MAT 627, Spring 2025, Stony Brook University

Topics in Complex Analysis: Quasiconformal Mappings

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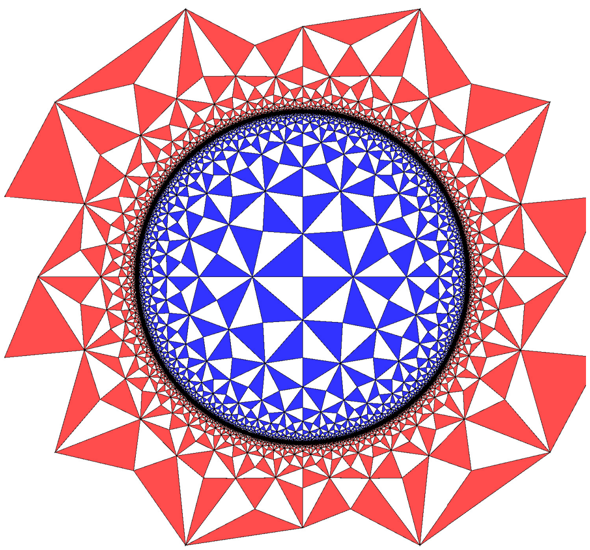
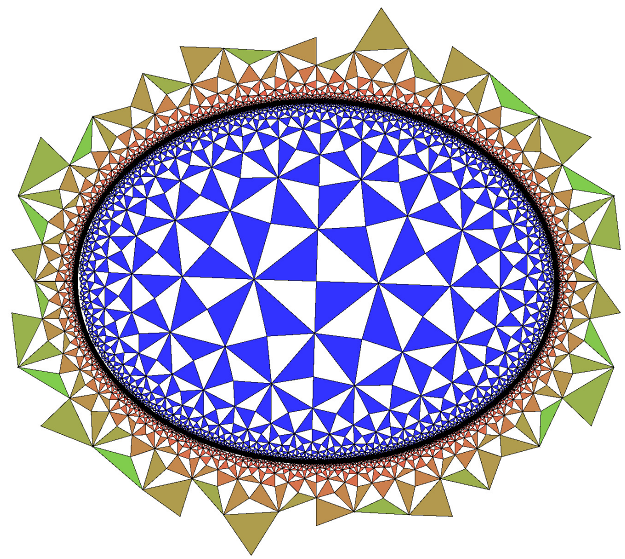




- Review of conformal mappings
- Extremal length and conformal modulus, log capacity, harmonic measure
- Definitions of quasiconformal mappings; geometric and analytic
- Geometric definition and basic properties
- Removable sets
- Analytic definition and measurable Riemann mapping theorem
- Conformal welding
- Further topics

Holomorphic and conformal mappings

A conformal map between planar domains is a C^1, orientation preserving diffeomorphism which preserves angles. Write $f(x, y) = (u(x, y), v(x, y))$. We can compute it derivative matrix

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Since f preserves orientation and angles, the linear map represented by this matrix must be an orientation preserving Euclidean similarity.

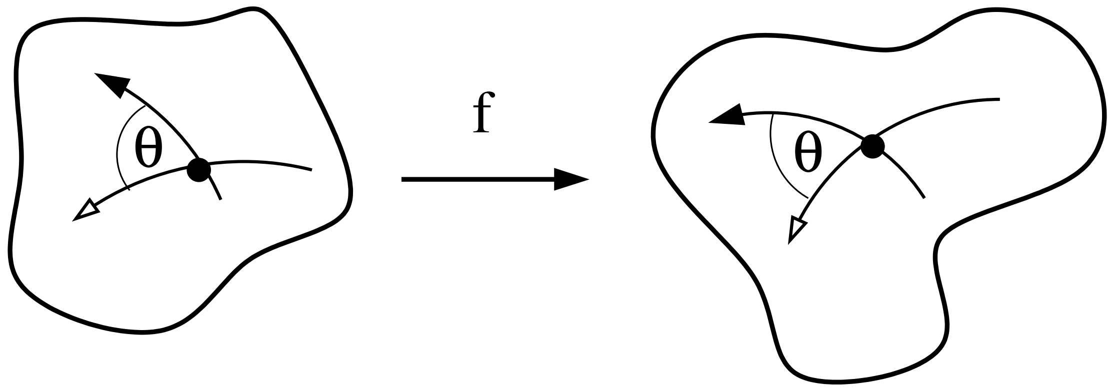
Thus it is a composition of a dilation and rotation and must have the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

which implies

$$u_x = v_y, \quad u_y = -v_x.$$

These are known as the Cauchy-Riemann equations. Thus f is conformal if it is C^1 diffeomorphism which satisfies the Cauchy-Riemann equations.



The simplest examples are the Euclidean similarities, and indeed, these are the only examples if we want maps $\mathbb{R}^2 \to \mathbb{R}^2$.

However, if we consider subdomains of $\mathbb{R}^2$, then there are many more examples. The celebrated Riemann mapping theorem says that any two simply connected planar domains (other that the whole plane) can be mapped to each other by a conformal map.

After the linear maps, the next simplest holomorphic maps are quadratic polynomials. If we take

