MAT 627, Spring 2025, Stony Brook University

Topics in Complex Analysis: Quasiconformal Mappings

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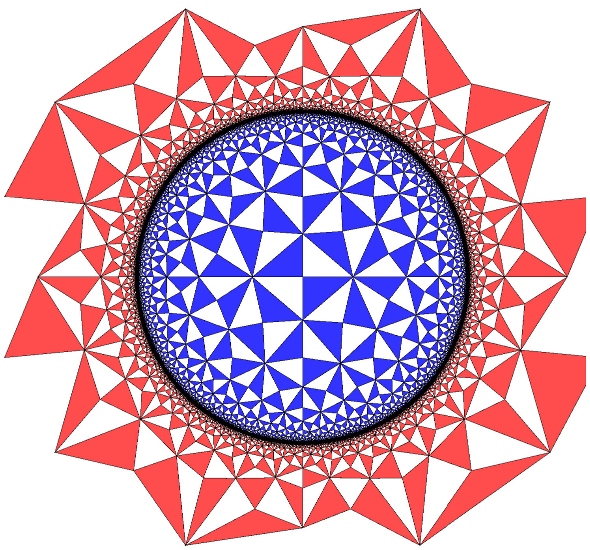
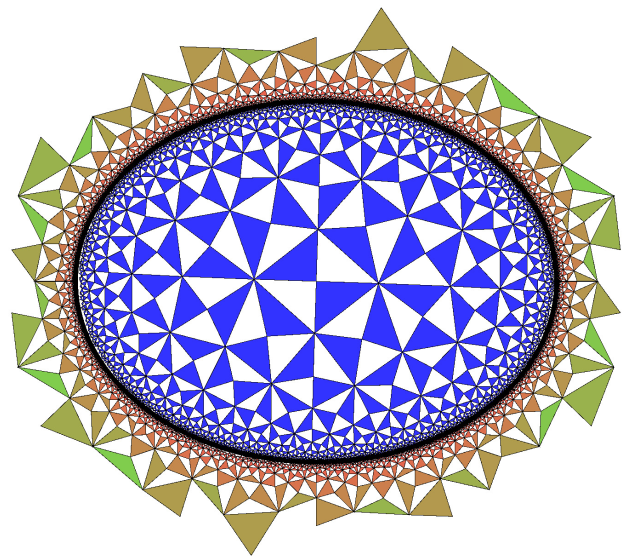




This semester I hope to cover the following topics:

- Review of conformal mappings
- Extremal length and conformal modulus, log capacity, harmonic measure
- Definitions of quasiconformal mappings; geometric and analytic
- Basic properties
- Quasisymmetric maps and boundary extension
- The measurable Riemann mapping theorem
- Removable sets
- Conformal welding
- David maps
- Astala's theorems on area and dimension distortion
- Quasiconformal maps on metric spaces
- Conformal dimension

Logarithmic Capacity

Measures and capacities both measure the size of sets. Measures are countably additive; capacities need not be.

Many capacities are associated to a function that blows up at the origin, such as $\log 1/|z|$ or $|z|^{-\alpha}$.

Many natural problems in analysis have answers given in terms of capacities. For example, a Brownian motion in the plane hits a set E with positive probability iff the set has positive log capacity.

Hausdorff content: given as set E, let $\{D_j\} = \{D(x_j, r_j)\}$, be a covering of E by disks and define the α-Hausdorff content

$$\mathcal{H}^\alpha_\infty(E) \inf \sum_j r_j^\alpha,$$

where the infimum is over all coverings of E.

Power r^α may be replaced by any increasing function $\phi(r)$.

Hausdorff measure: content is not a measure, but can be made into a measure by requiring covering disks to be small. We define

$$\mathcal{H}^\alpha(E) = \lim_{\delta \searrow 0} \inf \sum_j r_j^\alpha,$$

where the infimum is over all coverings with $\sup r_j \leq \delta$.

When $\alpha = 1$ this gives (a multiple of) Lebesgue measure on $\mathbb{R}$.

When $\alpha = 2$ this gives (a multiple of) Lebesgue measure on $\mathbb{R}^2$.

Hausdorff dimension: $\dim(E) = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\}$.

Standard Cantor set has dimension $\log 2 / \log 3$.

Von Koch Snowflake has dimension $\log 4 / \log 3$.

There are other dimensions: Minkowski, packing, Assouad,...

Lemma 3.1 (Frostman's Lemma)**.** *Let φ be a gauge function. Let $K \subset \mathbb{R}^d$ be a compact set with positive Hausdorff content, $\mathcal{H}^{\varphi}_{\infty}(K) > 0$. Then there is a positive Borel measure μ on K satisfying*

$$\mu(B) \leq C_d \varphi(|B|), \tag{3.6}$$

for all balls B and

$$\mu(K) \geq \mathcal{H}^{\varphi}_{\infty}(K).$$

Here C_d is a positive constant depending only on d.

For proof, see Chapter 3 of text by Bishop and Peres.

Let X be a Polish topological space (compatible complete, separable metric).

If Y is Polish, then a subset $E \subset Y$ is called **analytic** if there exists a Polish space X and a continuous map $f\colon X \to Y$ such that $E = f(X)$.

Analytic sets are also called Suslin sets in honor of Mikhail Yakovlevich Suslin. The analytic subsets of Y are often denoted by $A(Y)$ or $\Sigma_1^1(Y)$.

In any uncountable Polish space there exist analytic sets which are not Borel sets. For example see "Conformal removability is hard" by C. Bishop, or textbook by Bruckner, Bruckner and Thomson.

Lemma 3.2. *If X is Polish, then every Borel set $E \subset X$ is analytic.*

For a proof see Appendix B of text by Bishop and Peres.

Lebesgue famously (falsely) claimed continuous images of Borel sets are Borel.

Every analytic set is Lebesgue measurable.

A set function Ψ defined on all subsets of X is called a **Choquet capacity** if

(a) $\Psi(E_1) \leq \Psi(E_2)$ whenever $E_1 \subset E_2$;

(b) $\Psi(E) = \inf_{O \supset E \text{open}} \Psi(O)$ for all $E \subset X$;

(c) for all increasing sequences $\{E_n : n \in \mathbb{N}\}$ of sets in X,

$$\Psi\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n \to \infty} \Psi(E_n).$$

Given Ψ we can define a set function Ψ_* on all sets $E \subset X$ by

$$\Psi_*(E) = \sup_{F \subset E \text{compact}} \Psi(F).$$

A set E is called **capacitable** if $\Psi(E) = \Psi_*(E)$.

Theorem 3.3 (Choquet Capacitability Theorem)**.** *If* Ψ *is a Choquet capacity on a compact metric space* X*, then all analytic subsets of* X *are capacitable.*

For proof see Appendix B of text by Bishop and Peres.

later we will prove compact sets are capacitable.

