MAT 627, Spring 2025, Stony Brook University

Topics in Complex Analysis: Quasiconformal Mappings

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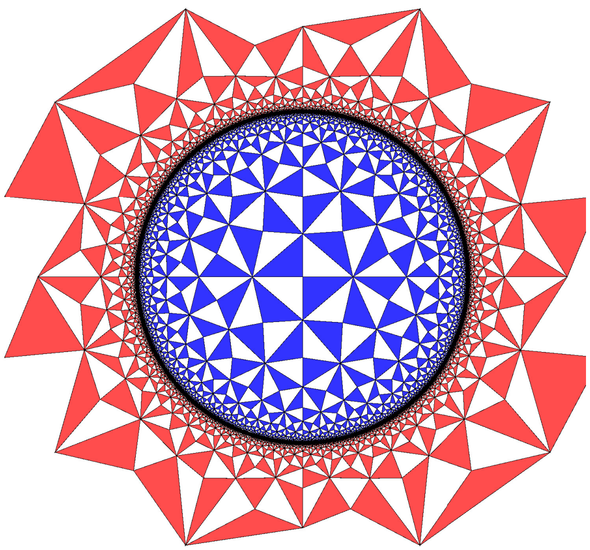
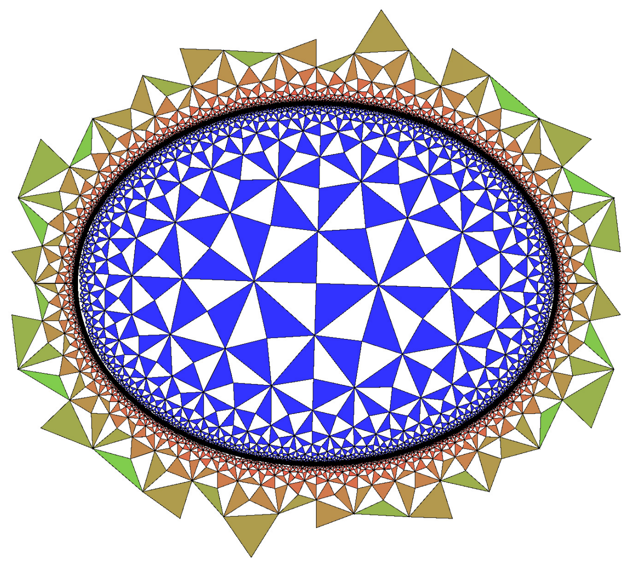




This semester I hope to cover the following topics:

- Review of conformal mappings
- Extremal length and conformal modulus, log capacity, harmonic measure
- Definitions of quasiconformal mappings; geometric and analytic
- Basic properties
- Quasisymmetric maps and boundary extension
- The measurable Riemann mapping theorem
- Removable sets
- Conformal welding
- David maps
- Astala's theorems on area and dimension distortion
- Quasiconformal maps on metric spaces
- Conformal dimension

Some Linear Algebra (QC linear maps)

Conformal maps preserves angles; quasiconformal maps can distort angles, but only in a controlled way.

To make this distinction more precise we must have a way to measure angle distortion and we start with a discussion of linear maps.

Consider the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax + by, cx + dy).$$

Let M^T denote the transpose of the real matrix M, i.e., its reflection over the main diagonal. Then

$$M^T \cdot M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \equiv \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is positive and symmetric and hence has two positive eigenvalues λ_1, λ_2, assuming M in non-degenerate.

The square roots $s_1 = \sqrt{\lambda_1}$, $s_2 = \sqrt{\lambda_2}$ are the singular values of A (without loss of generality we assume $s_1 \geq s_2$). Then

$$M = U \cdot \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \cdot V,$$

where U, V are rotations.

Thus M maps the unit circle to an ellipse whose major and minor axes have length s_1 and s_2.

Thus M preserves angles iff it maps the unit circle to a circle iff $s_1 = s_2$. Otherwise M distorts angles and we let $D = s_1/s_2$ denote the dilatation of the linear map M. This is the eccentricity of the image ellipse and is ≥ 1, with equality iff M conformal.

The inverse of a linear map with singular values $\{s_1, s_2\}$ has singular values $\{\frac{1}{s_2}, \frac{1}{s_1}\}$ and hence dilatation $D = (1/s_2)/(1/s_1) = s_1/s_2$. Thus the dilatation of a linear map and its inverse are the same.

Given two linear maps M, N with singular values $s_1 \geq s_2$ and $t_1 \geq t_2$ respectively, the singular values of the composition MN are trapped between $s_1 t_1$ and $s_2 t_2$ (this occurs for the maximum singular values since they give the operator norms of the matrices and these are multiplicative; a similar argument works for the minimum singular values and the inverse maps).

Thus the dilation is less than $(s_1 t_1)/(s_2 t_2)$ i.e., dilatations satisfy

$$D_{M \circ N} \leq D_M \cdot D_N.$$

The dilatation D can be computed in terms of a, b, c, d as follows.

The eigenvalues λ_1, λ_2 are roots of the

$$0 = \det(M^T \cdot M - \lambda I),$$

which is the same as

$$0 = (E - \lambda)(G - \lambda) - F^2 = EG - F^2 - (E + G)\lambda + \lambda^2.$$

Thus

$$
\begin{aligned}
\lambda_1\lambda_2 &= EG - F^2 \\
&= (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 \\
&= a^2b^2 + a^2d^2 + c^2b^2 + d^2c^2 - (a^2b^2 + 2abcd + c^2d^2) \\
&= a^2d^2 + c^2b^2 - 2abcd \\
&= (ad - bc)^2
\end{aligned}
$$

Similarly,

$$\lambda_1 + \lambda_2 = E + G = a^2 + b^2 + c^2 + d^2.$$

The values of λ_1, λ_2 can be found using the quadratic formula:

$$\begin{aligned} \{\lambda_1, \lambda_2\} &= \frac{1}{2}[E + G \pm \sqrt{(E+G)^2 - 4(EG - F^2)}] \\ &= \frac{1}{2}[E + G \pm \sqrt{(E-G)^2 + 4F^2)}]. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\lambda_1}{\lambda_2} &= \frac{E + G + \sqrt{(E-G)^2 + 4F^2}}{E + G - \sqrt{(E-G)^2 + 4F^2}} \\ &= \frac{(E + G + \sqrt{(E-G)^2 + 4F^2})^2}{(E+G)^2 - (E-G)^2 - 4F^2} \\ &= \frac{(E + G + \sqrt{(E-G)^2 + 4F^2})^2}{4(EG + F^2)}. \end{aligned}$$

and hence

$$D = \frac{s_1}{s_2} = \sqrt{\frac{\lambda_1}{\lambda_2}} = \frac{E + G + \sqrt{(E-G)^2 + 4F^2}}{2\sqrt{EG + F^2}}.$$

This formula can be made simpler by complexifying.

Think of the linear map M on $\mathbb{R}^2$ as a map f on $\mathbb{C}$:

$$x + iy \to ax + by + i(cx + dy) = u(x, y) + iv(x, y) = f(x + iy)$$

Then

$$M = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and we define

$$f_z = \frac{1}{2}(f_x - if_y) = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y),$$
$$f_{\overline{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y).$$

Some tedious arithmetic now shows that

$$\begin{aligned} 4|f_z|^2 &= (u_x+v_y)^2+(v_x-u_y)^2 \\ &= u_x^2+2u_xv_y+v_y^2+v_x^2-2v_xu_y+u_y^2 \end{aligned}$$

$$\begin{aligned} 4|f_{\overline{z}}|^2 &= (u_x-v_y)^2+(v_x+u_y)^2 \\ &= u_x^2-2u_xv_y+v_y^2+v_x^2+2v_xu_y+u_y^2 \end{aligned}$$

so

$$(|f_z|+|f_{\overline{z}}|)(|f_z|-|f_{\overline{z}}|)=|f_z|^2-|f_{\overline{z}}|^2 = u_xv_y-v_xu_y=s_1s_2=\det(M).$$

In particular, if we assume M is orientation preserving and full rank, then $\det(M) > 0$ and we deduce $|f_z| > |f_{\overline{z}}|$.

Similarly,

$$\begin{aligned}(|f_z| + |f_{\overline{z}}|)^2 + (|f_z| - |f_{\overline{z}}|)^2 &= 2(|f_z|^2 + |f_{\overline{z}}|^2) \\ &= u_x^2 + v_x^2 + u_y^2 + v_x^2 \\ &= E + G \\ &= \lambda_1 + \lambda_2 \\ &= s_1^2 + s_2^2.\end{aligned}$$

From these equations and the facts $s_1 \geq s_2$, $|f_z| > |f_{\overline{z}}|$ we can deduce

$$s_1 = |f_z| + |f_{\overline{z}}|, \qquad s_2 = |f_z| - |f_{\overline{z}}|,$$

and hence

$$D = \frac{s_1}{s_2} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}.$$

Note that $D \geq 1$ with equality iff f is a conformal linear map. It is often more convenient to deal with the complex number,

$$\mu = \frac{f_{\overline{z}}}{f_z},$$

which is called the **complex dilatation**.