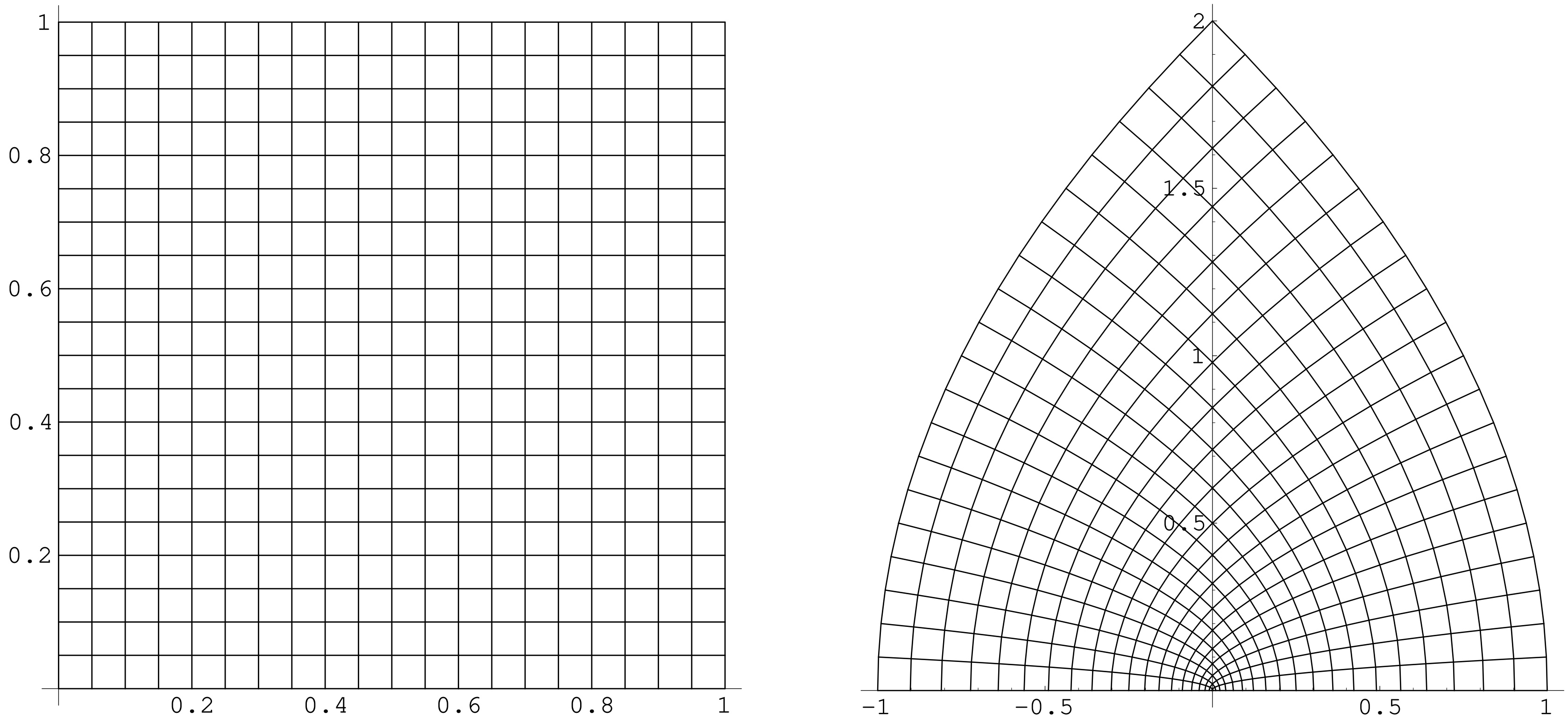
$$f(x, y) = (u(x, y), v(x, y)) = (x^2 - y^2, 2xy),$$

then we can easily check that

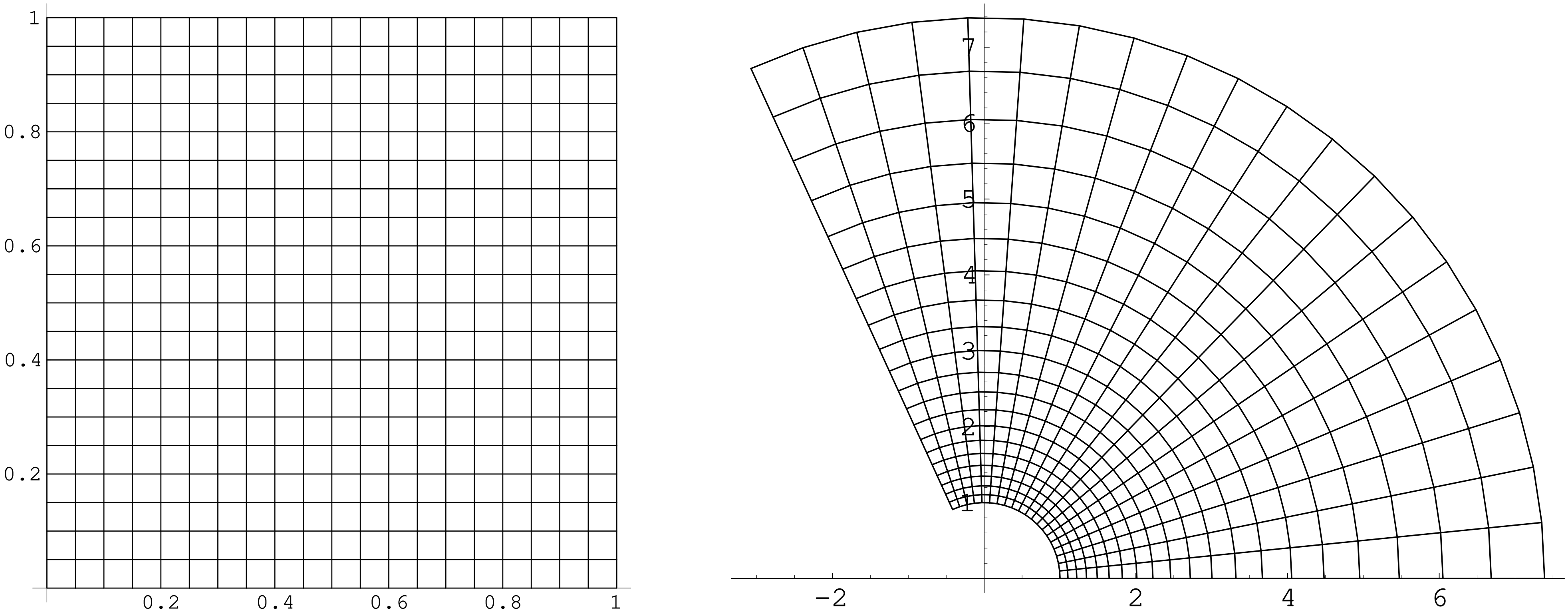
$$Df(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix},$$

so the Cauchy-Riemann equations are satisfied.