Suppose $\mu \geq 0$ is a finite Borel measure on $\mathbb{C}$. Define its potential function as

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w), z \in \mathbb{C}.$$

and its energy integral by

$$I(\mu) = \iint \log \frac{2}{|z-w|} d\mu(z) d\mu(w) = \int U_\mu(z) d\mu(z).$$

We put the "2" in the numerator so that the integrand is non-negative when $z, w \in \mathbb{T}$, however, this is a non-standard usage.

Measures energy needed to assemble particles with repelling force log.

Suppose E is Borel and μ is a positive measure that has its closed support inside E. We say μ is admissible for E if $U_\mu \leq 1$ on E and we define the **logarithmic capacity** of E as

$$\operatorname{cap}(E) = \sup\{\|\mu\| : \mu \text{ is admissible for } E\}$$

and we write $\mu \in \mathcal{A}(E)$.

Alternatively, the capacity of E is the infimum of $\sup U_\mu$ over all probability measures supported on E.

We define the **outer capacity** (or exterior capacity) as

$$\operatorname{cap}^*(E) = \inf\{\operatorname{cap}(V) : E \subset V, V \text{open}\}.$$

We say that a set E is **capacitable** if $\operatorname{cap}(E) = \operatorname{cap}^*(E)$.

The logarithmic kernel can be replaced by other functions, e.g., $|z-w|^{-\alpha}$, and there is a different capacity associated to each one.

To be precise, we should denote logarithmic capacity as $\text{cap}_{\log}$ or logcap, but to simplify notation we simply use "cap" and will often refer to logarithmic capacity as just "capacity". Since we do not use any other capacities in these notes, this abuse should not cause confusion.

WARNING: The logarithmic capacity that we have defined is **NOT** the same as is used in other texts such as Garnett and Marshall's book but it is related to what they call the Robin's constant of E, denoted $\gamma(E)$.

The exact relationship is $\gamma(E) = \frac{1}{\mathrm{cap}(E)} - \log 2$. Garnett and Marshall define the logarithmic capacity of E as $\exp(-\gamma(E))$.

The reason for doing this is that the logarithmic kernel $\log \frac{1}{|z-w|}$ takes both positive and negative values in the plane, so the potential functions for general measures and the Robin's constant for general sets need not be non-negative.

Exponentiating takes care of this. Since we are only interested in computing the capacity of subsets of the circle, taking the extra "2" in the logarithm gave us a non-negative kernel on the unit circle, and we defined a corresponding capacity in the usual way.

Sets of zero logarithmic capacity must be very small, indeed the following computations will show that they must have dimension zero.

Corollary 3.4. *If E has positive Hausdorff dimension, then it has positive logarithmic capacity.*

Proof. By Frostman's Lemma, if E has positive dimension then there is a measure μ supported on E such that $\mu(D(x,r)) \leq Cr^\alpha$ for all x and some $C < \infty$ and $\alpha > 0$.

We claim μ has bounded potential. Break the integral over the plane into dyadic annuli $A_n = \{2^{-n-1} < |z| \leq 2^{-n}\}$.

$$\begin{aligned} U_\mu(z) &= \int_{\mathbb{R}^2} \log \frac{d\mu(w)}{|z-w|} \\ &= \sum_n \int_{A_n} \log \frac{d\mu(w)}{|z-w|} \\ &\leq \sum_n 2^{-n\alpha} \log 2^{-(n+1)\alpha} \\ &= \log 2 \sum_n 2^{-n\alpha}(n+1)\alpha \\ &= C_\alpha. \end{aligned}$$

Since U_μ is bounded above by C_α, the log capacity of E is bounded below by $\|\mu\|/C_\alpha = \mathcal{H}^\alpha_\infty(E)/C_\alpha > 0$. □

Lemma 3.5. *U_μ is lower semi-continuous, i.e.,*

$$\liminf_{z \to z_0} U_\mu(z) \geq U_\mu(z_0).$$

Proof. Fatou's lemma. □

Recall that $\mu_n \to \mu$ weak-* if $\int f d\mu_n \to \int f d\mu$ for every continuous function f of compact support.

Lemma 3.6. *If* $\{\mu_n\}$ *are positive measures and* $\mu_n \to \mu$ *weak*, then* $\liminf_n U_{\mu_n}(z) \geq U_\mu(z)$.

Proof. If we replace $\varphi = \log \frac{2}{|z-w|}$ by the continuous kernel $\varphi_r = \max(r, \varphi)$ in the definition of U to get U^r, then weak convergence implies

$$\lim_n U^r_{\mu_n}(z) \to U^r_\mu(z).$$

So for any $\epsilon > 0$ we can choose N so that $n > N$ implies

$$U^r_{\mu_n}(z) \geq U^r_\mu(z) - \epsilon.$$

As $r \to \infty$, we have $U^r_{\mu_n} \nearrow U_{\mu_n}$, by the monotone convergence theorem (since the truncated kernels get larger). So for r large enough and $n > N$ we have

$$U_{\mu_n}(z) \geq U^r_{\mu_n}(z) \geq U_{\mu_n}(z) - \epsilon \geq U_\mu(z) - 2\epsilon.$$

Taking ϵ to zero proves the result. □

Lemma 3.7. *If* $\mu_n \to \mu$ *weak-*$*$*, then* $\liminf_n I(\mu_n) \geq I(\mu)$.

Corollary 3.8. *A probability measure minimizing the energy integral exists.*

Proof. The proof is almost the same as for the previous lemma, except that we have to know that if $\{\mu_n\}$ converges weak-$*$, then so does the product measure $\mu_n \times \mu_n$.

However, weak convergence of $\{\mu_n\}$ implies convergence of integrals of the form

$$\iint f(x)g(y)d\mu_n(x)d\mu_n(y).$$

The Stone-Weierstrass theorem implies that the finite sums of such product functions are dense in all continuous function on the product space.

Since weak-$*$ convergent sequences are bounded, the product measures $\mu_n \times \mu_n$ also have uniformly bounded masses, and hence convergence on a dense set of continuous functions of compact support implies convergence on all continuous functions of compact support. $\square$

Lemma 3.9. *Compact sets are capacitable.*

Proof. Since $\operatorname{cap}(E) \leq \operatorname{cap}^*(E)$ is obvious, we only have to prove the converse.

Set $U_n = \{z : \operatorname{dist}(z, E) < 1/n\}$ and choose a measure μ_n supported in U_n with $\|\mu_n\| \geq \operatorname{cap}(U_n) - 1/n$. Let μ be a weak accumulation point of $\{\mu_n\}$ and note

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w) \leq \int \log \frac{2}{|z-w|} d\mu_n(w) \leq 1$$

so μ is admissible in the definition of $\operatorname{cap}(E)$. Thus

$$\operatorname{cap}(E) \geq \limsup \|\mu_n\| = \lim \operatorname{cap}(U_n) = \lim \operatorname{cap}(U_n) = \operatorname{cap}^*(E) \quad \square.$$

Borel sets and even analytic sets are also capacitable.