Sometimes we abuse notation and just call thus the dilatation, if the meaning is clear from context.

Since $|f_{\overline{z}}| < |f_z|$, we have $|\mu| < 1$ and it is easy to verify that

$$D = \frac{1+|\mu|}{1-|\mu|}, \qquad |\mu| = \frac{D-1}{D+1},$$

so that either D or $|\mu|$ can be used to measure the degree of non-conformality.

We leave it to the reader to check that the map

$$x + iy \to (ax + by) + i(cx + dy)$$

can also be written as

$$(z, \overline{z}) \to \alpha z + \beta \overline{z},$$

where $z = x + iy$, $\overline{z} = x - iy$ and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, satisfy

$$\alpha_1 = \frac{a+d}{2}, \quad \alpha_2 = \frac{a-d}{2}, \quad \beta_1 = \frac{c-b}{2}, \quad \beta_2 = \frac{b+c}{2},$$

In this notation $\mu = \beta/\alpha$ and

$$D = \frac{|\beta| + |\alpha|}{|\alpha| - |\beta|}.$$

As noted above, the linear map f sends the unit circle to an ellipse of eccentricity D. What point on the circle is mapped furthest from the origin?

Since

$$s_1 = |f_z| + |f_{\overline{z}}|,$$

the maximum stretching is attained when $f_z z$ and $f_{\overline{z}}\overline{z}$ have the same argument, i.e., when

$$0 < \frac{f_z z}{f_{\overline{z}}\overline{z}} = \frac{z^2}{\mu|z|^2},$$

or

$$\arg(z) = \frac{1}{2}\arg(\mu),$$

Thus $|\mu|$ encodes the eccentricity of the ellipse and $\arg(\mu)$ encodes the direction of its major axis.

If we follow f by a conformal map g, then the same infinitesimal ellipse is mapped to a circle, so we must have $\mu_{g\circ f} = \mu_f$.

If f is preceded by a conformal map g, then the ellipse that is mapped to a circle is the original one rotated by $-\arg(g_z)$, so $\mu_{f\circ g} = (|g_z|/g_z)^2 \mu_f$.

To obtain the correct formula in general we need to do a little linear algebra. Consider the composition $g \circ f$ and let $w = f(z)$ so that the usual chain rule gives

$$(g \circ f)_z = (g_w \circ f) f_z + (g_{\overline{w}} \circ f)\overline{f}_z,$$
$$(g \circ f)_{\overline{z}} = (g_w \circ f) f_{\overline{z}} + (g_{\overline{w}} \circ f)\overline{f}_{\overline{z}}.$$

or in vector notation

$$\begin{pmatrix} (g \circ f)_z \\ (g \circ f)_z \end{pmatrix} = \begin{pmatrix} f_z & \overline{f}_z \\ f_{\overline{z}} & \overline{f}_{\overline{z}} \end{pmatrix} \begin{pmatrix} (g_w \circ f) \\ (g_{\overline{w}} \circ) \end{pmatrix}$$

The determinate of the matrix is

$$f_z \overline{f}_{\overline{z}} - \overline{f}_z f_{\overline{z}} = f_z \overline{f_z} - \overline{f_{\overline{z}}} f_{\overline{z}} = |f_z|^2 - |f_{\overline{z}}|^2 = J,$$

which is the Jacobian of f, so by Cramer's Rule,

$$\begin{aligned}(g_w \circ f) &= \frac{1}{J}[(g \circ f)_z \overline{f}_{\overline{z}} - (g \circ f)_{\overline{z}} \overline{f}_z],\\ (g_{\overline{w}} \circ f) &= \frac{1}{J}[(g \circ f)_{\overline{z}} f_z - (g \circ f)_z f_{\overline{z}}],\end{aligned}$$

so

$$\mu_g \circ f \;=\; \frac{(g \circ f)_{\overline{z}} f_z - (g \circ f)_z f_{\overline{z}}}{(g \circ f)_z \overline{f}_{\overline{z}} - (g \circ f)_{\overline{z}} \overline{f}_z} = \frac{\mu_{g \circ f} f_z - f_{\overline{z}}}{\overline{f}_{\overline{z}} - \mu_{g \circ f} \overline{f}_z} = \frac{f_z}{\overline{f_z}} \cdot \frac{\mu_{g \circ f} - \mu_f}{1 - \mu_{g \circ f} \overline{\mu_f}}.$$

Now set $h = g \circ f$ or $g = h \circ f^{-1}$ to get

$$\mu_{h \circ f^{-1}} \circ f \;=\; \frac{f_z}{\overline{f_z}} \frac{\mu_h - \mu_f}{1 - \mu_h \overline{\mu_f}}.$$

Thus if h and f have the same dilatation μ, then $g = h \circ f^{-1}$ is conformal. We will need this in the case when h is more general than an homeomorphism.

Geometric Definition of Quasiconformal Maps

A quadrilateral Q is a Jordan domain with two specified disjoint closed arcs on the boundary. (Equivalently, four distinct points and a choice of opposite edges.)

By the Riemann mapping theorem and Caratheodory's theorem, there is a conformal map from Q to a $1 \times m$ rectangle that extends continuously to the boundary with the two marked arcs mapping to the two sides of length a.

The ratio $M = M(Q) = 1/m$ is called the modulus of the four distinct marked on the boundary and is uniquely determined by Q.

The conjugate of Q is the same domain but with the complementary arcs marked. Its modulus is clearly the reciprocal of Q's modulus.

The geometric definition: A homeomorphism h, defined on a planar domain Ω, is K-quasiconformal if the

$$\frac{1}{K}M(Q) \leq M(h(Q)) \leq KM(Q),$$

for every quadrilateral $Q \subset \Omega$.

The following is a helpful sufficient condition. Many of the maps we use in practice are of this form.

The piecewise differentiable definition: h is K-quasiconformal on Ω if there are countable many analytic curves whose union is a closed set Γ of Ω such that h is continuously differentiable on each connected component of $\Omega' = \Omega \setminus \Gamma$ and $D_h \leq K$ on Ω'.

First we check that the piecewise definition implies the geometric definition.

A major goal for later is to replace piecewise differentiability with almost everywhere differentiability, but this requires some extra regularity assumptions.

Lemma 4.1. *Suppose h a homeomorphism of Ω such that there are countable many analytic curves whose union is a closed set Γ of Ω and h is continuously differentiable on each connected component of $\Omega' = \Omega \setminus \Gamma$ and $D_h \leq K$ on Ω'. Then h is K-quasiconformal.*

Proof. Using conformal maps, it suffices to consider the case when Ω and its image are both rectangles, say $\Omega = [0, a] \times [0, 1]$ and $h(\Omega) = [1, b] \times [0, 1]$.

By integrating over horizontal lines in the first rectangle, we see

$$b \leq \int_0^a (|f_z| + |f_{\overline{z}}|)dx.$$

We have used the piecewise analytic assumption here to break the integral into finitely many open segments where the fundamental theorem of calculus applies and then use the assumption that h is continuous at the endpoints to say the total integral is the sum of these sub-integrals.

Fact: if f continuous on $[a, b]$ and f' is continuous and bounded except at finitely many points, then $f(x) = \int_a^x f'(t)dt$.