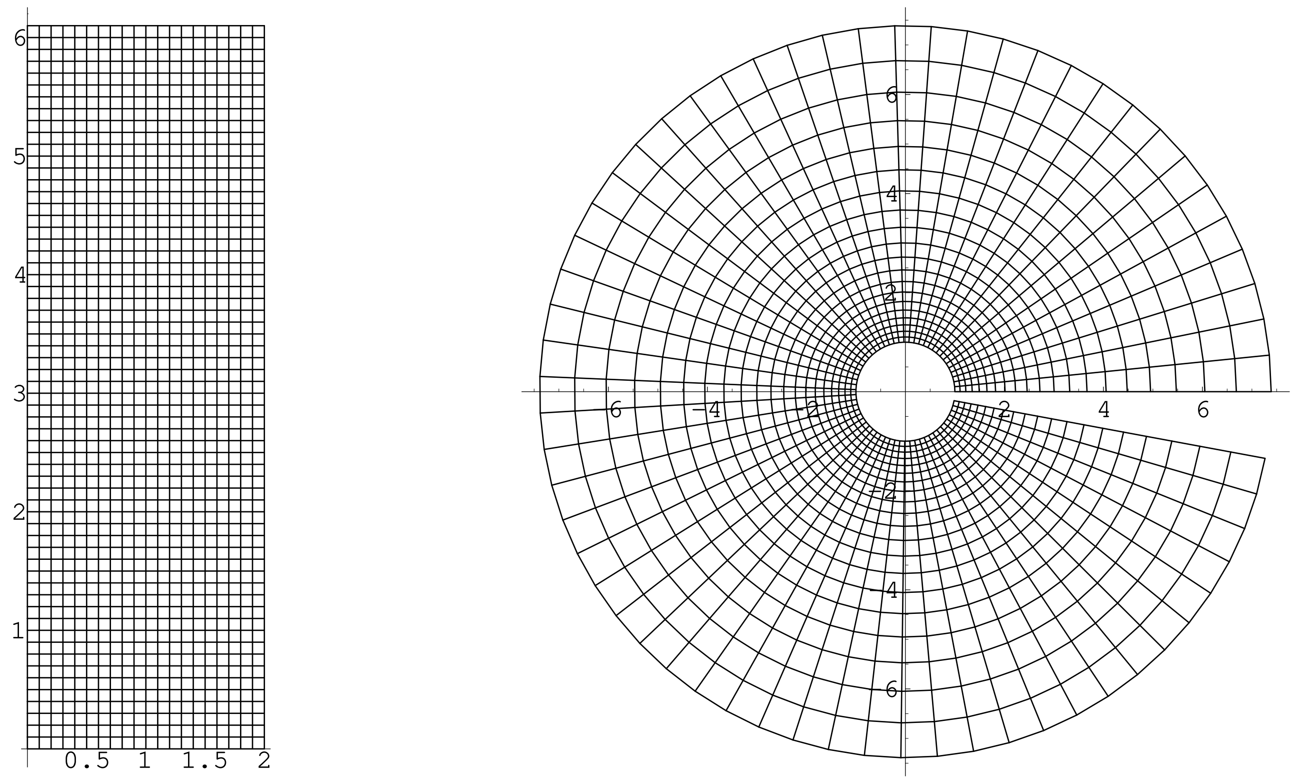
The map is not conformal on the plane since $f(-x, -y) = f(x, y)$ is 2-to-1 for $(x, y) \neq (0, 0)$ and Df vanishes at the origin. However, it is a conformal map if we restrict it to a domain (an open, connected set) where it is 1-to-1, such as the open square $[0, 1]^2$. The map sends this square conformally to a region in the upper half-plane.



This illustrates the map $z \to z^2$ or $(x, y) \to (x^2 - y^2, 2xy)$. The top left shows a grid in the square $[0, 1]^2$. The top right shows the image under squaring map.



The same square grid of $[0,2]^2$ and its image under e^z.



This illustrates the exponential map $e^z = e^r(\cos\theta + i\sin\theta)$.. We take the image of $[0, 2] \times [0, 6]$. The line at height 2π will be mapped into the positive real axis. The top edge of the grid is just below this, so the image stops just before it reaches the axis.

Cauchy's Integral Formula *Suppose* γ *is a cycle contained in a region* Ω *and suppose*

$$\int_\gamma \frac{d\zeta}{\zeta - a} = 0$$

for all $a \notin \Omega$. *If* f *is analytic on* Ω *and* $z \in \mathbb{C} \setminus \gamma$ *then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \cdot \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} d\zeta.$$

Cauchy's formula: Suppose Ω is bounded by a piecewise smooth curve γ and f is holomorphic on a neighborhood of $\overline{\Omega}$. Then

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-w} dz.$$

Pompeiu formula: Suppose Ω is bounded by a piecewise smooth curve and f is smooth on $\overline{\Omega}$.

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \iint_{\Omega} \frac{f_{\overline{z}}}{z-w} dxdy.$$

Möbius transformations

A linear fractional transformation (or Möbius transformation) is a map of the form $z \to (az + b)/(cz + d)$. This is a 1-1, onto, holomorphic map of the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ to itself.

The non-identity Möbius transformations are divided into three classes.

(1) Parabolic transformations have a single fixed point on $\mathbb{S}^2$ and are conjugate to the translation map $z \to z + 1$.

(2) Elliptic maps have two fixed points and are conjugate to the rotation $z \to e^{it}z$ for some $t \in \mathbb{R}$.

(3) The loxodromic transformations also have two fixed points and are conjugate to $z \to \lambda z$ for some $|\lambda| < 1$. If, in addition, λ is real, then the map is called hyperbolic.