It is clear from the definitions that logarithmic capacity is monotone

$$E \subset F \quad \Rightarrow \quad \operatorname{cap}(E) \leq \operatorname{cap}(F). \tag{3.7}$$

and satisfies the regularity condition

$$\operatorname{cap}(E) = \sup\{\operatorname{cap}(K) : K \subset E, K \text{ compact}\}. \tag{3.8}$$

Lemma 3.10 (Sub-additive)**.** *For any sets* $\{E_n\}$,

$$\operatorname{cap}(\cup E_n) \leq \sum \operatorname{cap}(E_n). \tag{3.9}$$

Proof. We can write any $\mu = \sum \mu_n$ as a sum of mutually singular measures so that μ_n gives full mass to E_n.

Restrict each μ_n to a compact subset K_n of E_n so that $\mu_n(K_n) \geq (1-\epsilon)\mu(E_n)$.

These restrictions are admissible for each E_n and hence

$$\sum \operatorname{cap}(E_n) \geq \sum \mu_n(K_n) \geq (1-\epsilon) \sum \mu_n(E_n) = (1-\epsilon)\|\mu\|.$$

Taking $\epsilon \to 0$ proves the result. □

Corollary 3.11. *A countable union of zero capacity sets has zero capacity.*

Corollary 3.12. *Outer capacity is also sub-additive.*

Proof. Given a sequence of sets $\{E_n\}$, choose open sets $V_n \supset E_n$ so that

$$\text{cap}(V_n) \leq \text{cap}^*(E_n) + \epsilon 2^{-n}.$$

By the sub-additivity of capacity

$$\text{cap}^*(\cup E_n) \leq \text{cap}(\cup V_n) \leq \sum \text{cap}(V_n) \leq \epsilon + \sum \text{cap}^*(E_n).$$

Taking $\epsilon \to 0$ proves the result. $\square$

Lemma 3.13. *If μ has bounded potential, then* $\operatorname{cap}(E) = 0 \Rightarrow \mu(E) = 0$.

Proof. If $\mu(E) > 0$ then μ restricted to E also has bounded potential function and proves that E has positive capacity. $\square$

Lemma 3.14. *If E is compact has positive capacity, then there exists an admissible μ that attains the maximum mass in the definition of capacity and $U_\mu(z) = 1$ everywhere on E, except possible a set of capacity zero.*

Proof. Let μ_n be a sequence of probability measures on E so that $\|\mu_n\| \to R$ where $R = \inf I(\mu)$ over all probability measures supported on E.

This is finite since E has positive capacity.

By the Banach-Alaoglu theorem there is a weak-$*$ convergent subsequence with limit μ, and by Lemma 3.7,

$$I(\mu) \leq \liminf_n I(\mu_n) = R.$$

We claim that $U_\mu \geq R$ except possibly on a set of zero capacity.

Otherwise let $T \subset E$ be a set of positive capacity on which $U_\mu < 1 - \epsilon$ and let σ be a non-zero, positive measure on T which potential bounded by 1. Define

$$\mu_t = (1-t)\mu + t\sigma.$$

This is a measure on E so that

$$\begin{aligned}
I(\mu_t) &\leq \int \log \frac{1}{|z-w|}((1-t)d\mu + td\sigma)((1-t)d\mu + td\sigma) \\
&\leq (1-t)^2 I(\mu) + 2t \int U_\mu d\sigma + t^2 I(\sigma) \\
&\leq I(\mu) - 2tI(\mu) + 2t \int U_\mu d\sigma + O(t^2) \\
&\leq I(\mu) - 2tI(\mu) + 2t(1-\epsilon)\|\sigma\| + O(t^2) \\
&< I(\mu),
\end{aligned}$$

if $t > 0$ is small enough. This contradicts minimality of μ, proving the claim.

Next we show that $U_\mu \le 1$ everywhere on the closed support of μ.

By previous step, $U_\mu \ge 1$ except on capacity zero (hence μ-measure zero).

If there is a point z in the support of μ such that $U_\mu(z) > 1$, then by lower semi-continuity of potentials, U_μ is $> 1 + \epsilon$ on some neighborhood of z and this neighborhood has positive μ measure (since z is in the support of μ) and thus $I(\mu) = \int U_\mu d\mu > \|\mu\|$, a contradiction.

Finally, let $\sigma = \mu/R$. Then the potential function of σ is bounded by 1 everywhere, so σ is admissible for E and hence $\|\sigma\| \le \text{cap}(E)$.

If ν is any other admissible measure for E, then $\nu(\{z \in E : \sigma(z) < 1\}) = 0$ by Lemma 3.13. Hence

$$\|\nu\| = \int 1 d\nu = \int U_\sigma d\nu = \int U_\nu d\sigma \leq \int 1 d\sigma = \|\sigma\|,$$

and thus $\|\sigma\| \geq \mathrm{cap}(E)$.

Thus $\mathrm{cap}(E) = \|\sigma\| = \|\mu/R\| = 1/R$. Hence $R = 1/\mathrm{cap}(E)$ is the Robin's constant of E. Since $\|\sigma\| \geq I(\sigma) = I(\mu)/R^2 = 1/R$ $\square$

Pfluger's Theorem

Pfluger's theorem connects logarithmic capacity and extremal length.

Suppose $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of K. Let K^* be the reflection of K across $\mathbb{T}$.

For any $E \subset \mathbb{T}$ that is a finite union of closed intervals, let Ω be the connected component of $\mathbb{C} \setminus (E \cup K \cup K^*)$ that has E on its boundary.

Let $h(z)$ be the harmonic function in Ω with boundary values 0 on K and K^* and boundary value 1 on E.

All boundary points are regular for the Dirichlet problem (since all boundary components are non-degenerate continua). Hence h extends continuously to the boundary with the correct boundary values.

h is symmetric with respect to $\mathbb{T}$, so its normal derivative on $\mathbb{T} \setminus E$ is 0.

Let $D(h) = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy$.

Let Γ_E denote the paths in $\mathbb{D} \setminus K$ that connect K to E.

Lemma 2.15. *With notation as above,* $M(\Gamma_E) = D(h)$.

Proof. Clearly $|\nabla h|$ is an admissible metric for Γ_E, so

$$M(\Gamma_E) \leq D(h) \equiv \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Thus we need only show the other direction.

