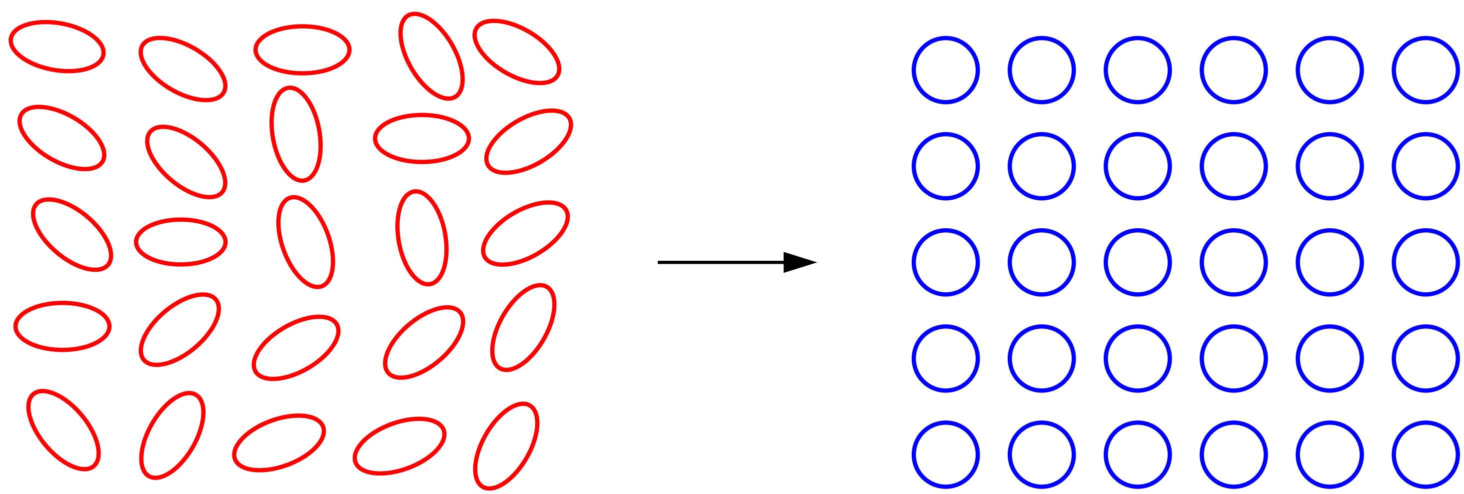

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Ellipses determined by dilatation $\mu = f_{\overline{z}}/f_z$ with $f_{\overline{z}}, f_z = \frac{1}{2}(f_x \pm i f_y)$.

$$|\mu| = \frac{K-1}{K+1} < 1, \qquad \arg(\mu) \text{ gives major axis.}$$

$f$ is QC $\Leftrightarrow \|\mu\|_\infty < 1$. $\quad f$ is conformal = 1-1 holomorphic $\Leftrightarrow \mu \equiv 0$.

**Measurable Riemann Mapping Theorem:** Given any measurable $\mu$ on the plane with $\|\mu\|_\infty = k < 1$ there is a $K$-quasiconformal map $f$ with $\mu = f_{\overline{z}}/f_z$ almost everywhere. Here $K = (k+1)/(k-1)$.

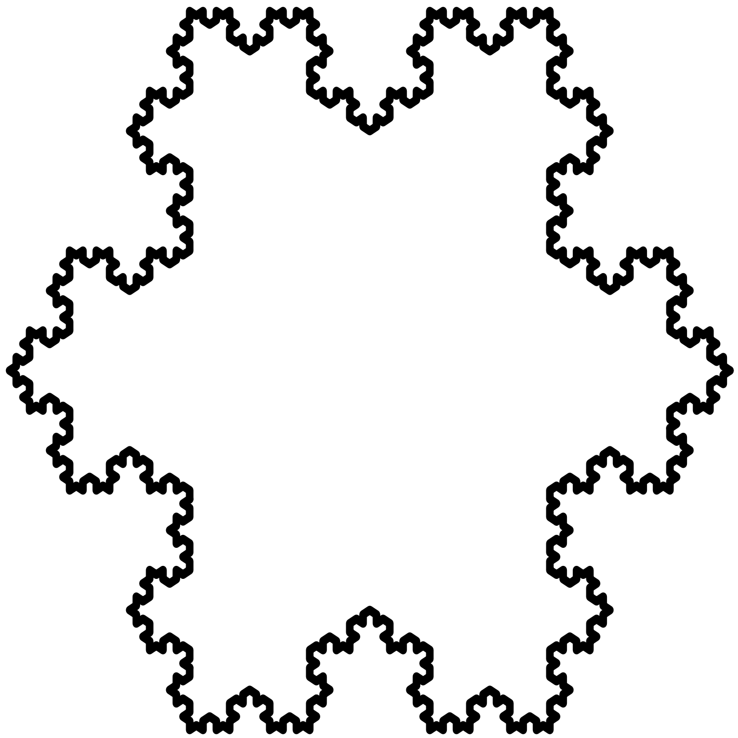
Very important theorem in applications, e.g., in dynamics.