Given two sets of three distinct points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ there is a unique Möbius transformation that sends $w_k \to z_k$ for $k = 1, 2, 3$. This map is given by the formula

$$\tau(z) = \frac{w_1 - \zeta w_3}{1 - \zeta},$$

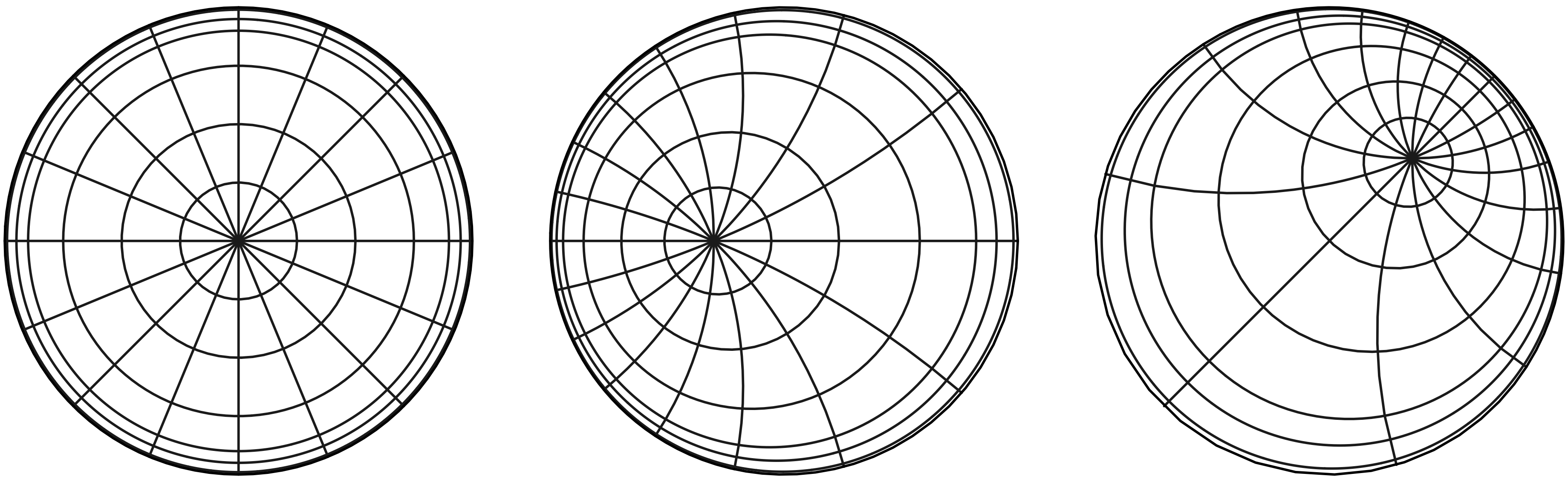
where

$$\zeta = \frac{(w_2 - w_1)}{(w_2 - w_3)} \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

A Möbius transformation sends the unit disk 1-1, onto itself iff it is if the form

$$z \to \lambda \frac{z-a}{1-\bar{a}z},$$

for some $a \in \mathbb{D}$ and $|\lambda| = 1$. In this case, any loxodromic transformation must actually be hyperbolic.



A polar grid in the disk and some images under Möbius transformations that preserve the unit disk.

Given four distinct points a, b, c, d in the plane we define their cross ratio as

$$\mathrm{cr}(a, b, c, d) = \frac{(d-a)(b-c)}{(c-d)(a-b)}.$$

Note that $\mathrm{cr}(a, b, c, z)$ is the unique Möbius transformation which sends a to 0, b to 1 and c to ∞.

This makes it clear that cross ratios are invariant under Möbius transformations; that $\mathrm{cr}(a, b, c, d)$ is real valued iff the four points lie on a circle; and is negative iff in addition the points are labeled in counterclockwise order on the circle.

Möbius transformations form a group under composition. If we identity the transformation $(az+b)/(cz+d)$ with the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then composition of maps is the same as matrix multiplication.

For any non-zero λ, the translations $(\lambda az + \lambda b)/(\lambda cz + \lambda d)$ are all the same, but correspond to different matrices.

We can choose one to represent the transformation, say the one with determinate $ad - bc = 1$, and this identifies the group of transformations the the group $\mathrm{SL}(2, \mathbb{C})$ of two by two matrices of determinate 1.

If $ad = bc$, then

$$\frac{az + b}{cz + d} = \frac{adz + bd}{cdz + d^2} = \frac{bcz + bd}{cdz + d^2} = \frac{b}{d}\frac{cz + d}{cz + d} = \frac{b}{d},$$

is constant and not a Möbius transformation.

The mapping

$$z \to \frac{az+b}{cz+d},$$

can be written as a composition of the maps

$$z \to cz+d, \quad z \to \frac{1}{z}, \quad z \to \frac{a}{c} + \frac{bc-ad}{c}z,$$

which equivalent to claiming

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (bc-ad) & a \\ 0 & c \end{pmatrix}.$$

Either claim follows by a direct computation.

The linear maps have the property that circles map to circles an lines map to lines. The inversion also has this property, although it may interchange the two types of sets.

Lemma 1.1. *Möbius transformations map circles to circles, assuming the convention that lines are considered as circles through infinity.*

It is enough to check this for $1/z$. The equation

$$x^2 + y^2 + \alpha x + \beta y + \gamma = 0 \tag{1.1}$$

defines a circle in the plane, depending on the choice of α, β, γ. If we set $z = x + iy \neq 0$ and $\frac{1}{z} = u + iv$, then

$$u = \mathrm{Re}\left(\frac{x - iy}{x^2 + y^2}\right) = \frac{x}{x^2 + y^2},$$

$$v = \mathrm{Im}\left(\frac{x - iy}{x^2 + y^2}\right) = \frac{-y}{x^2 + y^2},$$

$$x = \frac{u}{u^2 + v^2}, \qquad y = \frac{-v}{u^2 + v^2},$$

So (1.1) becomes

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\frac{\alpha u}{(u^2+v^2)^2}+\frac{-\beta v}{(u^2+v^2)^2}+\gamma=0.$$

After simplifying this becomes

$$\frac{1}{(u^2+v^2)^2}+\frac{\alpha u}{u^2+v^2}+\frac{-\beta v}{u^2+v^2}+\gamma=0,$$

$$1+\alpha u-\beta v+\gamma(u^2+v^2)=0,$$

which is the equation of a circle or line (depending on whether $\gamma \neq 0$ or $\gamma = 0$).