Green's theorem states that

$$\iint_{\Omega} u\Delta v - v\Delta u dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} ds. \tag{2.10}$$

Using this and the fact that $h = 1$ on E, we have

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = -\int_{\mathbb{T}} \frac{\partial h}{\partial n} ds = -\int_{E} \frac{\partial h}{\partial n} ds = -\int_{E} h\frac{\partial h}{\partial n} ds.$$

Continuing,

$$\begin{aligned}\int_{\partial K} \frac{\partial h}{\partial n} ds &= -\frac{1}{2}\int_E \frac{\partial (h^2)}{\partial n} ds \\ &= \frac{1}{2}\int_{\mathbb{T}\setminus E} \frac{\partial (h^2)}{\partial n} ds + \frac{1}{2}\int_{\partial K} \frac{\partial (h^2)}{\partial n} ds + \frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dx dy.\end{aligned}$$

The first term is zero because h has normal derivative zero on $\mathbb{T} \setminus E$, and hence the same is true for h^2.

The second term is zero because h is zero on K and so $\frac{\partial}{\partial n} h^2 = 2h \frac{\partial h}{\partial n} = 0$.

To evaluate the third term, we use the identity

$$\begin{aligned}\Delta(h^2) &= 2h_x \cdot h_x + 2h \cdot h_{xx} + 2h_y \cdot h_y + 2h \cdot h_{yy} \\ &= 2h\Delta h + 2\nabla h \cdot \nabla h \\ &= 2h \cdot 0 + 2|\nabla h|^2 \\ &= 2|\nabla h|^2,\end{aligned}$$

to deduce

$$\frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Therefore,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Thus the tangential derivative of h's harmonic conjugate has integral $D(h)$ around ∂K and therefore $2\pi h/D(h)$ is the real part of a holomorphic function g on $\mathbb{D} \setminus K$.

Then $f = \exp(g)$ maps $\mathbb{D} \setminus K$ into the annulus

$$A = \{z : 1 < |z| < \exp(2\pi/D(h))\}$$

with the components of E mapping to arcs of the outer circle and the components of $\mathbb{T} \setminus E$ mapping to radial slits.

The path family Γ_E maps to the path family connecting the inner and outer circles without hitting the radial slits, and our earlier computations show the modulus of this family is $D(h)$. $\square$

Theorem 2.16 (Pfluger's theorem)**.** *If* $K \subset \mathbb{D}$ *is a compact connected set with smooth boundary with* 0 *in the interior of* K*. Then there are constants* C_1, C_2 *so that following holds. For any* $E \subset \mathbb{T}$ *that is a finite union of closed intervals,*

$$\frac{1}{\operatorname{cap}(E)} + C_1 \leq \pi\lambda(\Gamma_E) \leq \frac{1}{\operatorname{cap}(E)} + C_2,$$

where Γ_E *is the path family connecting* K *to* E*. The constants* C_1, C_2 *can be chosen to depend only on* $0 < r < R < 1$ *if* $\partial K \subset \{r \leq |z| \leq R\}$*.*

Later we will extend this to compact sets $E \subset \mathbb{T}$.

Proof. Using Lemma 2.15, we only have to relate $D(h)$ to the logarithmic capacity of E.

Let μ be the equilibrium probability measure for E. We know in general that $U_\mu = \gamma$ where $\gamma = 1/\text{cap}(E)$ almost everywhere on E (since sets of zero capacity have zero measure) and is continuous off E, but since U_μ is harmonic in $\mathbb{D}$ and equals the Poisson integral of its boundary values, we can deduce $U_\mu = \gamma$ everywhere on E.

Let $v(z) = \frac{1}{2}(U_\mu(z) + U_\mu(1/\overline{z}))$. Then since ∂K has positive distance from 0, there are constants C_1, C_2 so that

$$v + C_1 \leq 0, \qquad v + C_2 \geq 0,$$

on ∂K. Note that $C_1 \geq -\gamma$ by the maximum principle and $C_2 \geq 0$ trivially.

Moreover, since μ is a probability measure supported on the unit circle, given $0 < r < R < 1$, U_μ is uniformly bounded on both the annulus $\{r \leq |z| \leq R\}$ and its reflection across the unit circle, since these both have bounded, but positive distance from the unit circle.

This proves that C_1, C_2 can be chosen to depend on only these numbers, as claimed in the final statement of the theorem.

The following inequalities are easy to check on K, K^* and E,

$$\frac{v(z)+C_1}{\gamma+C_1} \le h(z) \le \frac{v(z)+C_2}{\gamma+C_2}.$$

and hence hold on Ω by the maximum principle.

Since we have equality on E, we also get

$$\frac{\partial}{\partial n}\left(\frac{v(z)+C_1}{\gamma+C_1}\right) \le \frac{\partial h}{\partial n} \le \frac{\partial}{\partial n}\left(\frac{v(z)+C_2}{\gamma+C_2}\right)$$

for $z \in E$.

When we integrate over E, the middle term is $-D(h)$ (we computed this above) and by Green's theorem

$$-\int_E \frac{\partial}{\partial n} \frac{v(z) + C_1}{\gamma + C_1} ds = \frac{1}{\gamma + C_1} \int_{\mathbb{D}} \Delta(v) dx dy = \frac{\pi}{\gamma + C_1}$$

because v is harmonic except for a $\frac{1}{2} \log \frac{1}{|z|}$ pole at the origin.

A similar computation holds for the other term and hence

$$\frac{\pi}{\gamma + C_1} \leq D(h) = M(\Gamma_E) \leq \frac{\pi}{\gamma + C_2},$$

since $D(h) = \int_E \frac{\partial h}{\partial n} ds$.

Hence

$$\gamma + C_1 \leq \pi\lambda(\Gamma_E) \leq \gamma + C_2.$$

This completes the proof of Pfluger's theorem for finite unions of intervals. $\square$

To extend Pfluger's theorem to all compact subsets of $\mathbb{T}$. First we need a continuity property of extremal length.

Recall that an extended real-valued function is lower semi-continuous if all sets of the form $\{f > \alpha\}$ are open.

Lemma 2.17. *Suppose $E \cap \mathbb{T}$ is compact, $K \subset \mathbb{D}$ is compact, connected and contains the origin, and Γ_E is the path family connecting K and E in $\mathbb{D} \setminus K$. Fix an admissible metric ρ for Γ_E and for each $z \in \mathbb{T}$, define $f(z) = \inf \int_\gamma \rho ds$ where the infimum is over all paths in Γ_E that connect K to z. Then f is lower semi-continuous.*

Proof. Suppose $z_0 \in \mathbb{T}$ and use Cauchy-Schwarz to get

$$\begin{aligned}\int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r} \rho ds \right)^2 dr &\leq \int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r} \rho^2 ds \right) dr \left(\int_{|z-z_0|=r} 1 ds \right) dr \\ &\leq \int_{2^{-n-1}}^{2^{-n}} r \int_0^{2\pi} \rho^2 r d\theta dr \\ &\leq \pi 2^{-n} \int_{2^{-n-1}<|z-z_0|<2^{-n}} \rho^2 dx dy \\ &= o(2^{-n}).\end{aligned}$$

Thus there are circular cross-cuts $\{\gamma_n\} \subset \{z : 2^{-n-1} < |z - z_0| < 2^{-n}\}$ of $\mathbb{D}$ centered at z_0 and with ρ-length ϵ_n tending to 0. By taking a subsequence we may assume $\sum \epsilon_n < \infty$.