Proving this is major goal of this class.

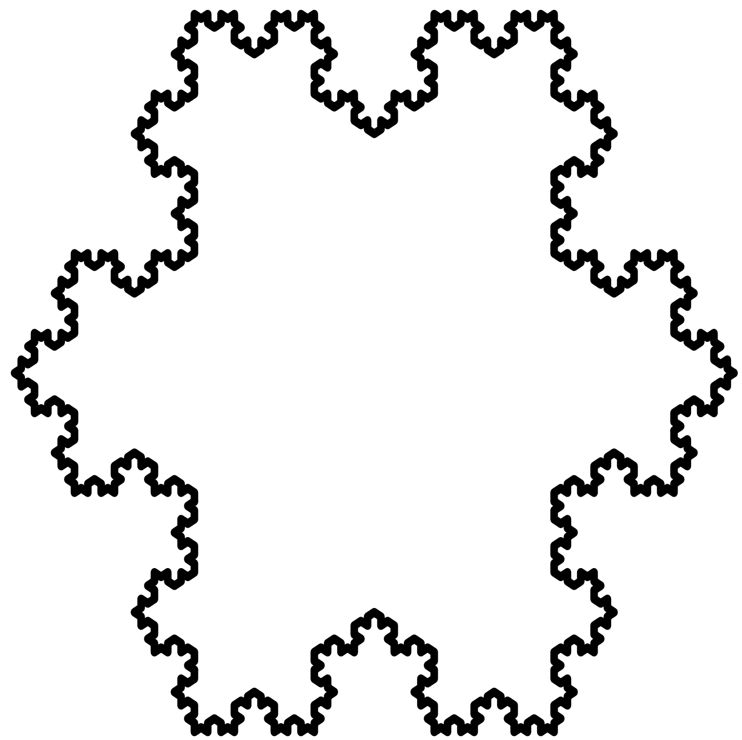
Quasiconformal maps preserve sets of zero area.

QC maps also preserve sets of zero dimension or dimension 2.

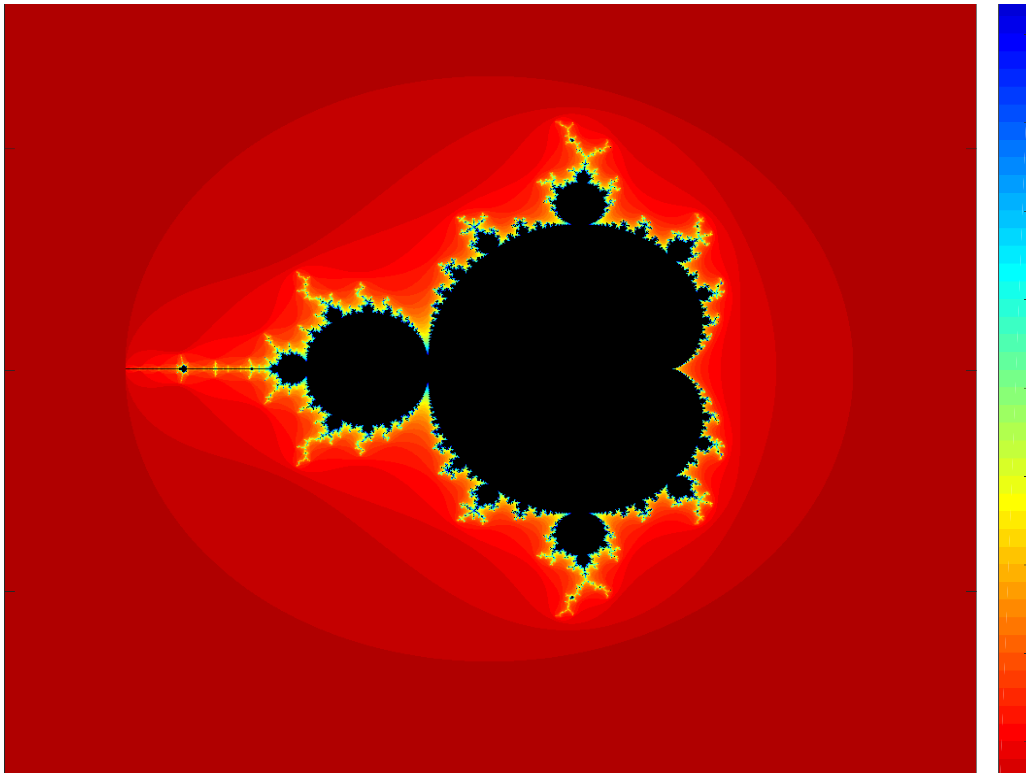
Intermediate dimensions can change, e.g., segment can map to fractal curve.