Thus $z \to \frac{1}{z}$ sends a circle missing the origin to a circle, and sends a circle though 0 to a line (which is the same as a circle passing through ∞).

The reflection through a circle $|z-c| = r$ is defined by $\arg(w^* - c) = \arg(w - c)$ and $|w - c| \cdot |w^* - c| = r^2$. Möbius transformation preserve reflections, i.e., if τ is a linear fractional transformation that send circle (or line) C_1 to circle (or line) C_2 then pairs of symmetric points for C_1 are mapped by τ to symmetric points for C_2.

Lemma 1.2. *Every Möbius transformation can be written as a even number of compositions of circle and line reflections.*

The proof is left to the reader.

In higher dimensions, reflections through planes and spheres still makes sense. In this case, Möbius transformations are defined as the group generated by any even number of compositions of such maps (even so that the result is orientation preserving).

The hyperbolic metric

The hyperbolic metric on $\mathbb{D}$ is given by

$d\rho_{\mathbb{D}} = 2|dz|/(1-|z|^2)$. This means that

the hyperbolic length of a rectifiable curve γ in $\mathbb{D}$ is defined as

$$\ell_\rho(\gamma) = \int_\gamma \frac{2|dz|}{1-|z|^2},$$

and the hyperbolic distance between two points $z, w \in \mathbb{D}$ is the infimum of the lengths of paths connecting them (we shall see shortly that there is an explicit formula for this distance in terms of

Corresponding metric on upper half-plane is ds/t.

This metric has constant curvature -4. Some sources use $d\rho_{\mathbb{D}} = |dz|/(1-|z|^2)$, which has curvature -1.

On the disk it is convenient to define the pseudo-hyperbolic metric

$$\rho(z, w) = |\frac{z - w}{1 - \bar{w}z}|.$$

The hyperbolic metric between two points can then be expressed as

$$\psi(w, z) = \log \frac{1 + \rho(w, z)}{1 - \rho(w, z)}.$$

On the upper half-plane the corresponding function is

$$\rho(z, w) = |\frac{z - w}{w - \bar{z}}|,$$

and ψ is given as before. A hyperbolic ball in the disk is also a Euclidean ball, but the hyperbolic and Euclidean centers are different (unless they are both the origin).

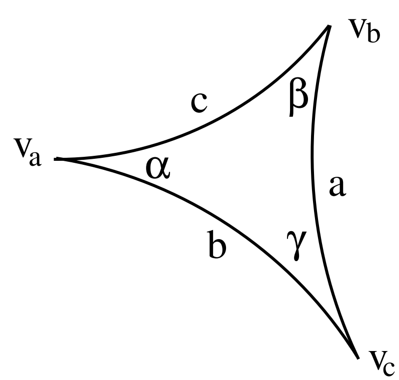
The orientation preserving isometries of the hyperbolic disk are exactly the Möbius transformations that map the disk to itself. All of these have the form

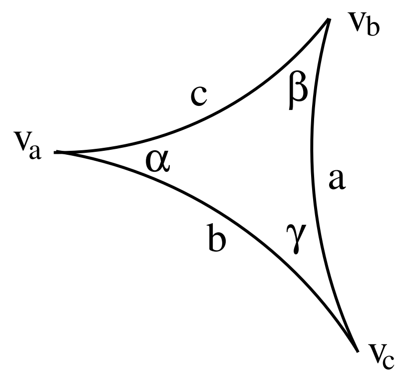
$$e^{i\theta}\frac{z-a}{1-\bar{a}z},$$

where θ is real and $a \in \mathbb{D}$.

Recall the sine and cosine rules for hyperbolic geometry (e.g., see page 148 of Beardon's book "The geometry of discrete groups".

Let T denote a hyperbolic triangle with angles α, β, γ and opposite side lengths denoted by a, b, c.





Then we have the Sine Rule,

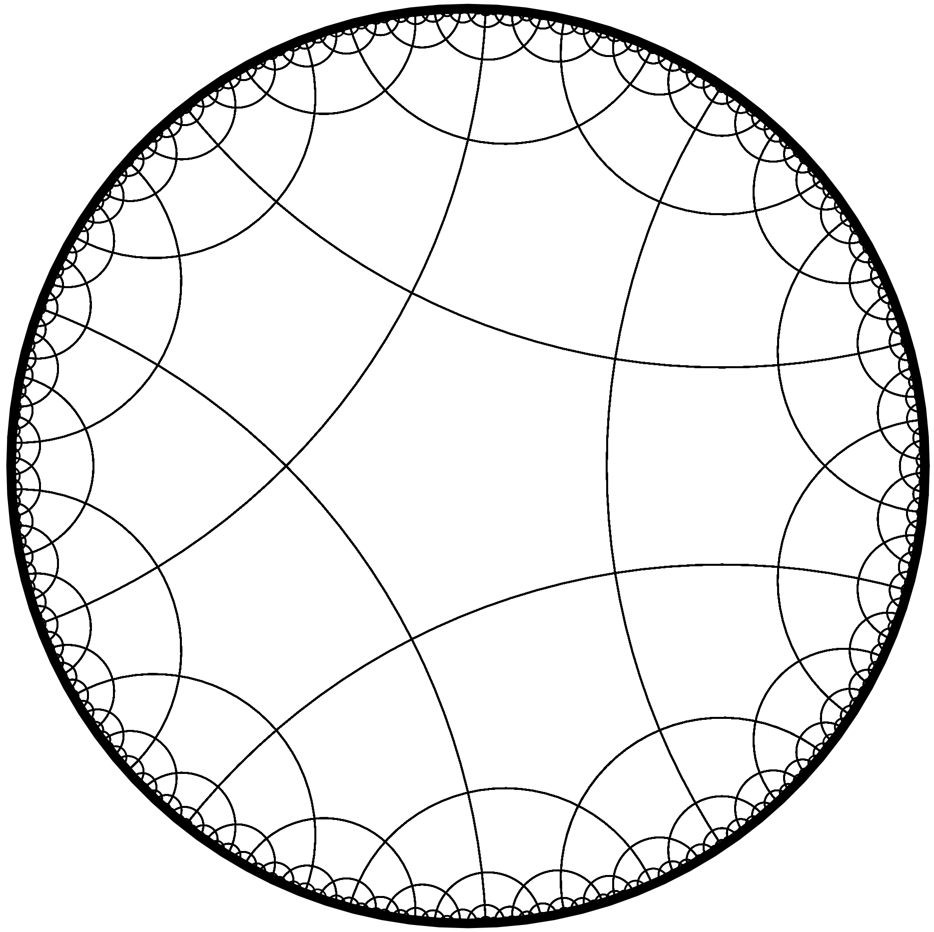
$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma} \tag{1.2}$$