Now choose $z_n \to z_0$, with z_n separated from 0 by γ_n, and so that

$$f(z_n) \to \alpha \equiv \liminf_{z \to z_0} f(z).$$

We claim there is a path from K to z_0 whose ρ-length is $\leq \alpha + \epsilon$.

Let c_n be the infimum of ρ-lengths of paths connecting γ_n and γ_{n+1}.

By considering a path connecting K to z_n, we see that $\sum_1^n c_k \leq f(z_n)$, for all n and hence $\sum_1^\infty c_n \leq \alpha$.

Next choose $\epsilon > 0$ and n so that we can connect K to z_n (and hence to γ_n) by a path of ρ-length less than $\alpha + \epsilon$.

We can then connect γ_n to z_0 by a infinite concatenation of arcs of γ_k, $k > n$ and paths connecting γ_k to γ_{k+1} that have total length $\sum_n^\infty (\epsilon_n + c_n) = o(1)$.

Thus K is connected to z_0 by a path of ρ-length as close to α as we wish. $\square$

Corollary 2.17. *Suppose $E \subset \mathbb{T}$ is compact and $\epsilon > 0$. Then there is a finite collection of closed intervals F so that $E \subset F$ and*

$$\lambda(\Gamma_E) \leq \lambda(\Gamma_F) + \epsilon,$$

where the path families are defined as above.

Proof. Choose an admissible ρ so that $\int \rho^2 dxdy \leq M(\Gamma_E) + \epsilon$. Set

$$r = (\frac{M(\Gamma_E) + \epsilon}{M(\Gamma_E) + 2\epsilon})^{1/2} < 1.$$

By Lemma 2.17, $V = \{z \in \mathbb{T} : f(z) > r\}$ is open, and therefore we can choose a set F of the desired form inside V. Then ρ/r is admissible for Γ_F, so

$$M(\Gamma_F) \leq \int (\frac{\rho}{r})^2 dxdy = \frac{M(\Gamma_E) + 2\epsilon}{M(\Gamma_E) + \epsilon} \int \rho^2 dxdy \leq M(\Gamma_E) + 2\epsilon.$$

Thus an inequality in the opposite direction holds for extremal length. □

Corollary 2.18. *Pfluger's theorem holds for all compact sets in* $\mathbb{T}$*.*

Proof. Suppose E is compact. Using Corollary 2.17 and Lemma 3.9 we can choose nested sets $E_n \searrow E$ that are finite unions of closed intervals and satisfy

$$\lambda(\mathcal{F}_{E_n}) \to \lambda(\mathcal{F}_E),$$

and

$$\operatorname{cap}(E_n) \to \operatorname{cap}(E).$$

Thus the inequalities in Pfluger's theorem extend to E. □

Gehring, Hayman and Carathéodory

The boundary of a simply connected domain need not be a Jordan curve, nor even locally connected, and such examples arise naturally in complex dynamics as the Fatou components of various polynomials and entire functions.

If the boundary is locally connected, then the conformal map from the disk extends continuously to the boundary.

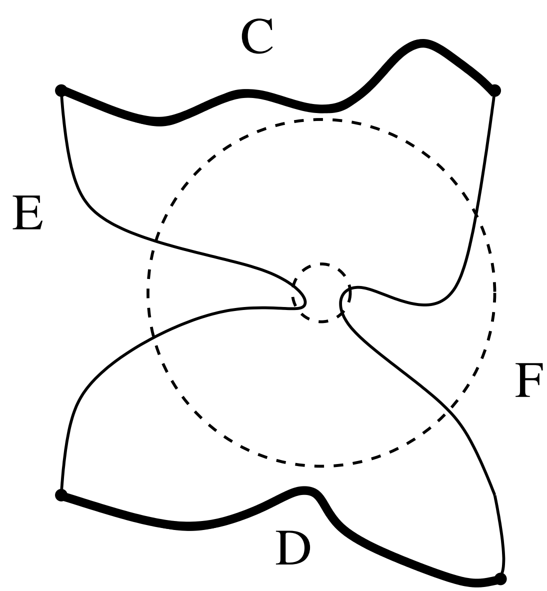
Even for general simply connected domains, the boundary values exist in some sense at most points. We will make this precise.

Lemma 2.19. *Suppose Q is a quadrilateral with opposite pairs of sides E, F and C, D. Assume*

(1) *E and F can be connected in Q by a curve σ of diameter $\leq \epsilon$,*

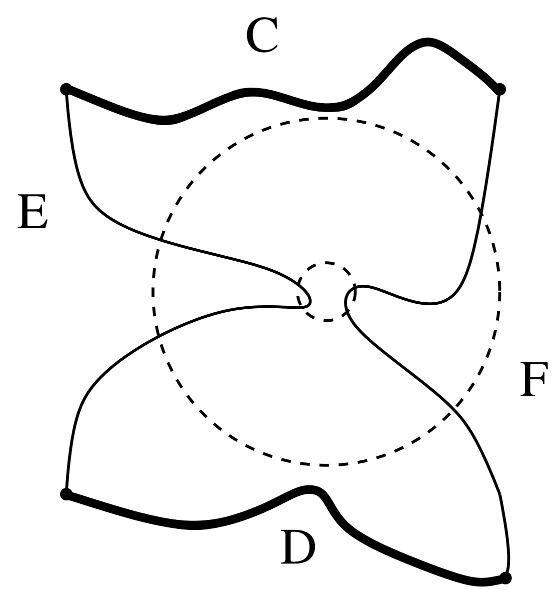
(2) *any curve connecting C and D in Q has diameter at least 1.*

Then the modulus of the path family connecting E and F in Q is larger than $M(\epsilon)$ where $M(\epsilon) \to \infty$ as $\epsilon \to 0$.



Proof. Define a metric on Q by $\rho(z) = \frac{1}{2}|z-a|^{-1}/\log(1/2\epsilon)$ for $\epsilon < |z-a| < 1/2$. Any curve γ connecting C and D must cross σ and since γ has diameter ≥ 1 it must leave the annulus where ρ is non-zero.

This shows that the modulus of the path family in Q separating E and F is small, hence the modulus of the family connecting them is large. $\square$



The following fundamental fact says that hyperbolic geodesics are almost the same as Euclidean geodesics.