Integrating in the other variable,

$$b \leq \int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|)dxdy.$$

By Cauchy-Schwarz,

$$\begin{aligned}
b^2 &\leq (\int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|)(|f_z| - |f_{\overline{z}}|)dxdy)(\int_0^1 \int_0^a \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}dxdy) \\
&\leq (\int_0^1 \int_0^a (|f_z|^2 - |f_{\overline{z}}|^2)dxdy)(\int_0^1 \int_0^a \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}dxdy) \\
&\leq (\int_0^1 \int_0^a J_f dxdy)(\int_0^1 \int_0^a D_f dxdy) \\
&\leq baK,
\end{aligned}$$

and so $b \leq Ka$. The other direction follows by repeating the argument for vertical lines instead of horizontal ones. □

In order for the proof to work we need two things:

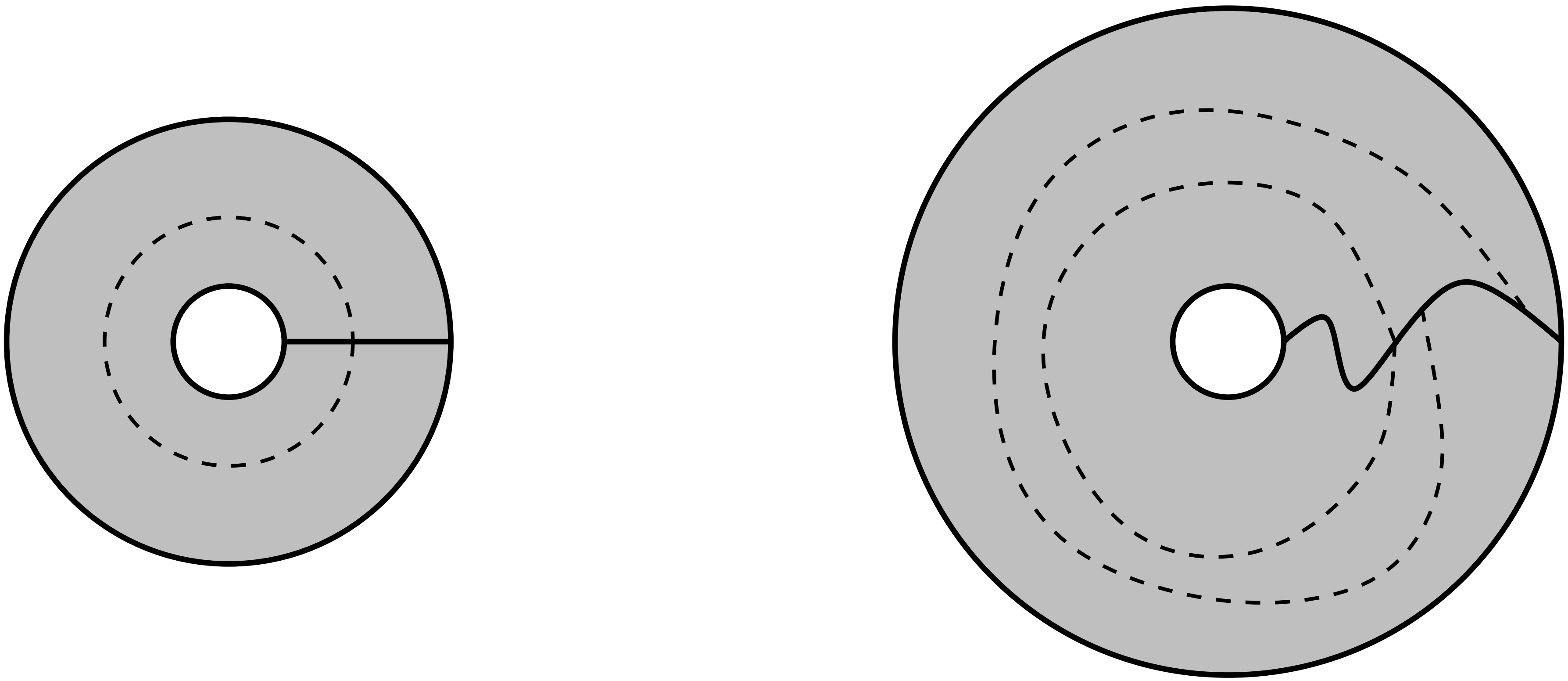
(1) the area of the range to be bounded above by integrating the Jacobian over the domain and,

(2) each horizontal line segment S to have an image whose length is bounded above by the integral of $|f_z| + |f_{\overline{z}}|$ over S.

These certainly hold if f_z and $f_{\overline{z}}$ are piecewise continuous on a partition of the plane given by countable many analytic curves, as we have assumed, but it holds much more generally.

The geometric definition of quasiconformality actually implies that the map h has partials almost everywhere and is absolutely continuous on almost every line. This, in turn, implies the necessary estimates holds. This will be discussed later.

Corollary 4.2. *If we have a piecewise differentiable K-quasiconformal map f between annuli $A_r = \{1 < |z| < r\}$ and $A_R = \{1 < |z| < R\}$ with dilatation $\leq K$, then* $\frac{1}{K} \log r \leq \log R \leq K \log r$.



Proof. Slit A_r with $[1, r]$ to get a quadrilateral $Q \subset A_r$ and let $Q' = f(Q) \subset A_R$.

Then $M(A_R) \leq M(Q') \leq KM(Q) = M(A_r)$.

The first inequality occurs because of monotonicity of modulus (Lemma 2.2); every separating curve for the annulus connects opposite sides of Q' (but there are connecting curves that don't correspond to closed loops).

The other direction follows by considering the inverse map. $\square$

Compactness of K-quasiconformal maps

Theorem 10.5, Arzela-Ascoli Theorem: *A family $\mathcal{F}$ of continuous functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if*

(1) $\mathcal{F}$ is equicontinuous on Ω, and

(2) there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

This result is usually proven in MAT 532 (Chap 4 of Folland's book).

We want to verify K-quasiconformal maps satisfy the Arzela-Ascoli theorem.

Lemma 4.3. *Suppose* $\Omega \subset \mathbb{C}$ *is open and simply connected and* $D \subset \Omega$ *is a topological closed disk. If* f *is* K*-quasiconformal on* Ω *and* $x, y, z \in K$ *with* $|x - y| \leq |x - z|$*. Then*

$$|f(x) - f(y)| \leq M|f(z) - f(y)|,$$

where M *depends on* Ω*,* D *and* K*, but not on* x*,* y *or* z*.*

Proof. After renormalizing by conformal linear maps we may assume $y = f(y) = 0$ and $z = f(z) = 1$.

Then x is in the half-plane H that lies to the left of the bisector of 0 and 1 and it suffices to show that $|f(x)|$ is bounded depending only on K, D and Ω.

Connect 1 to $\partial\Omega$ by a real segment $\sigma \subset \Omega \cap \mathbb{R}$; then $D \setminus \sigma$ is connected and there is an $\epsilon > 0$ so that and 0 can be connected to any point of $H \cap D$ by a path in D that is at least distance ϵ from σ.

Connect 0 to x by such a curve γ. Then $A = \Omega \setminus (\gamma \cup \sigma)$ is a topological annulus and $\rho = 1/\epsilon$ on $\{|z| \leq \epsilon + \mathrm{diam}(D)\}$ is admissible for the path family connecting γ and σ in Ω.

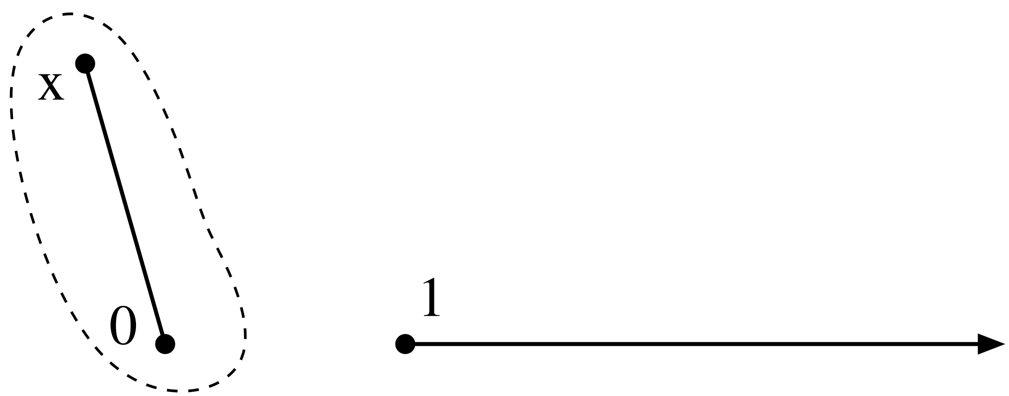
Therefore the modulus of A, which is the modulus of the family separating the two curves is greater than $\epsilon^2/(\epsilon + \text{diam}(D))^2 > 0$.

Moreover, the modulus of A differs by at most a factor of K from the modulus of $B = F(A)$. However, if $|f(x)| \gg 1$, then by considering the metric $\rho(z) = 1/|z|$ on the annulus $\{z : 1 < |z| < |f(x)|\}$, we see that B has modulus tending to zero as $|f(x)| \nearrow \infty$.