the First Cosine Rule,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma \tag{1.3}$$

and the Second Cosine Rule

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta} \tag{1.4}$$



Normal families

A collection, or family, $\mathcal{F}$ of continuous functions on a region $\Omega \subset \mathbb{C}$ is said to be **normal on $\boldsymbol{\Omega}$** provided every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω.

• The family $\mathcal{F}_1 = \{f_c(z) = z + c : |c| < 1\}$ is normal in $\mathbb{C}$ but not countable.

• The family $\mathcal{F}_2 = \{z^n : n = 0, 1, \ldots\}$ is normal in $\mathbb{D}$ but the only limit function, the zero function, is not in $\mathcal{F}_2$.

• The sequence z^n converges uniformly on each compact subset of $\mathbb{D}$, but does not converge uniformly on $\mathbb{D}$.

• The family $\mathcal{F}_3 = \{g_n\}$, where $g_n \equiv 1$ if n is even and $g_n \equiv 0$ if n is odd, is normal but the sequence $\{g_n\}$ does not converge.

Definition: A family of functions $\mathcal{F}$ defined on a set $E \subset \mathbb{C}$ is

(1) **equicontinuous at $\boldsymbol{w \in E}$** if for each $\epsilon > 0$ there exist a $\delta > 0$ so that if $z \in E$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$.

(2) **equicontinuous on $\boldsymbol{E}$** if it is equicontinuous at each $w \in E$.

(3) **uniformly equicontinuous on $\boldsymbol{E}$** if for each $\epsilon > 0$ there exists a $\delta > 0$ so that if $z, w \in E$ with $|z - w| < \delta$ then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$.

The Arzela-Ascoli Theorem: *A family $\mathcal{F}$ of continuous functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if*

(1) *$\mathcal{F}$ is equicontinuous on Ω, and*

(2) *there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.*

This result is usually proven in MAT 532 (Chap 4 of Folland's book).

Definition: A family $\mathcal{F}$ of continuous functions is said to be **locally bounded** on Ω if for each $w \in \Omega$ there is a $\delta > 0$ and $M < \infty$ so that if $|z - w| < \delta$ then $|f(z)| \leq M$ for all $f \in \mathcal{F}$.

Theorem: *The following are equivalent for a family $\mathcal{F}$ of analytic functions on a region Ω.*

(1) *$\mathcal{F}$ is normal on Ω.*

(2) *$\mathcal{F}$ is locally bounded on Ω.*

(3) *$\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is locally bounded on Ω and there is a $z_0 \in \Omega$ so that $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.*

Montel's Theorem: *A family $\mathcal{F}$ of meromorphic functions on a region Ω that omits three distinct fixed values $a, b, c \in \mathbb{C}^*$ is normal in the chordal metric.*

Picard's Great Theorem *If f is meromorphic in $\Omega = \{z : 0 < |z - z_0| < \delta\}$, and if f omits three (distinct) values in $\mathbb{C}^*$, then f extends to be meromorphic in $\Omega \cup \{z_0\}$.*

• An equivalent formulation of Picard's great theorem is that an analytic function omits at most one complex number in every neighborhood of an essential singularity.

• $f(z) = e^{1/z}$ does omit the values 0 and ∞ in every neighborhood of the essential singularity 0, so that Picard's theorem is the strongest possible statement.

• The weaker statement that a non-constant entire function can omit at most one complex number is usually called **Picard's little theorem**.

See Emile Picard

Normal families can be used to prove results like:

Koebe: *There is a* $K > 0$ *so that if* f *is analytic and one-to-one on* $\mathbb{D}$ *with* $f(0) = 0$ *and* $f'(0) = 1$*, then* $f(\mathbb{D}) \supset \{z : |z| < K\}$.

A sharper version is known and called the Koebe 1/4-theorem.

Theorem (Koebe 1/4-theorem): Assume $f(z) = z + a_2 z^2 + \dots$ is univalent on $\mathbb{D}$. Then $|a_2| \leq 2$ and

$$\operatorname{dist}(0, \partial f(\mathbb{D})) \geq \frac{1}{4}$$

.

Theorem (Koebe's estimate): Suppose f is a conformal map from $\mathbb{D}$ to a simply connected region Ω. Then for all $z \in \mathbb{D}$,