Theorem 2.20 (Gehring-Hayman inequality)**.** *There is an absolute constant* $C < \infty$ *to that the following holds. Suppose* $\Omega \subset \mathbb{C}$ *is hyperbolic and simply connected. Given two points in* Ω*, let* γ *be the hyperbolic geodesic connecting these two points and let* σ *be any other curve in* Ω *connecting them. Then* $\mathrm{len}(\gamma) \leq \mathrm{Clen}(\sigma)$*.*

Proof. Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that γ is the image of $I = [0, r] \subset \mathbb{D}$ for some $0 < r < 1$. Without loss of generality we may assume $r = r_N 1 - 2^{-N}$ for some N. Let

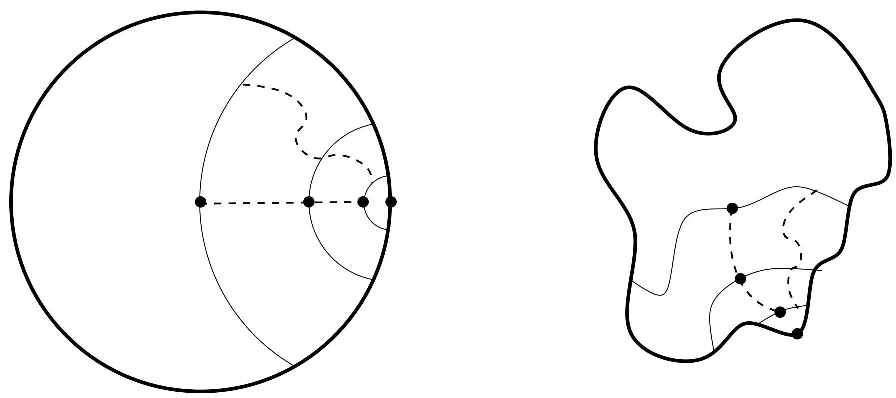
$$Q_n = \{z \in \mathbb{D} : 2^{-n-1} < |z - 1| < 2^{-n}\},$$
$$\gamma_n = \{z \in \mathbb{D} : |z - 1| = 2^{-n}\},$$
$$z_n = \gamma_n \cap [0, 1).$$

Let $Q'_n \subset Q_n$ be the sub-quadrilateral of points with $|\arg(1 - z)| < \pi/6$. Each Q'_n has bounded hyperbolic diameter and, by Koebe's theorem, its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart.

In particular, this means that any curve in $f(Q_n)$ separating $f(\gamma_n)$ and $f(\gamma_{n+1})$ must cross $f(Q'_n)$ and hence has diameter $\gtrsim d_n$. Since Q_n has bounded modulus, so does $f(Q_n)$ and so Lemma 2.19 says that the shortest curve in $f(Q_n)$ connecting γ_n and γ_{n+1} has length $\ell_n \simeq d_n$.

Thus any curve γ in Q connecting γ_n and γ_{n+1} has length at least ℓ_n, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\sigma)). \quad \square$$



If $f : \mathbb{D} \to \Omega$ is conformal define

$$a(r) = \text{area}(\Omega \setminus f(r \cdot \mathbb{D}).$$

If Ω has finite area (e.g., if it is bounded), then clearly $a(r) \searrow 0$ as $r \nearrow 1$.

Lemma 2.21. *There is a $C < \infty$ so that the following holds. Suppose $f : \mathbb{D} \to \Omega$ and $\frac{1}{2} \leq r < 1$. Let $E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(rx)| \geq \delta$ for some $r < s < 1\}$. Then the extremal length of the path family $\mathcal{P}$ connecting $D(0, r)$ to E is bounded below by $\delta^2/Ca(r)$.*

Proof. Let $z = f(sx)$ and suppose $w \in f(D(0,r))$. By the Gehring-Hayman estimate, the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them.

But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ.

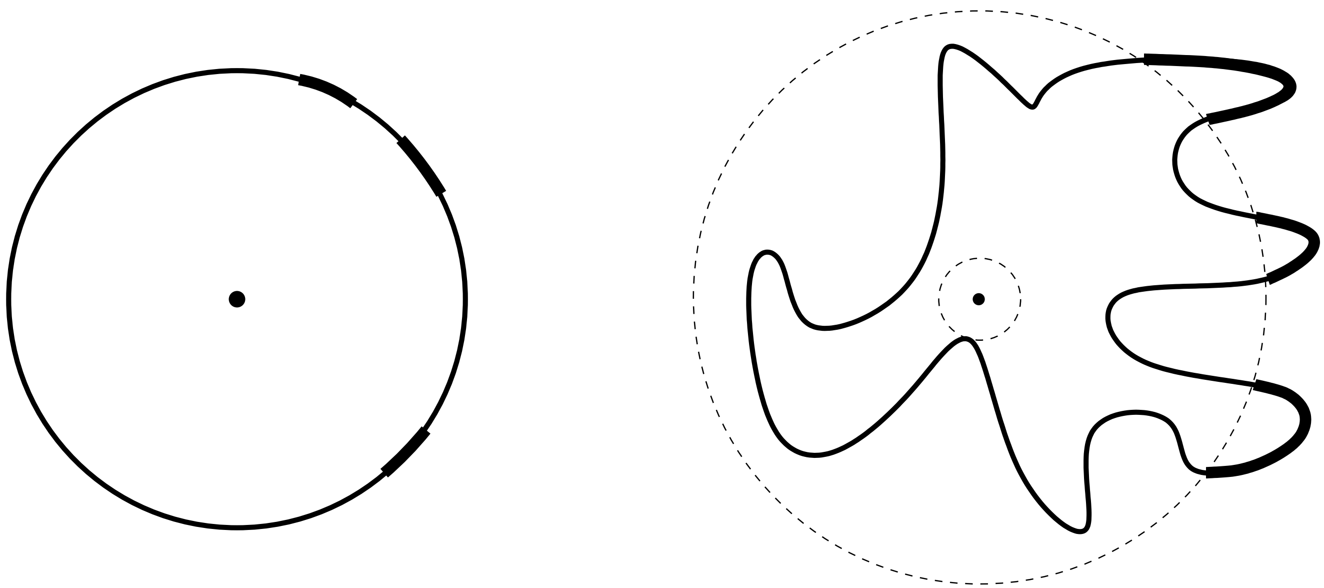
Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then ρ is admissible for $f(\mathcal{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$.

Thus $\lambda(\mathcal{P}) \geq \frac{\delta^2}{C^2 a(r)}$. $\square$

Lemma 2.22. *Suppose $f : \mathbb{D} \to \Omega$ is conformal, and for $R \geq 1$,*

$$E_R = \{x \in \mathbb{T} : |f(x) - f(0)| \geq R \operatorname{dist}(f(0), \partial\Omega)\}.$$

Then E_R has capacity $O(1/\log R)$ if R is large enough.



Proof. Assume $f(0) = 0$ and $\mathrm{dist}(0, \partial\Omega) = 1$ and let $\rho(z) = |z|^{-1}/\log R$ for $z \in \Omega \cap \{1 < |z| < R\}$. Then ρ is admissible for the path family Γ connecting $D(0, 1/2)$ to $\partial\Omega \setminus D(0, R)$ and $\iint \rho^2 dxdy \leq 2\pi/\log R$.