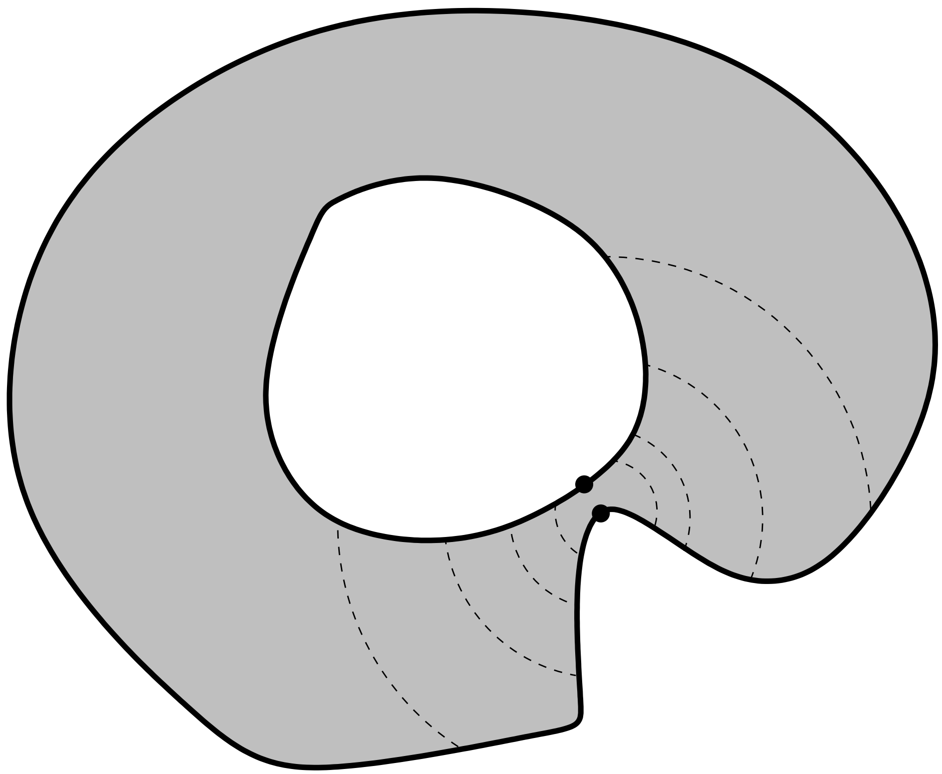
Thus $|f(x)|$ is bounded in terms of K and the modulus of A, which, in turn, depends only on D and Ω. $\square$

Corollary 4.4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map that fixes both 0 and 1. Then $|f(x)|$ is bounded with an estimate depending on $|x|$ and K, but not on f.*

Proof. Take $\Omega = \mathbb{C}$ and $D = \{|z| < |x| + 1\}$ in Lemma 4.3. □



Lemma 4.5. *Suppose $\Omega \subset \mathbb{C}$ is a topological annulus of modulus M whose boundary consists of two Jordan curves γ_1, γ_2 with γ_2 separating γ_1 from ∞. Then $\mathrm{diam}(\gamma_1) \leq (1-\epsilon)\mathrm{diam}(\gamma_2)$ where $\epsilon > 0$ depends only on M.*



Proof. Rescale so $\operatorname{diam}(\gamma_2) = \operatorname{diam}(\Omega) = 1$ and suppose $\operatorname{diam}(\gamma_1) > 1 - \epsilon$.

Then there are points $a \in \gamma_1$ and $b \in \gamma_2$ with $|a - b| \leq \epsilon$. Let ρ be the metric on Ω defined by $\rho(z) = \frac{1}{|z-a|\log(1/2\epsilon)}$ for $\epsilon < |z - a| < 1/2$.

Then any curve $\gamma \subset \Omega$ that separates γ_1 and γ_2 satisfies $\int_\gamma \rho ds \geq 1$ and

$$\int \rho^2 dxdy \leq \frac{\pi}{4} \log^{-2} \frac{1}{2\epsilon}.$$

Thus the modulus of the path family separating the boundary components is bounded above by the right hand side, and the modulus of the reciprocal family connecting the boundary components is bounded below by $\frac{\pi}{4} \log^2 \frac{1}{2\epsilon}$.

Thus $\epsilon \geq \frac{1}{2} \exp(-\sqrt{\pi M/4})$. $\square$

A function f is α**-Hölder continuous** on a set E if there is a $C < \infty$ so that

$$|f(x) - f(y)| \leq C|x - y|^{\alpha},$$

for all $x, y \in E$.

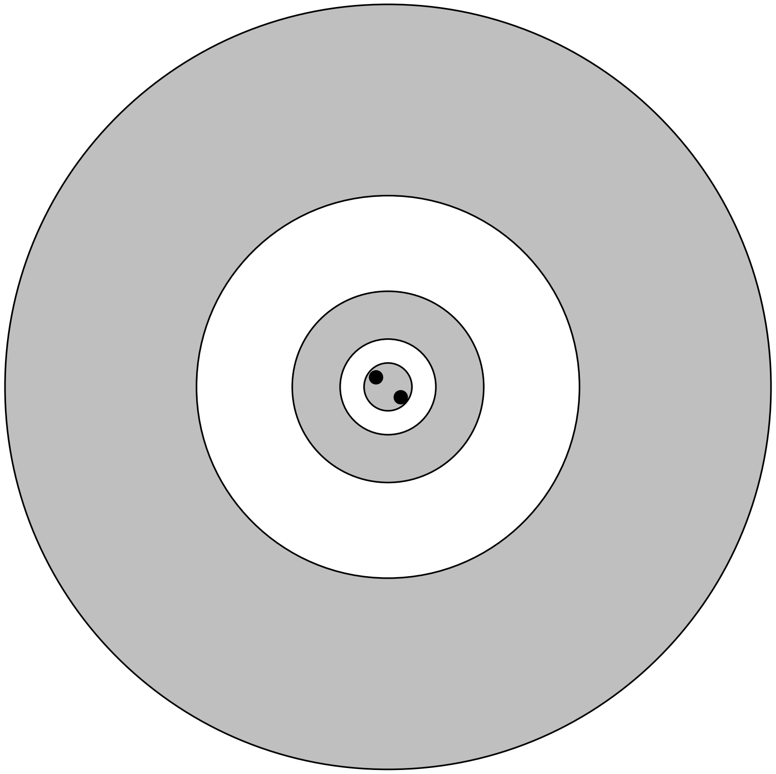
We say f is Hölder continuous on E if this holds for some $\alpha > 0$.

We say f is locally α-Hölder on an open set Ω if each point of Ω has a neighborhood on which f is α-Hölder. This implies that f is α-Hölder on any compact set of Ω, although the multiplicative constant may depend on the set.

f is bi-Hölder if both f and f^{-1} are Hölder.

Theorem 4.6. *A K-quasiconformal map of an open set Ω is locally α-Hölder continuous for some $\alpha > 0$ that only depends on K.*

Later we will compute the actual Hölder exponent as $\alpha = 1/K$.



Proof. It is enough to show that f is Hölder on any disk D so that $3D \subset \Omega$.

Without loss of generality, assume $D = D(0, r)$, $f(0) = 0$ and $x, y \in D(0, r)$.

By Lemma 4.3, $D(0, 2r)$ is mapped into $D(0, R)$ for some $R = R(r, K)$. Surround $\{x, y\}$ by $N = \lfloor \log_2 \frac{r}{|x-y|} \rfloor$ annuli $\{A_j\}$ of modulus $\log 2$.

The image annuli $\{f(A_j)\}$ have moduli bounded away from zero, and hence $\operatorname{diam}(f(A_{j+1})) \leq (1-\epsilon)\operatorname{diam}(f(A_j))$ by Lemma 4.5. Therefore

$$\begin{aligned} |f(x) - f(y)| &\leq R(1-\epsilon)^N \leq R2^{\log_2(1-\epsilon)(1+\log_2 R - \log_2 |x-y|)} \\ &\leq C(R)|x-y|^{\log_2(1-\epsilon)}. \quad \square \end{aligned}$$

The proof can generalized to a slightly bigger class than the quasiconformal maps where the dilatation is allowed to grow to ∞ sufficiently slowly.

There have been a number of excellent papers written on explicit bounds for this kind of result, but we will only need the "soft" version above.

Theorem 4.7. *If f is piecewise differentiable and the dilatation μ satisfies certain estimates of the form*

$$\max_{|x|\leq R, r>1/R} \frac{1}{r^2} \int_{D(x,r)} D_f(z)dxdy \leq \phi(R),$$

then f has modulus of continuity that depends only on ϕ if $\phi \nearrow \infty$ slowly enough as $R \to \infty$.

Proof. Repeat the proof of Theorem 4.6, only now the moduli of the image annuli can tend to zero. However, as long as ϕ grows slowly enough, then

$$\mathrm{diam}(f(A_j)) \leq \prod_{j=1}^{N} (1 - \epsilon(\phi(R)))$$

where $\epsilon(K)$ is as in Lemma 4.5. $\square$

We want to show that K-quasiconformal maps have continuous boundary extensions.

This essentially follows from the fact they are Hölder continuous, but our proof of that fact is only local and may give a multiplicative constant that blows up as we approach the boundary.