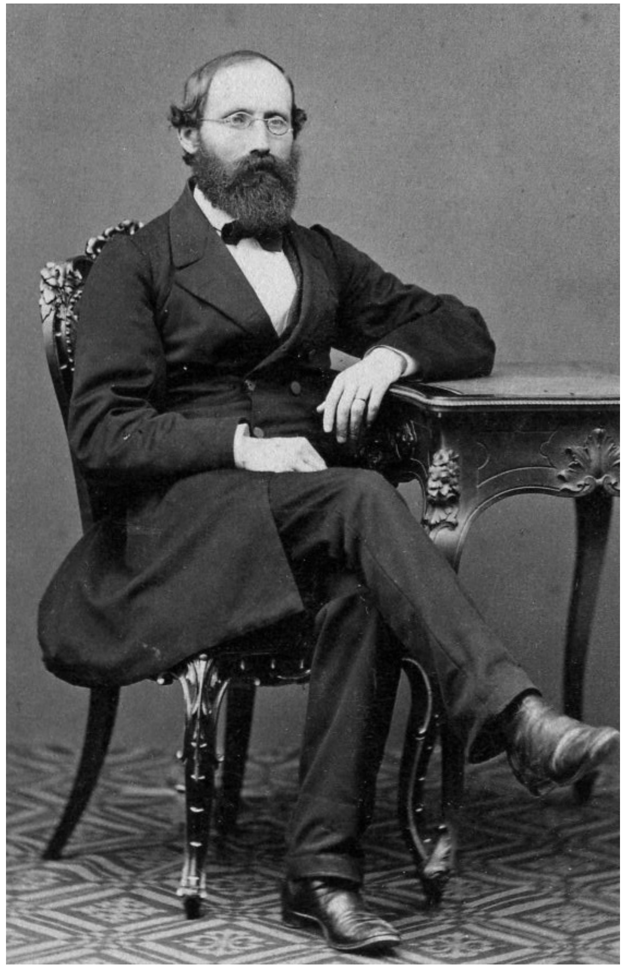
$$\frac{1}{4}|f'(z)|(1 - |z|^2) \leq \operatorname{dist}(f(z), \partial\Omega) \leq |f'(z)|(1 - \mathbf{z}|^2)$$

The Riemann mapping theorem

The Riemann Mapping Theorem *Suppose* $\Omega \subset \mathbb{C}$ *is simply-connected and* $\Omega \neq \mathbb{C}$*. Then there exists a one-to-one analytic map* f *of* Ω *onto* $\mathbb{D} = \{z : |z| < 1\}$*. If* $z_0 \in \Omega$ *then there is a unique such map with* $f(z_0) = 0$ *and* $f'(z_0) > 0$*.*

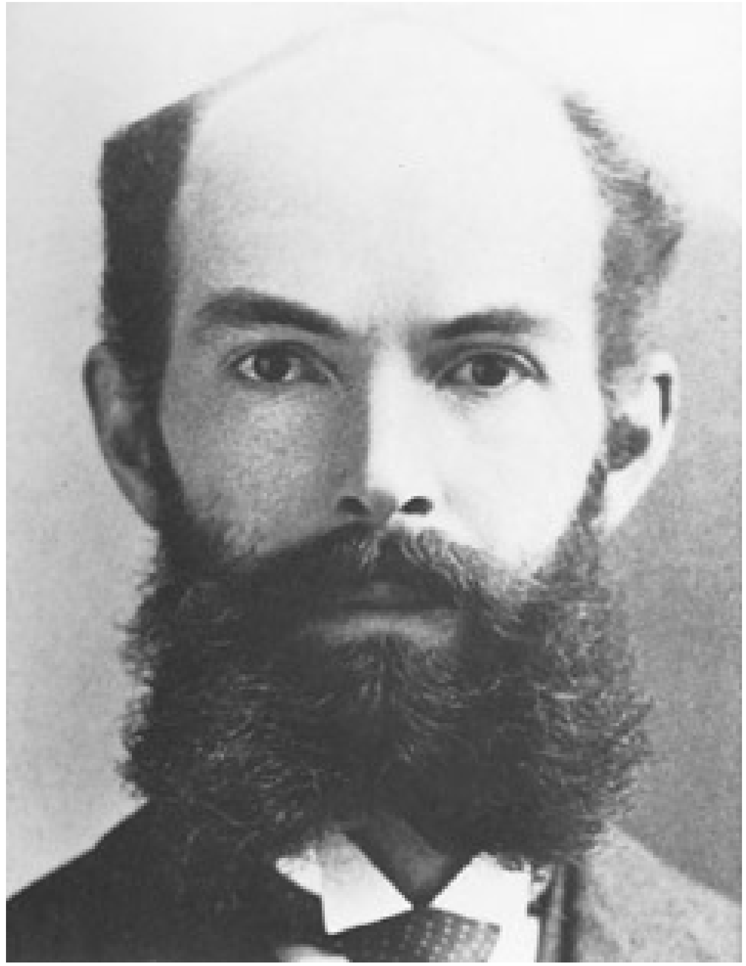
Idea of proof:

- Show there is a conformal map of Ω into $\mathbb{D}$ so that $f(z_0) = 0$ and $f'(z_0) > 0$.
- Among all such maps, choose one maximizing $f'(z_0)$. (uses normality)
- Prove this map is 1-1 and onto $\mathbb{D}$.



Georg Friedrich Bernhard Riemann

Stated RMT in 1851



William Fogg Osgood

First proof of RMT, Trans. AMS, vol. 1, 1900

The proof of Osgood represented, in my opinion, the "coming of age" of mathematics in America. Until then, numerous American mathematicians had gone to Europe for their doctorates, or for other advanced study, as indeed did Osgood. But the mathematical productivity in this country in quality lagged behind that of Europe, and no American before 1900 had reached the heights that Osgood then reached.

J.L. Walsh, "History of the Riemann mapping theorem", Amer. Math. Monthly, 1973.

Schwarz-Christoffel Formula: *Suppose* Ω *is a bounded simply-connected region whose positively oriented boundary* $\partial\Omega$ *is a polygon with vertices* $v_1, ..., v_n$. *Suppose the tangent direction on* $\partial\Omega$ *increases by* $\pi\alpha_j$ *at* v_j, $-1 < \alpha_j < 1$. *Then there exists* $x_1 < x_2 < \cdots < x_n$ *and constants* c_1, c_2 *so that*

$$f(z) = c_1 \int_{\gamma_z} \prod_{j=1}^{n} (\zeta - x_j)^{-\alpha_j} d\zeta + c_2$$

is a conformal map of $\mathbb{H}$ *onto* Ω, *where the integral is along any curve* γ_z *in* $\mathbb{H}$ *from* i *to* z.