By definition $M(\Gamma) \leq 2\pi/\log R$ and $\lambda(\Gamma) \geq (\log R)/2\pi$. By the Koebe theorem $f^{-1}(D(0, 1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω.

By Pfluger's theorem (Theorem 2.16),

$$\mathrm{cap}(E_r) \leq \frac{2}{-2C_2 + \log R}.$$

□

Corollary 2.23. *If* $f : \mathbb{D} \to \Omega$ *is conformal, then* f *has radial limits except on a set of zero capacity (and hence has finite radial limits a.e. on* $\mathbb{T}$*).*

Proof. Let $E_{r,\delta} \subset \mathbb{T}$ be the set of $x \in \mathbb{T}$ so that $\operatorname{diam}(f(rx, x)) > \delta$, and let $E_\delta = \bigcap_{0<r<1} E_{r,\delta}$.

If f does not have a radial limit at $x \in \mathbb{T}$, then $x \in E_\delta$ for some $\delta > 0$, and this has zero capacity by Lemma 2.21.

Taking the union over a sequence of δ's tending to zero proves the result. The set where f has a radial limit ∞ has zero capacity by Lemma 2.22, so we deduce f has finite radial limits except on zero capacity. $\square$

Combining the last two results proves

Corollary 2.24. *Given* $\epsilon > 0$ *there is a* $C < \infty$ *so that the following holds. If* $f : \mathbb{D} \to \Omega$ *is conformal,* $z \in \mathbb{D}$ *and* $I \subset \mathbb{T}$ *is an arc that satisfies* $|I| \geq \epsilon(1 - |z|)$ *and* $\text{dist}(z, I) \leq \frac{1}{\epsilon}(1 - |z|)$, *then* I *contains a point* w *where* f *has a radial limit and* $|f(w) - f(z)| \leq C\text{dist}(f(z), \partial\Omega)$.

Theorem 2.25 (Carathéodory)**.** *Suppose that $f : \mathbb{D} \to \Omega$ is conformal, and that $\partial\Omega$ is compact and locally path connected (for every $\epsilon > 0$ there is a $\delta > 0$ so that any two points of $\partial\Omega$ that are within distance δ of each other can be connected by a path in $\partial\Omega$ of diameter at most ϵ). Then f extends continuously to the boundary of $\mathbb{D}$.*

Lasse Rempe has pointed out this is actually due to Carathéodory's student Marie Torhorst. See Rempe's article On prime ends and local connectivity.

Proof. Suppose $\eta > 0$ is small. Since $\partial\Omega$ is compact $\Omega \setminus f(\{|z| < 1 - \frac{1}{n}\})$ has finite area that tends to zero as $n \nearrow \infty$. Thus if n is sufficiently large, this region contains no disk of radius η.

Choose $\{z_j\}$ to be n equally spaced points on the unit circle and using Lemma 2.24 choose interlaced points $\{w_j\}$ so that f has a radial limit $f(w_j)$ at w_j and this limit satisfies $|f(w_j) - f(rw_j)| \leq C\eta$ where $r = 1 - 1/n$. Then

$$\begin{aligned} |f(w_j) - f(w_{j+1})| &\leq |f(w_j) - f(rw_j)| \\ &\quad + |f(rw_j) - f(rw_{j+1})| \\ &\quad\quad + |f(rw_{j+1}) - f(w_{j+1})| \\ &\leq C\delta. \end{aligned}$$

The center term is bounded by Koebe's theorem and the others by definition.

Fix $\epsilon > 0$ and choose $\delta > 0$ as in the definition of locally connected.

Thus if η is so small that $C\eta < \delta$, then the shorter arc of $\partial\Omega$ with endpoints $f(w_j)$ and $f(w_{j+1})$ can be connected in $\partial\Omega$ by a curve of diameter at most ϵ.

Thus the image under f of the Carleson square with base I_j (the arc between w_j and w_{j+1}) has diameter at most $C\eta + \epsilon$. This implies f has a continuous extension to the boundary. $\square$

Uniform convergence on compact subsets of $\mathbb{D}$ does not imply uniform convergence on the boundary.

However, it is true that the conformal boundary values will converge if the image domains have some parameterizations that converge.

In other words, if a sequence of simply connected domains have boundaries with continuous parameterizations that converge uniformly to the continuous parameterization of the limiting domain, then we also get uniform convergence for the conformal parameterizations of the boundaries.

This is analogous to Carathédory's theorem: if a domain boundary has any continuous parameterization, then the conformal parameterization is also continuous.

Lemma 2.26. *Suppose* $\{f_n\}$ *are conformal maps of* $\mathbb{D} \to \Omega_n$ *that converge uniformly on compact subsets of* $\mathbb{D}$ *to a conformal map* $f : \mathbb{D} \to \Omega$. *Suppose that the boundary of each* Ω_n *is the homeomorphic image* $\partial\Omega_n = \sigma_n(\mathbb{T})$ *and that* $\{\sigma_n\}$ *converges uniformly on* $\mathbb{T}$ *to a homeomorphism* $\sigma : \mathbb{T} \to \partial\Omega$. *Then* $f_n \to f$ *uniformly on the* $\overline{\mathbb{D}}$.

Proof. Fix $\epsilon > 0$ and choose n so large that if we divide $\mathbb{T}$ into n equal sized intervals $\{J_j\}_1^n$, then σ maps each of them to a set I_j of diameter at most $\epsilon/2$.

Let $I_j^k = f_k(J_j)$. Because $\sigma_k \to \sigma$ uniformly, the sets I_j all have diameter at most ϵ, if k is large enough.

Next choose $\eta > 0$ so small that if $k, m > 1/\eta$ and $\sigma_m(J_j)$ and $\sigma_k(J_i)$ contain points at most distance $C\eta$ apart, then J_i and J_k are the same or adjacent to each other.

We can do this because of the uniform convergence and the fact that σ is 1-to-1. By passing to the limit the same property holds if we replace σ_m by σ.

Next choose m so large that $f(\mathbb{D}) \setminus f(\{|z| < 1 - \frac{1}{m}\})$ is contained in an η-neighborhood of $\partial\Omega$.

Choose m points $\{z_j\}$ equally spaced on the circle $|z| = 1 - \frac{1}{m}$,and let $K_j^m \subset \mathbb{T}$ be the arc centered at $z_j/|z_j|$ of length $4\pi/m$. Fix a small number $\delta > 0$ (δ will be determined below, depending only on η).