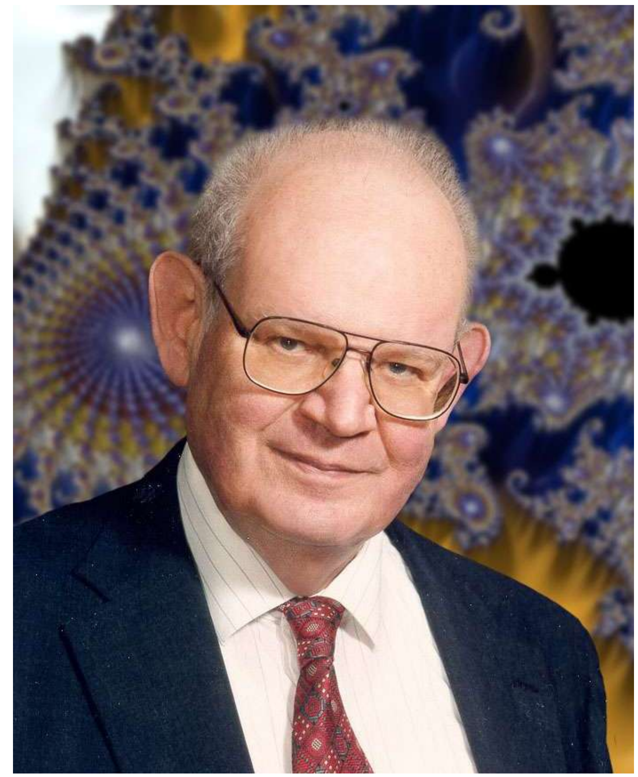
Von Koch snowflake is a quasicircle, i.e., quasiconformal image of a circle.



There is a simple characterization of quasicircles. Many equivalent characterizations. See book by Gehring and Hag.