Elwin Bruno Christtoffel

Hermann Amandus Schwarz

$$f(z) = c_1 \int_{\gamma_z} \prod_{j=1}^{n} (\zeta - x_j)^{-\alpha_j} d\zeta + c_2$$

The exponents $\{\alpha_j\}$ are known from the target polygon, but the $\{x_j\}$ are not.

- The points are the preimages of the vertices under the conformal map.

- Finding these points numerically is challenging: there are several heuristics that work in practice, but are not proven to work, e.g., SC-Toolbox program by T. Driscoll.

- A provably correct algorithm is given in the paper Conformal mapping in linear time and explained in the recorded lecture Fast conformal mapping via computational and hyperbolic geometry.

A **Jordan region** is simply-connected region in $\mathbb{C}^*$ whose boundary is a Jordan curve.

Carathéodory-Tohorst Theorem: *If* φ *is a conformal map of* $\mathbb{D}$ *onto a Jordan region* Ω*, then* φ *extends to be a homeomorphism of* $\overline{\mathbb{D}}$ *onto* $\overline{\Omega}$*. In particular* $\varphi(e^{it})$ *is a parameterization of* $\partial\Omega$*.*

Although usually called “Carathéodory’s theorem, the result actually appears in the 1917 Bonn thesis of Marie Torhorst, a student of Carathéodory.

For a discussion of the history, see On prime ends and local connectivity by Lasse Rempe. Torhorst did not become an academic mathematician, but eventually became Minister of Education for the state of Thüringen in communist East Germany following WWII.

A compact set K is called "locally connected" if whenever U is a relatively open subset of K and $z \in U \subset K$, there is a relatively open subset of K that is connected and such that $z \in V \subset U$.

This is equivalent to K being a continuous image of $[0, 1]$.

Carathéodory's extends to say that a conformal map $f : \mathbb{D} \to \Omega$ has a continuous extension to the boundary iff $\partial\Omega$ is locally connected.

We will prove the theorem later in the course. Proof uses "length-area" method which is closely connected to extremal length and quasiconformal maps.

The uniformization theorem

Suppose W is a Riemann surface and $p \in W$.

The Green's function on W with pole at p_0 is a positive function $G(z, p_0)$ that is harmonic on $W \setminus \{p\}$, has a logarithmic pole at p_0 and tends to zero at ∞.

For example, $\log \frac{1}{|z|}$ is the Green's function for $\mathbb{D}$ with pole at 0.

Some Riemann surfaces have a Green's function; some do not.

Very important distinction. Many different characterizations of two cases.

A Riemann surface has a Green's function iff several other conditions hold.

(1) Brownian motion is recurrent.

(2) Geodesic flow on the unit tangent bundle of W is ergodic.

(3) Poincare series of covering group Γ diverges.

(4) Γ has the Mostow rigidity property (conjugating circle homeomorphisms are Möbius or singular).

(5) Γ has the Bowen's property (corresponding limit sets are either a circle or have dimension > 1).

(6) Almost every geodesic ray is recurrent. Equivalently, the set of escaping geodesic rays from a point $p \in W$ has zero (visual) measure.

The Uniformization, Case 1: *If W is a simply-connected Riemann surface then the following are equivalent:*

$$g_W(p, p_0) \text{ exists for some } p_0 \in W$$

$$g_W(p, p_0) \text{ exists for all } p_0 \in W,$$

$$\text{There is a one-to-one analytic map } \varphi \text{ of } W \text{ onto } \mathbb{D}.$$

Moreover if g_W exists, then

$$g_W(p_1, p_0) = g_W(p_0, p_1),$$

and $g_W(p, p_0) = -\log|\varphi(p)|$, where $\varphi(p_0) = 0$.

Paul Koebe

Proved uniformization theorem in 1907.

The dipole Green's function has two logarithmic poles with opposite signs, e.g.,

$$\log \left| \frac{z-a}{z-b} \right|$$

on the plane. This has two opposite poles and tends to 0 at infinity.

The next lemma says that a dipole Green's function always exists.

For surfaces with Green's function this is easy: take $G(z,p) - G(z,q)$ for $p \neq q$.

The Uniformization Theorem, Case 2 *Suppose* W *is a simply-connected Riemann surface for which Green's function does not exist.*
If W *is compact, then there is a one-to-one analytic map of* W *onto* $\mathbb{C}^*$.
If W *is not compact, there is a one-to-one analytic map of* W *onto* $\mathbb{C}$.

The Uniformization Theorem: *Suppose W is a simply-connected Riemann surface.*

(1) *If Green's function exists for W, then there is a one-to-one analytic map of W onto $\mathbb{D}$.*
(2) *If W is compact, then there is a one-to-one analytic map of W onto $\mathbb{C}^*$.*
(3) *If W is not compact and if Green's function does not exist for W, then there is a one-to-one analytic map of W onto $\mathbb{C}$.*

Theorem: *If* $U = \mathbb{C}^*$, $\mathbb{C}$, *or* $\mathbb{D}$ *and if* $\mathbb{G}$ *is a properly discontinuous group of LFTs of* U *onto* U, *then* $U/\mathbb{G}$ *is a Riemann surface. A function* f *is analytic, meromorphic, harmonic, or subharmonic on* $U/\mathbb{G}$ *if and only if there is a function* h *defined on* U *which is (respectively) analytic, meromorphic, harmonic, or subharmonic on* U *satisfying* $h \circ \tau = h$ *for all* $\tau \in \mathbb{G}$ *and* $h = f \circ \pi$ *where* $\pi : U \to U/\mathbb{G}$ *is the quotient map. Every Riemann surface is conformally equivalent to* $U/\mathbb{G}$ *for some such* U *and* $\mathbb{G}$.

Properly discontinuous: every point has a neighborhood U so $U \cap g(U) \neq \emptyset$ implies $g = \mathrm{Id}$.

The only Riemann surface covered by the $\mathbb{C}^*$ is $\mathbb{C}^*$ (Proposition 16.2).

The only surfaces covered by $\mathbb{C}$ are $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, and tori (Proposition 16.3).

Any other Riemann surface is covered by the disk $\mathbb{D}$.