We will prove that this does not happen if the boundary itself is nice enough, e.g., a circle:

Theorem 4.8. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and onto, then φ is α-Hölder on $\mathbb{D}$, where $\alpha > 0$ only depends on K. Thus φ extends continuously to a homeomorphism of $\mathbb{T} = \partial\mathbb{D}$ to itself.*

The proof is very similar to the Hölder estimates for quasiconformal maps in the plane, however, we will also need a trick for converting certain quadrilaterals in the disk into annuli in the plane by reflecting across the circle. The precise statement is:

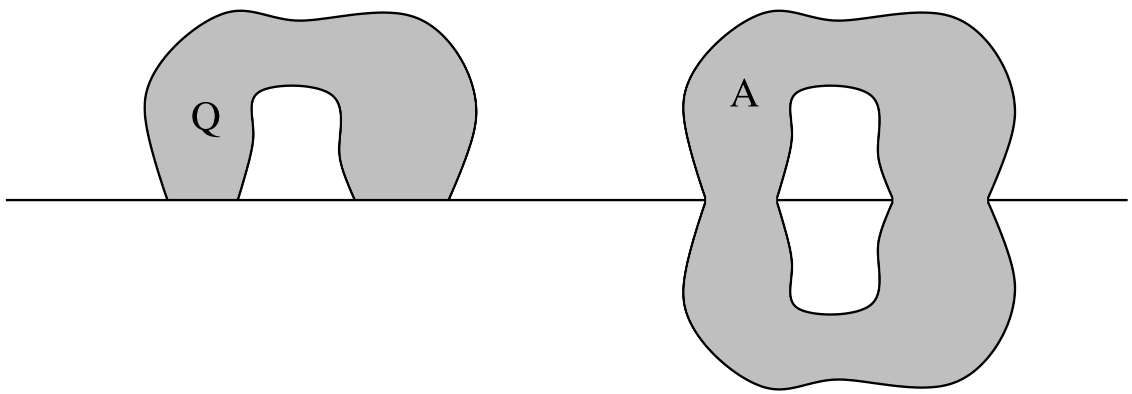
Lemma 4.9. *Suppose $Q \subset \mathbb{H}$ is a quadrilateral with a pair of opposite sides being intervals $I, J \subset \mathbb{R}$. Let A be the topological annulus formed by taking $Q \cup I \cup J \cup Q^*$ (where Q^* is the reflection of Q across $\mathbb{R}$. Then $M(A) = \frac{1}{2}M(Q)$ (here the modulus of Q refers to the modulus of the path family connecting the two sides of Q that line on the unit circle).*

Proof. Using conformal invariance, assume Q is in the upper half-plane and A is obtained by reflecting Q across the real line.

Consider the path family Γ_A in A that connects the two boundary components of A, and the path family Γ_Q in Q that separate the boundary arcs $Q \cap \mathbb{R}$. Then $(\Gamma_A)_+ = \Gamma_Q$ (notation as in Lemma 2.9), so by the Symmetry Rule

$$M(\Gamma_A) = 2M((\Gamma_A)_+) = 2M(\Gamma_Q).$$

The desired moduli are the reciprocals of these, so the result follows. □



Proof of Theorem 4.8. We may assume $f(0) = 0$; the general case then follows after composing with a Möbius transformation.

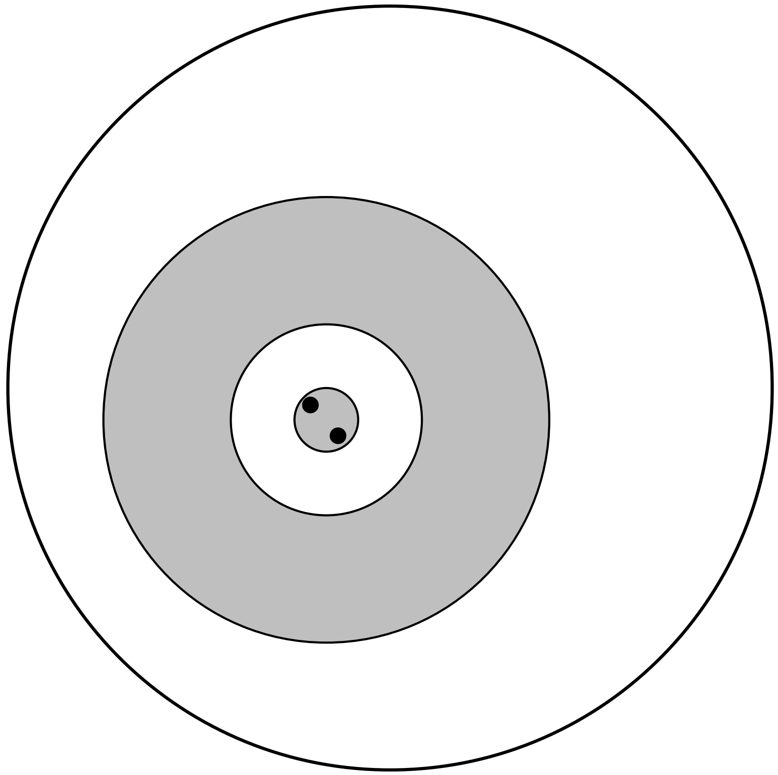
We first suppose φ extends continuously to the boundary. This may seem a bit circular given the final statement of the theorem, but our plan is to prove φ is $\alpha(K)$-Hölder for assuming continuity, and then use a limiting argument to remove the continuity assumption.

More precisely, suppose $w, z \in \mathbb{D}$. We will show that

$$|\varphi(z) - \varphi(w)| \leq C|z - w|^{\alpha},$$

for constants $C < \infty$, $\alpha > 0$ that depend only on the quasiconstant K of f. This implies f is uniformly continuous and hence has a continuous extension to the boundary of $\mathbb{D}$.

Let $d = |z - w|$ and $r = \min(1 - |z|, 1 - |w|)$. There are several cases depending on the positions of the points z, w and the relative sizes of d and r.



To start, note that if $|z-w| \geq \frac{1}{10}$ we can just take $C = 20$ and $\alpha = 1$. So from here on, we assume $|z - w| < 1/10$.

Suppose $r > 1/4$, so $z, w \in \frac{3}{4}\mathbb{D}$. Surround the segment $[z, w]$ by $N \simeq \log d$ annuli with moduli $\simeq 1$. Then just as in the proof of Theorem 4.6, the image annuli have moduli $\simeq 1$ (with a constant depending on K) and hence

$$|f(z) - f(w)| \leq (1 - \epsilon(K))^N = O(|z - w|^\alpha),$$

for some $\alpha > 0$ depending only on K.

Next suppose $|z| \geq 3/4$ and $d > r$. Then separate $[z, w]$ from 0 by $N \simeq \log d$ disjoint quadrilaterals with a pair of opposite sides being arcs of $\mathbb{T}$, and all with moduli $\simeq 1$. Since $f(0) = 0$ and the image quadrilaterals have moduli $\simeq 1$, there diameters shrink geometrically, so

$$|z - w| = (1 - \epsilon(K))^N = O(d^\alpha),$$

as desired.

Finally, if $r \leq d$ we combine the two previous ideas: we start by separating $[z, w]$ from 0 by $\simeq \log d$ quadrilaterals with as above.

The smallest quadrilateral then bounds a region of diameter approximately r containing $[z, w]$ and we then construct $\simeq \log r/d$ disjoint annuli with moduli $\simeq 1$ that each separate $[z, w]$ from this smallest quadrilateral.

The same arguments as before now show

$$|z - w| = (1 - \epsilon(K))^{-\log r}(1 - \epsilon(K))^{\log r/d} = O(d^{\alpha}) = O(|z - w|^{\alpha}).$$

This proves the theorem assuming φ extends continuously to the boundary. Now we have to remove this extra assumption. Assume φ is any K-quasiconformal of $\mathbb{D}$ onto itself, such that $\varphi(0) = 0$. Take r close to 1 and let $\Omega_r = \varphi(\{|z| < r\})$

Then Ω_r is a Jordan domain that satisfies

$$\{|z| < 1 - \delta\} \subset \Omega_r \subset \mathbb{D},$$

with $\delta \to 0$ as $r \nearrow 1$. Let $f_r : \Omega_r \to \mathbb{D}$ be the the conformal map so that $f_r(0) = 0$ and $f_r'(0) > 0$.

By Caratheodory's theorem f_r is a homeomorphism from the closure of Ω_r to the closed unit disk, hence the K-quasiconformal map $g_r = f_r \circ \varphi$ is a homeomorphism from the closed unit disk to itself. Thus the previous argument applies to g_r, and we deduce g_r is α-Hölder.