The Mandelbrot set

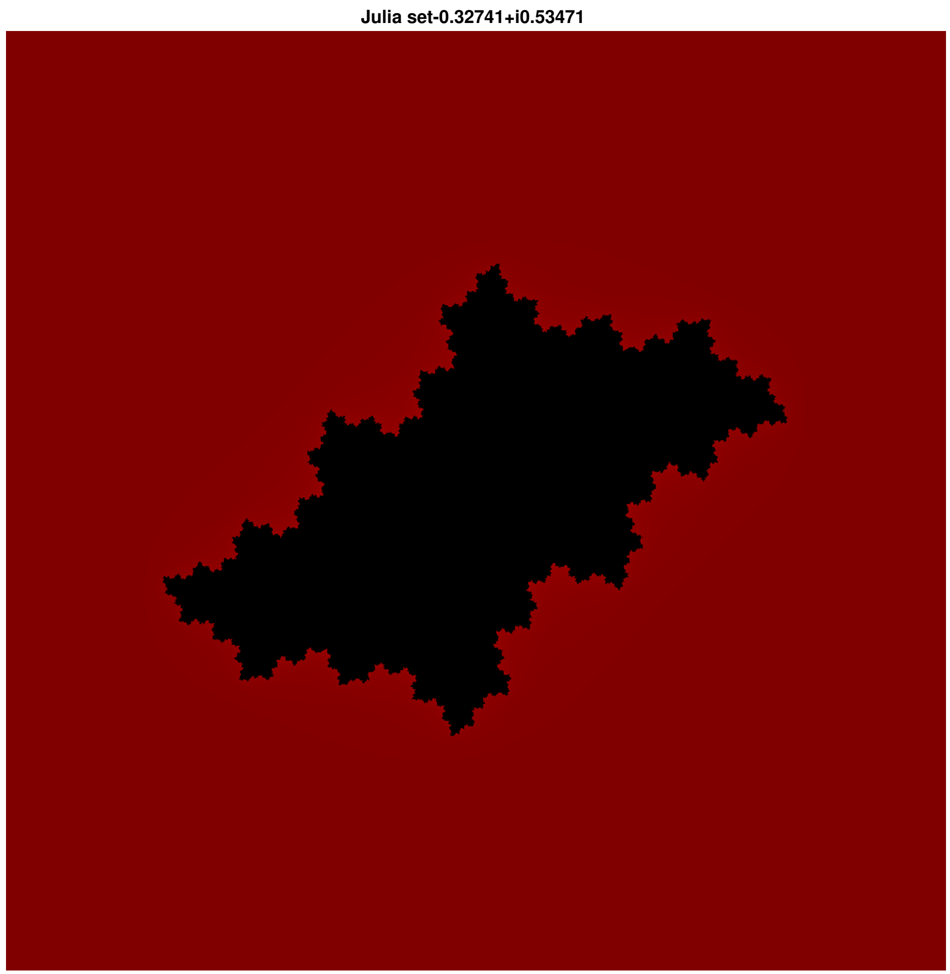


The Mandelbrot

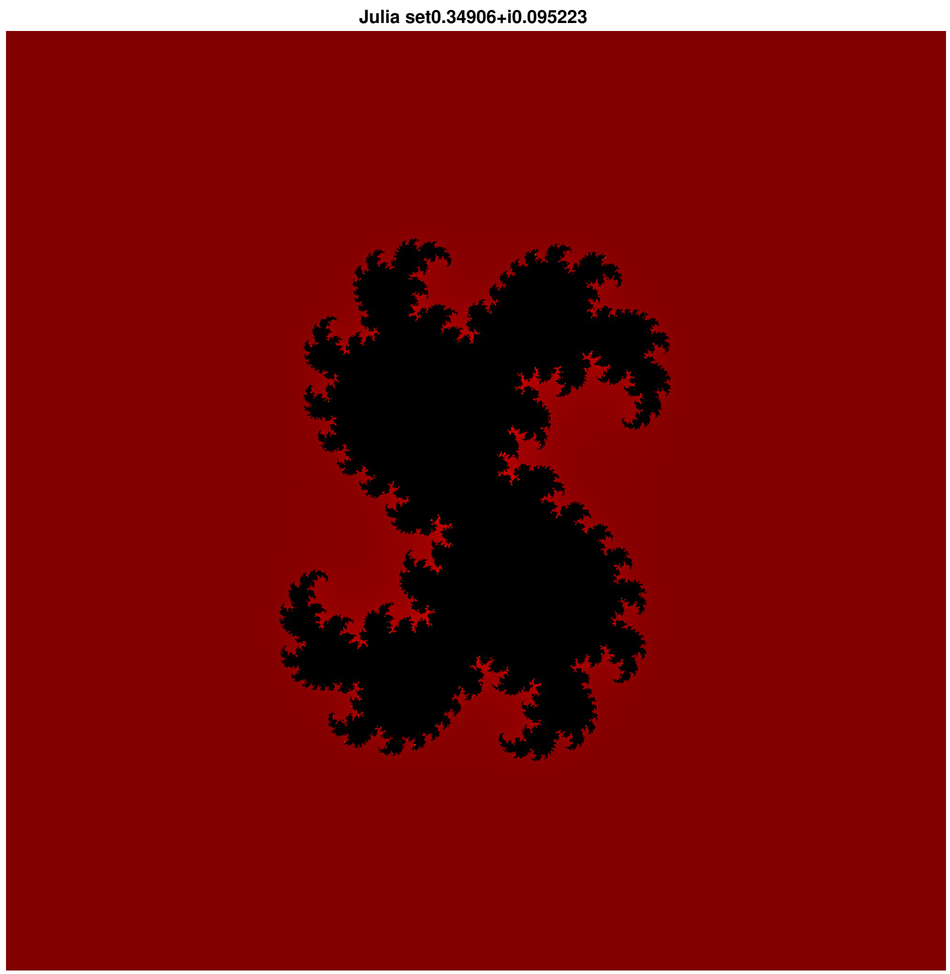
The Brooks-Matelski set

Robert Brooks

Who discovered the Mandelbrot set? Scientific American, 2009




Julia set from the main cardioid.




Another Julia set from the main cardioid.

**Astala's theorem:** If $f$ is $K$-quasiconformal then

$$|F(E)| \leq C(K) \cdot |E|^{1/K},$$

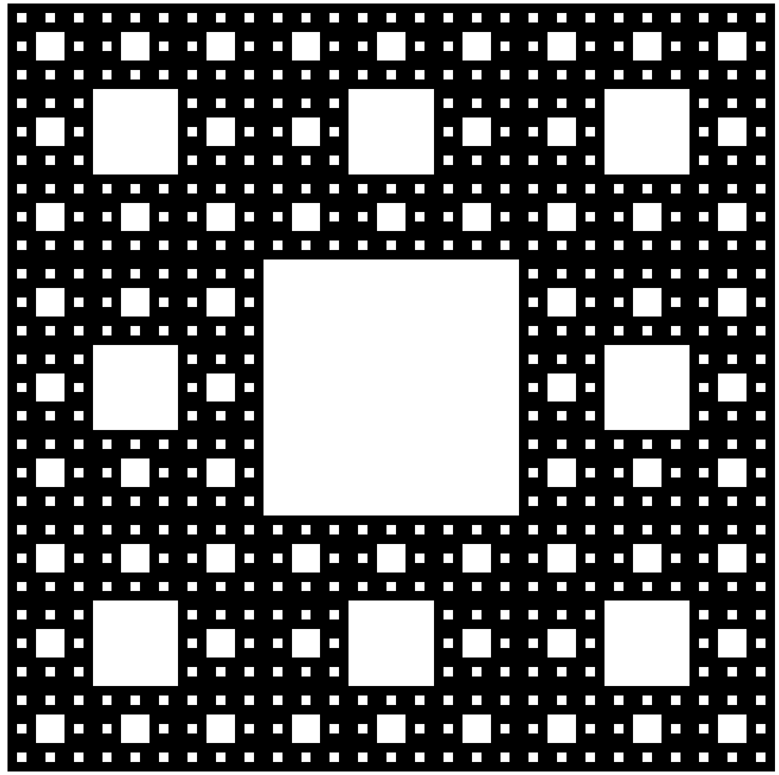
$$\dim(f(E)) \leq \frac{2K \dim(E)}{2 + (K-1)\dim(E)}.$$

**Smirnov's theorem:** a $k$-quasicircle has dimension $< 1 + k^2$.

Conformal dimension of a set $E$ is the infimum of the Hausdorff dimensions of all quasiconformal images of $E$.

Recent progress computing this for some well known fractals.

Graph of Brownian motion has conformal dim $= 1$, (Binder-Hakobyan-Li).



Conformal dimension of Sierpinski carpet is unknown.

Dimension can be lower by mapping into a metric space (Keith and Laakso)

Conformal dimension is $\geq 1 + \log 2/\log 3$ (Kwapisz, 2022)

For all $\alpha \geq 1$ there is a $E \subset \mathbb{R}^d$ with conformal dimension $= \alpha$ (B.-Tyson).

If $E \subset \mathbb{R}$ has dimension $< 1$ then it conformal dimension $= 0$ (Kovalev).

A compact set $K$ is conformally removable if any homeomorphism of the plane that is conformal off $K$ is conformal everywhere.

This is same as saying that any homeomorphism of the plane that is quasiconformal off $K$ is quasiconformal everywhere.

Some sufficient conditions known. Some necessary conditions known.

No characterization yet. Seem very hard. Applications in dynamics.

If $\Omega$ is bounded by a Jordan curve $\Gamma$, then the conformal map $f : \mathbb{D} \to \Omega$ extends continuously to a homeomorphism $\mathbb{T} \to \Gamma$.

If $f, g$ are conformal maps from inside and outside of unit circle to inside and outside of $\Gamma$, then $h = g^{-1} \circ f$ is homeomorphism of $\mathbb{T}$ to itself.

Called "conformal weldings". Not every circle homeomorphism is a welding.

A circle homeomorphism is quasisymmetric if adjacent intervals of equal length map to intervals of comparable length.

A circle homeomorphism extends to a QC map of disk iff quasisymmetric.

A circle homeomorphism is a welding of a quasicircle iff it is quasisymmetric.

No characterization of which circle homeomorphisms are weldings.

Do weldings form a Borel subset of circle homeomorphisms?

Some of my own theorems proven using quasiconformal mappings.

**Theorem:** The conformal map from the disk onto a $n$-sided polygon can be computed to accuracy $\epsilon$ in time $O(n\sqrt{\log 1/\epsilon \log\log 1/\epsilon})$.

**Theorem:** Any simple polygon can be meshed with quadrilaterals with all new angles between $60^\circ$ and $120^\circ$. These bounds are sharp.

**Theorem:** There are entire functions that have bounded singular set and also have wandering domains.

**Theorem:** A minimal surface in hyperbolic 3-space has scalar curvature in $L^2$ iff its boundary curve has arclength parameterization in the Sobolev space $H^{3/2}$.

**Theorem:** Any planar continuum can be approximated by the Julia set of a polynomial with a finite post-critical orbit.

What is a Quasiconformal Map? by Juha Heinonen

A Historical Survey of Quasiconformal Maps by Olli Lehto

Quasiconformal Mappings by Seppo Rickman