As $r \nearrow 1$, both f_r and f_r^{-1} tend to the identity on compact subsets of $\mathbb{D}$. In particular, for $z, w \in \mathbb{D}$, we have

$$\begin{aligned} |\varphi(z) - \varphi(w)| &= \lim_{r \nearrow 1} |f_r^{1-}(g_r(z)) - f_r^{-1}(g_r(w))| \\ &= \lim_{r \nearrow 1} |g_r(z) - g_r(w)| \\ &\leq C(K)|z - w|^\alpha. \end{aligned}$$

By the Schwarz Lemma $g_r(z)$ and $g_r(w)$ remain in a compact subset of $\mathbb{D}$ as $r \nearrow 1$.

Thus φ is α-Hölder as well. $\square$

We have now verified that normalized K-quasiconformal maps satisfy the Arzela-Ascoli theorem, so they form a pre-compact family. To prove compactness, we need to prove:

Theorem 4.10. *If $\{f_n\}$ is a sequence of K-quasiconformal maps on Ω that converge uniformly on compact subsets to a homeomorphism f, then f is K-quasiconformal.*

This is immediate from the following result (proven earlier):

Theorem 4.11. *Suppose* $\{h_n\}$ *are homeomorphisms defined on a domain* Ω *and* $Q \subset \Omega$ *is a generalized quadrilateral that is compactly contained in* Ω*. If* $\{h_n\}$ *converge uniformly on compact sets to a homeomorphism* h *on* Ω*, then* $M(h_n(Q)) \to M(h(Q))$

Proof of Theorem 4.10. Any quadrilateral $Q \subset \Omega$ has compact closure in Ω so $f(Q) = \lim_n f_n(Q)$ is a quadrilateral in $f(\Omega)$ and

$$M(f(Q)) = \lim_n M(f_n(Q)) \leq K \lim_n M(Q)$$

by Lemma 2.26. The opposite inequality follows by considering the inverse maps, so we see that f is K-quasiconformal. □

Lemma 4.12. *Suppose* $f : \mathbb{C} \to \mathbb{C}$ *is a* K*-quasiconformal map that fixes both* 0 *and* 1*. Then there is a constant* $0 < C < \infty$*, depending only on* K *so that if* $|z| < 1/C$*, then*

$$C^{-1}|z|^K \leq |f(z)| \leq C|z|^{1/K}.$$

Proof. Since normalized K-quasiconformal maps form a compact family, there here is a constant $A = A(K)$ so that

$$f(\{|z| = 1\}) \subset \{\frac{1}{A} < |z| < A\}.$$

By rescaling we also get that for any $0 < r < \infty$

$$f(\{|z| = r\}) \subset \{\frac{|f(r)|}{A} < |z| < A|f(r)|\}.$$

Thus if $r < A^{-2}$,

$$\{A|f(r)| < |z| < \frac{1}{A}\}\} \subset f(\{r < |z| < 1\}) \subset \{\frac{|f(r)|}{A} < |z| < A\}\}.$$

Comparing moduli in the first inclusion we get

$$\frac{1}{2\pi}\log\frac{1}{A^2|f(r)|} \leq M(f(\{r < |z| < 1\})) \leq \frac{K}{2\pi}\log\frac{1}{r},$$

which gives

$$|f(r)| \geq r^K/A^2.$$

The second inclusion similarly gives

$$\frac{1}{2\pi}\log\frac{A^2}{|f(r)|} \geq M(f(\{r < |z| < 1\})) \geq \frac{1}{2\pi K}\log\frac{1}{r},$$

which implies $|f(r)| \leq A^2 r^{1/K}$. Taking $C = A^2$ proves the lemma. □

Corollary 4.13. *For each $K \geq 1$ there is a $C = C(K) < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and γ is a circle, then there is $w \in \mathbb{C}$ and $r > 0$ so that $f(\gamma) \subset \{z : r \leq |z - w| \leq Cr\}$.*

Proof. Without loss of generality, we can pre and post-compose so that γ is the unit circle and f fixes $0, 1$. By Lemma 4.12, $f(\gamma)$ is then contained in an annulus $\{\frac{1}{C} \leq |z| \leq C\}$, and this gives the result. $\square$

The following is then immediate.

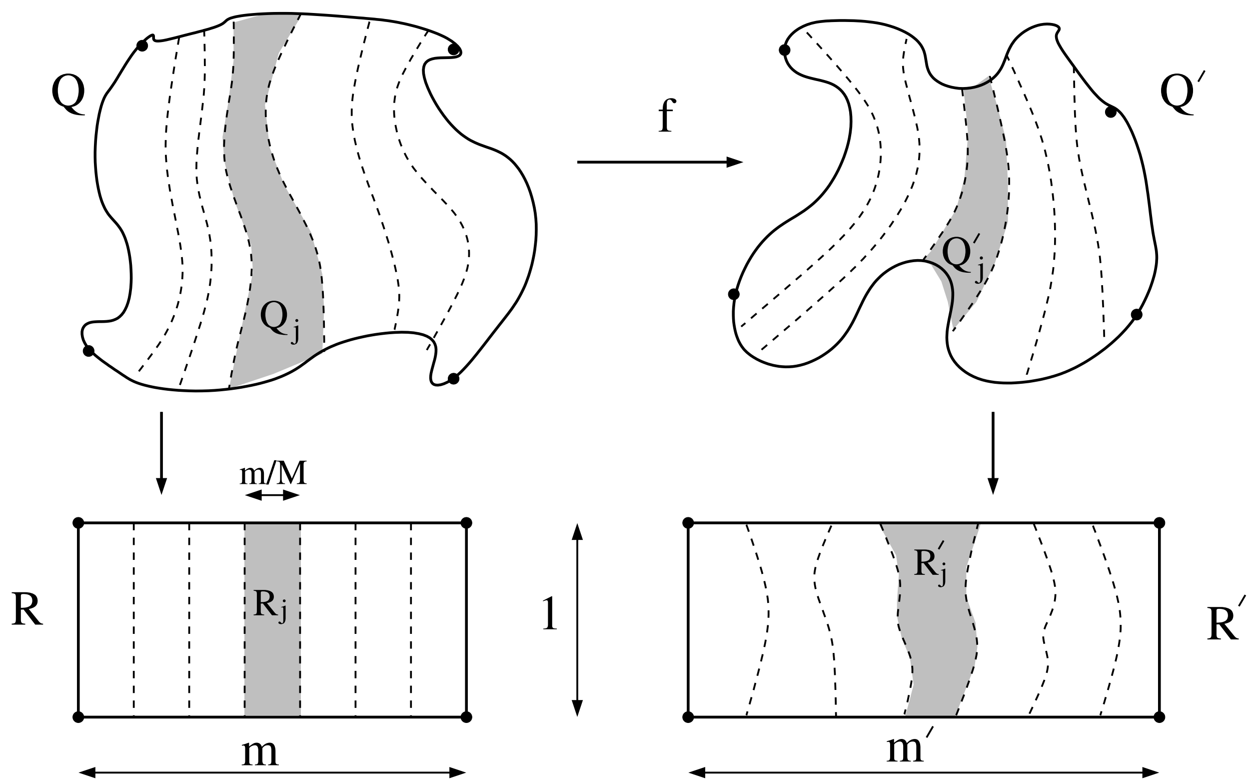
Corollary 4.14. *If f is a K-quasiconformal mapping of the plane and D is a disk, then* $\operatorname{diam}(f(D))^2 \simeq \operatorname{area}(f(D))$, *with constants that depend only on K.*

Quasiconformality is local

The definition of quasiconformality requires us to check the moduli of all quadrilaterals. In this section we prove that it is enough to verify the definition just on all sufficiently small quadrilaterals.

Lemma 3.15. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ that is K-quasiconformal in a neighborhood of each point of Ω, then f is K-quasiconformal on Ω.*

Proof. Suppose $Q \subset \Omega$ is a quadrilateral that is conformally equivalent via a map φ to a $1 \times m$ rectangle R and $Q' = f(Q)$ is conformally equivalent a $1 \times m'$ rectangle R'. Divide R into M equal vertical strips $\{S_j\}$ of dimension $1 \times m/M$.



We have to choose M sufficiently large that two things happen.

First choose $\delta > 0$ so that f^{-1} is K-quasiconformal on any disk of radius δ centered at any point of Q' (we can do this since Q' has compact closure in Ω).

Next, note that the closure of Q' is a union of Jordan arcs γ corresponding via $f \circ \varphi^{-1}$ to vertical line segments in R. By the continuity of $f \circ \varphi^{-1}$ there is an $\eta > 0$ so that if $z \in R$ then $f(\varphi^{-1}(D(z, \eta)))$ has diameter $\leq \delta$. By the continuity of the inverse map, there is an $\epsilon > 0$ so that $x, y \in Q'$ and $|x - y| < \epsilon$ implies $|\varphi(f^{-1}(x)) - \varphi(f^{-1}(y))| \leq \eta$.

Thus for any $\delta > 0$ there is an $\epsilon > 0$ so that if $x, y \in \gamma \subset Q'$ are at most distance ϵ apart, then the arc of γ between then has diameter at most δ (and ϵ is independent of which γ we use).

Choose M so large that each region $Q'_j = f(\varphi^{-1}(R_j))$ contains a disk of radius at most ρ, where ρ will be chosen small depending on ϵ.

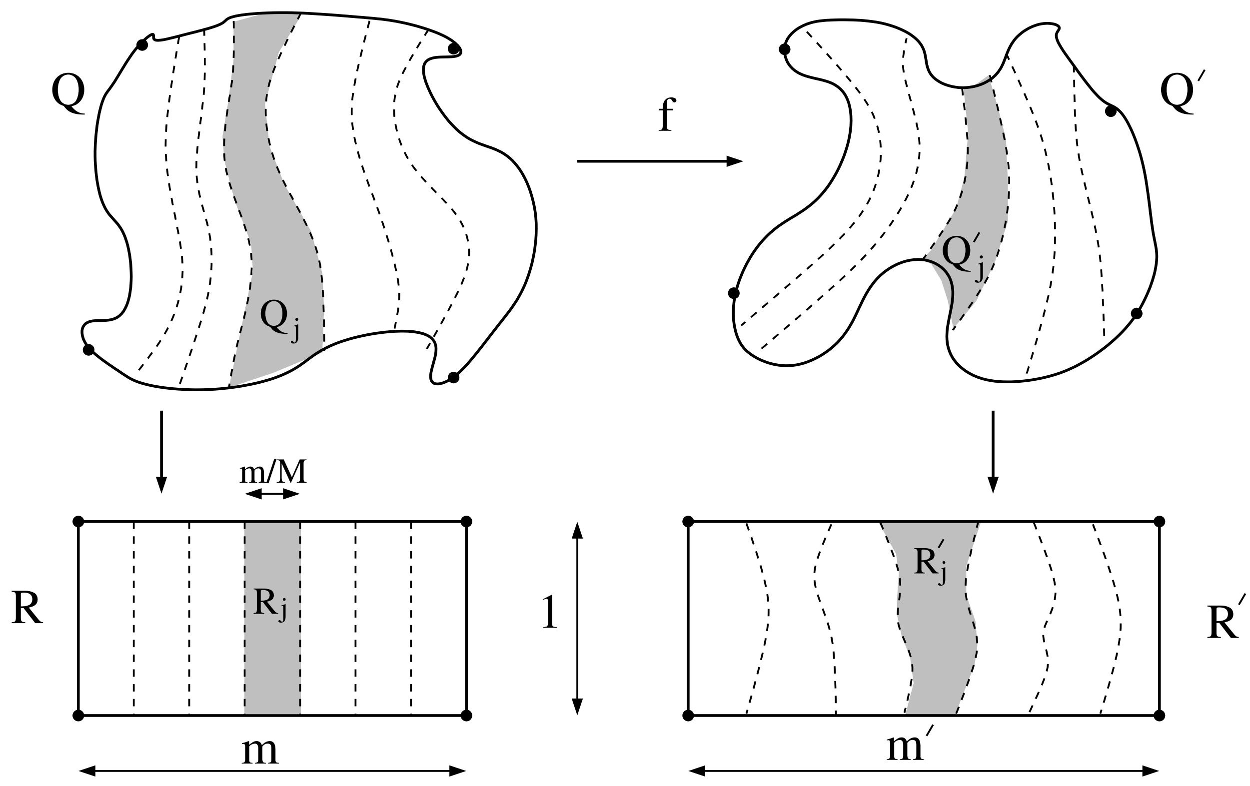
Map Ω_j conformally to a $1 \times m'_j$ rectangle S'_j. Note that this rectangle is conformally equivalent to the region $R'_j = \psi(f(\varphi^{-1}(R_j))) \subset R_j$, both with the obvious choice of vertices.

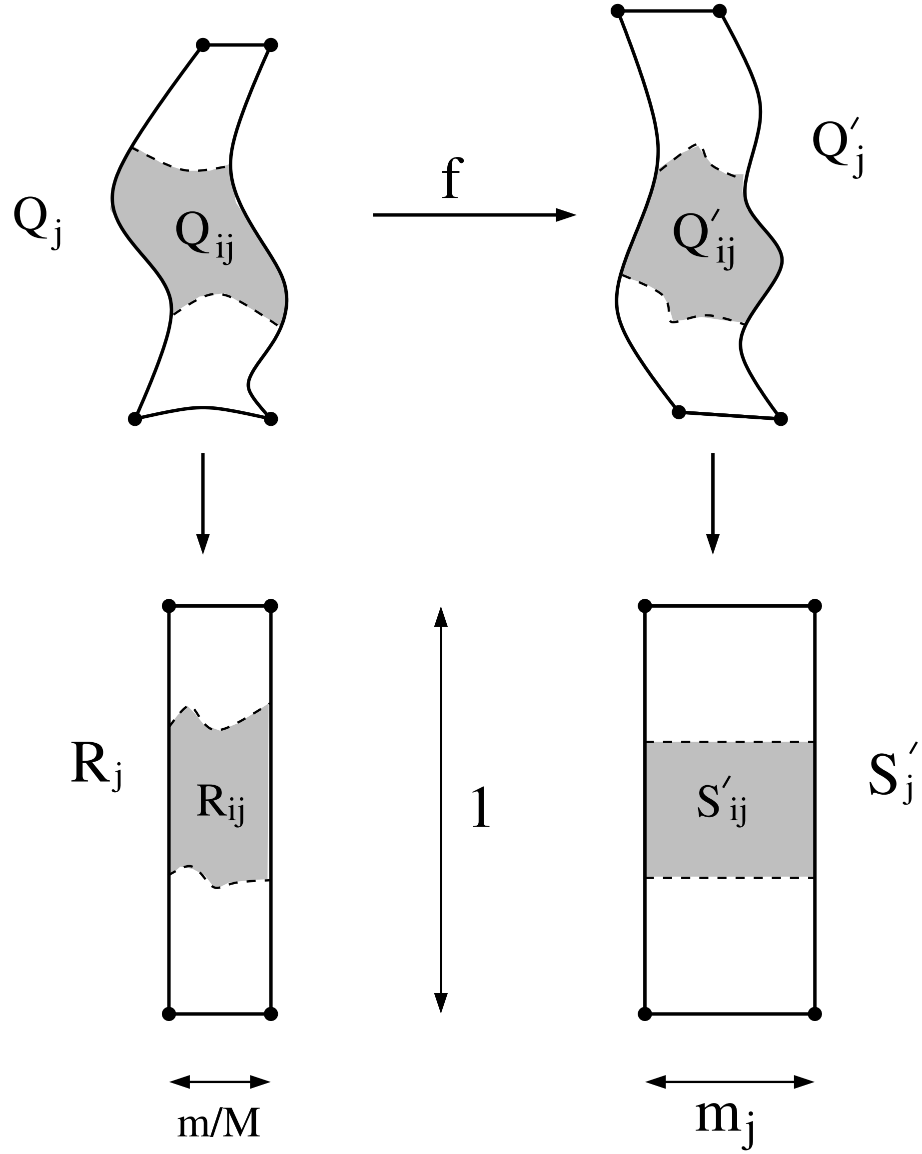
By Lemma 2.24 there is an absolute constant C so that every for every $y \in [0,1]$, there is a $t \in (0,1)$ with $|t-y| \leq Cm_j$ and so that the horizontal cross-cut of R'_j at height t maps via φ_j^{-1} to a Jordan arc of length $\leq C\rho$.

Thus we can divide R'_j by horizontal cross-cuts into rectangles $\{R'_{ij}\}$ of modulus $m'_{ij} \simeq 1$ so that the preimages of these rectangles under ϕ_j are quadrilaterals with two opposite sides of length $\leq C\rho$ and which can be connected inside the quadrilateral by a curve of length $\leq C\rho$.

Taking δ as above, choose ϵ as above corresponding to $\delta/4$ and choose ρ so that $3C\rho < \min(\epsilon, \delta/4)$.

Then all four sides of the quadrilateral Q'_{ij} have diameter $\leq \delta/4$ and hence Q'_{ij} has diameter less than δ and hence lies in a disk where f^{-1} is K-quasiconformal. Let m_{ij} be the modulus of corresponding preimage quadrilateral $Q_{ij} = f^{-1}(Q'_{ij})$.





Q_j
Q_ij
f
Q′_j
Q′_ij
R_j
R_ij
1
S′_ij
S′_j
m/M
m_j

In S'_j consider the path family Γ'_j that connects the "top" and "bottom" sides of this rectangle and let m'_j denote the modulus of this path family (so $1/m'_j$ is its extremal length).

Let m_{ij} denote the modulus of the path family in the subrectangles S'_{ij} (again we take the path family connecting the top and bottom edges). These are conformally equivalent to path families connecting opposite sides of Q'_{ij} and via f^{-1} to path families in Q_{ij} whose modulus is denoted m_{ij}.

Since these quadrilaterals were chosen small enough to fit inside neighborhoods where f is K quasiconformal, we have

$$\frac{m_{ij}}{K} \leq m'_{ij} \leq K m_{ij}.$$

Finally, let Γ_j be the path family that connects the top and bottom of R_j and let Γ'_j be the family that connects the left and right sides of R'.

By the Series Rule

$$\frac{M}{m} = \lambda(\Gamma_j) \geq \sum_i \lambda(\Gamma_{ij}) = \sum_i \frac{1}{m_{ij}}.$$

Similarly,

$$m' = \lambda(\Gamma') \geq \sum_j \lambda(\Gamma'_j) = \sum_j m'_j.$$

We get equality in the Series Rule when a rectangle is cut by vertical lines, so

$$\frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}.$$

Hence

$$\frac{M}{m} \geq \sum_i \frac{1}{m_{ij}} \geq \frac{1}{K} \sum_i \frac{1}{m'_{ij}} = \frac{1}{Km'_j}$$

or

$$\frac{m}{M} \leq Km'_j$$

for every j. Thus

$$m \leq \sum_{j=1}^{M} \frac{m}{M} \leq \sum_j Km'_j \leq Km'.$$

Applying the same result to the inverse map shows f is K-quasiconformal. $\square$

If $K = 1$, then $m = m'$ the last line of the above proof becomes

$$m' = m \leq \sum_j \frac{m}{M} \leq \sum_j m'_j \leq m'.$$

so we deduce

$$\sum_j m'_j = m',$$

whereas in general, we only have $\sum_j m'_j \leq m'$.

We want to use this to deduce that 1-quasiconformal map must be conformal.

We start with:

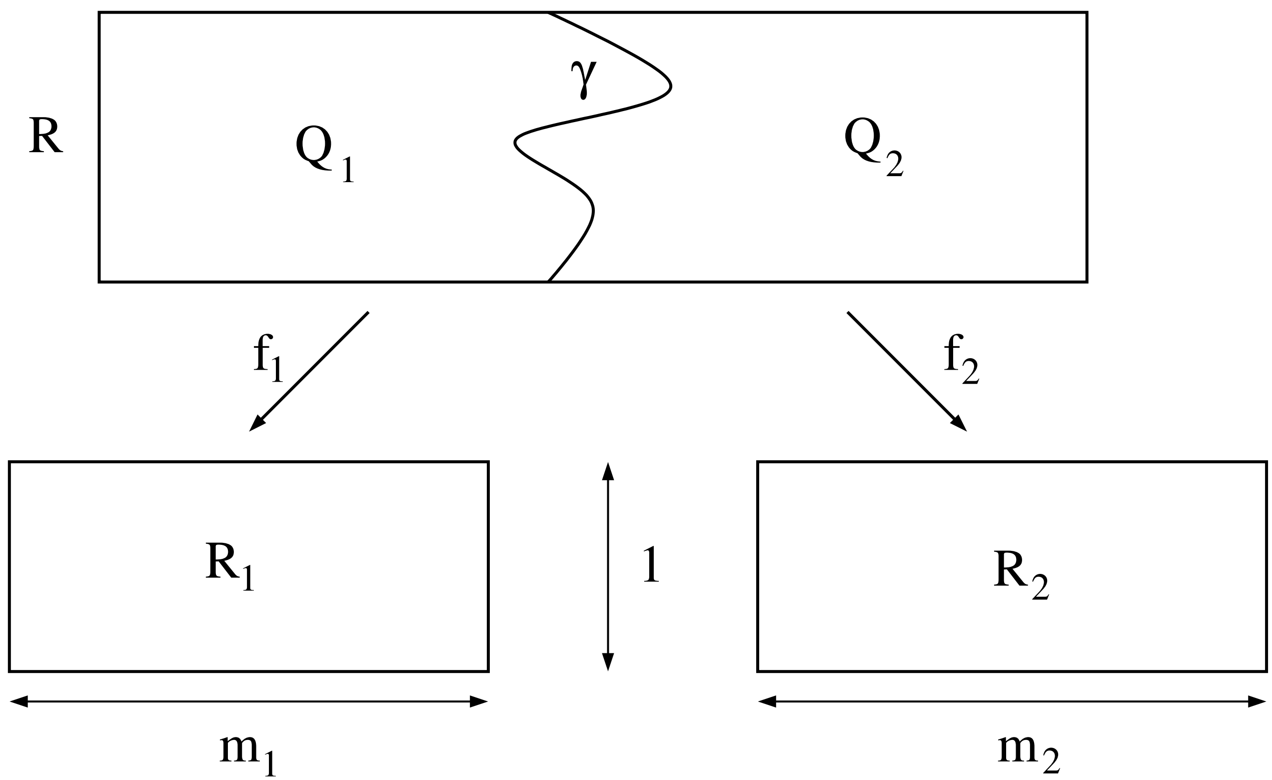
Lemma 3.16. *Consider a* $1 \times m$ *rectangle* R *that is divided into two quadrilaterals* Q_1, Q_2 *of modulus* m_1 *and* m_2 *by a Jordan arc* γ *the connects the top and bottom edges of* R*. If* $m = m_1 + m_2$*, then the curve* γ *is a vertical line segment.*

Proof. Let φ_1, φ_2 be the conformal maps of Q_1, Q_2 onto $1 \times m_1$ and $1 \times m_2$ rectangles R_1, R_2 respectively.

Set $\rho = |f_1'|$ on Q_1 and $\rho = |f_2'|$ in Q_2 and zero elsewhere. Then each horizontal line is cut by γ into pieces one of which connects the left vertical edge of R to γ, and another that connect γ to the right edge of R.

The images of these connect the vertical edges of R_1 and R_2 respectively.

Thus the images have lengths at least m_1 and m_2 respectively, there length of the image of the entire horizontal segment in Q is $\geq m_1 + m_2$.



If we integrate over all horizontal segments in Q, we see

$$\int_R (\rho - 1) dx dy \geq m_1 + m_1 - m = 0.$$

Similarly,

$$\begin{aligned} \int_R (\rho^2 - 1) dx dy &= \text{area}(f_1(Q_1) + \text{area}(f_2(Q_2)) - \text{area}(R) \\ &= (m_1 + m_2) - m \leq 0 \end{aligned}$$

(we would have equality if we knew γ had zero area). Thus

$$\int_Q (\rho - 1)^2 dx dy = \int_Q (\rho^2 - 1) - 2(\rho - 1) dx dy \leq 0.$$

Since $(\rho - 1)^2 \geq 0$, this implies the integral equals zero and hence that that $\rho = 1$ almost everywhere, i.e., f_1 and f_2 are most linear and the curve γ is a vertical line segment. □

Lemma 3.17. *If f is 1-quasiconformal on Ω, then it is conformal on Ω.*

Proof. If f is 1-quasiconformal in the proof of Theorem 3.15, then as noted before Lemma 3.16, we must have

$$\frac{M}{m} = \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}, \qquad m' = \sum_j m'_j,$$

Thus the map $\psi = \varphi' \circ f \circ \varphi^{-1}$ between identical rectangles must be the identity map. Thus $f = (\varphi')^{-1} \circ \varphi$ is a composition of conformal maps, hence conformal. □

Lemma 3.18. *For any* $\delta > 0$ *and and any* $r > 0$ *there is an* $\epsilon > 0$ *so that the following holds. If* $f : \mathbb{C} \to \mathbb{C}$ *is* $(1+\epsilon)$*-quasiconformal and* f *fixes* 0 *and* 1*, then* $|z - f(z)| \leq \delta$ *for all* $|z| < r$*.*

Proof. If not, there is a sequence of $(1+\frac{1}{n})$-quasiconformal maps that all fix 0 and 1 and points $z_n \in D(0,r)$ so that $|z_n - f_n(z_n)| > \delta$.

However, there is a subsequence that converges uniformly on compact subsets of the plane to a 1-quasiconformal map that fixes 0 and 1 and that moves some point by at least δ.

However a 1-quasiconformal map is conformal on $\mathbb{C}$, hence of form $az + b$ and since it fixes both 0 and 1, it is the identity and hence doesn't move any points, a contradiction. □