By Lemma 2.22 choose a point $w_j \in K_j^m$ so that $|w_j - z_j| \leq 2/m$ and

$$|f(w_j) - f(w_j(1 - \frac{1}{m}))| \leq C\delta.$$

Similarly, choose points $w_j^k \in K_j^m$ so that

$$|f_k(w_j^k) - f_k(z_j)| \leq 2C\delta.$$

This is possible since $f_k \to f$ uniformly on the compact set $\{|z| \leq 1 - \frac{1}{m}\}$ and thus $\partial f_k(\mathbb{D})$ is contained in a 2δ-neighborhood of $\partial\Omega$ for k large enough and $\partial\Omega_k$ is contained in a δ-neighborhood of $\partial\Omega$ because of the uniform convergence of the parameterizations.

By taking m larger, if necessary, we can also arrange that each I_j contains at least one of the points $f(z_m/|z_m|)$.

Thus each $f(K_j^m)$ is mapped into the union of at most 2 of the I_j and hence its image has diameter at most 2ϵ.

Also, the points $f(w_p^k)$ and $f(w_{p+1}^k)$ are at most $C\delta$ apart, so belong to the same or adjacent sets I_j. Thus $f_k(K_p)$ is a union of at most 4 such adjacent sets and hence has diameter $O(\epsilon)$.

For each w_p^k there is an arc J_j so that $f_k(w_p^k) \subset \sigma_k(J_j)$. Similarly, there is an arc J_i so that $f(w_p) \in I_i = \sigma(J_i)$.

Since $f_k \to f$ uniformly on the finite set $\{z_n\}$, we have, for k sufficiently large

$$\begin{aligned}
|f_k(w_n^k) - f(w_n)| &\leq |f_k(w_n^k) - f_k(z_n)| \\
&\qquad +|f_k(z_n) - f(z_n)| \\
&\qquad\qquad +|f(z_n) - f(w_n)| \\
&\leq (2C + 1 + C)\delta.
\end{aligned}$$

This is less than η if δ is small enough. Since I_i and I_j each have diameter at most ϵ, there union has diameter $< 2\epsilon$ and the union

of the intervals adjacent to these is at most 4ϵ. Similarly for I_i^k and J_j^k. Thus $f_k(K_p)$ and $f(K_p)$ are contained in $O(\epsilon)$-neighborhoods of each other.

Thus $f_k \to f$ uniformly on $\mathbb{T}$. By the maximum principle, this implies uniform convergence on the closed disk, as desired. $\square$

Corollary 2.27. *If* $\{f_n\}$ *are homeomorphisms that converge uniformly to a homeomorphism* f *then* $M(f_n(Q)) \to M(f(Q))$

Proof. If v is a vertex of Q and $v_n \to v$ are vertices of Q_n, then the uniform convergence of $f_n : \mathbb{D} \to Q_n$ to $f : \mathbb{D} \to Q_n$ (the normalized conformal maps) implies that preimages of v_n under f_n must converge to the preimage of v under f. Since this holds for all four vertices and modulus on $\mathbb{D}$ is a continuous function of the four vertices, this proves the corollary. $\square$

Harmonic measure

Suppose Ω is a planar Jordan domain bounded, $z \in \Omega$, and $E \subset \partial\Omega$ is Borel.

Suppose $f : \mathbb{D} \to \Omega$ is conformal and $f(0) = z$ (use Riemann mapping theorem).

By Carathéodory's theorem, f extends continuously (even homeomorphically) to the boundary, so $f^{-1}(E) \subset \mathbb{T}$ is also Borel. We define "the harmonic measure of the set E for the domain Ω, with respect to the point z" as

$$\omega(z, E, \Omega) = |E|/2\pi,$$

where $|E|$ denotes the Lebesgue 1-dimensional measure of E.

This depends on the choice of the Riemann map f, but any two maps, both sending 0 to z, will differ only by a pre-composition with a rotation.

Thus the two possible pre-images of E differ by a rotation and hence have the same Lebesgue measure. If we fix E and Ω, then $\omega(z, E, \Omega)$ is a harmonic function of z, giving rise the name "harmonic measure".

Since we always have $0 \leq \omega(z, E, \Omega) \leq 1$, we can deduce that if E has harmonic measure with respect to one point z in Ω then it has zero harmonic measure with respect to all points.

There are several alternate definitions:

- Hitting distribution of Brownian motion.
- Normal derivative of Green's function (need smooth boundary).
- Solution of Dirichlet problem.
- Measure minimizing log-energy (for base point ∞).

If $\partial\Omega$ is merely locally connected, then Carathéodory's theorem still implies that the Riemann map f has a continuous extension to the boundary, so the same definition of harmonic measure works.

We can define harmonic measure for general simply connected domains, by taking an increasing union of domains with Jordan boundaries, but we will postpone this discussion until later, as we will postpone the discussion of harmonic measure on multiply connected domains (defined via covering maps).

For the moment, Jordan domains and locally connected sets will provide sufficiently many interesting examples.

We want estimate harmonic measure in terms of extremal length. We have already seen how to relate extremal length to logarithmic capacity, and the following relates the latter to harmonic measure:

Lemma 2.28. *For any compact* $E \subset \mathbb{T}$*,*

$$\operatorname{cap}(E) \geq \frac{1}{1 + \log 2 + \pi + \log \frac{1}{|E|}}.$$

If $E \subset \mathbb{T}$ *has positive Lebesgue measure, then it has positive capacity. So, if* $E \subset \mathbb{T}$ *is an arc, then*

$$\operatorname{cap}(E) \leq \frac{1}{\log 4 + \log \frac{1}{|E|}}.$$

For arcs of small measure, the two bounds are comparable.

Proof. Let μ be Lebesgue measure restricted to E and let $x \in E$. Let I be the arc centered at x and with length $|E|$. If $y \in \mathbb{T}$ and t is the arclength distance between x and y, then $\frac{2}{\pi}t \leq |x-y| \leq t$, so

$$U_\mu(x) = \int_E \log \frac{2}{|x-y|} dy \leq \int_I \log \frac{1}{|x-y|} dy$$

$$\leq 2 \int_0^{|E|/2} \log \frac{\pi dt}{2t} = |E| \log \frac{2}{|E|} + \pi |E|$$

Thus the log-capacity of E is at least

$$\|\mu\| / \sup U_\mu \leq |E| / |E| \log \frac{2}{|E|} + \pi |E| = \frac{1}{\pi + \log 2 + \log 1/|E|}$$

If E is an arc, then the center x of the arc is at most distance $|E|/2$ from any other point of the arc, and so

$$U_\mu(x) \geq \log \frac{2}{|E|/2} = \log \frac{4}{|E|} = \log \frac{1}{|E|} + \log 4,$$

for any probability measure supported on E. This gives the desired estimate. $\square$

The following is the fundamental estimate for harmonic measure, from which all other estimates flow (at least, all the ones that we will use).

Theorem 2.29. *Suppose* Ω *is a Jordan domain,* $z_0 \in \Omega$ *with* $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ *and* $E \subset \partial\Omega$*. Let* Γ *be the family of curves in* Ω *which connects* $D(z_0, 1/2)$ *to* E*. Then*

$$\omega(z_0, E, \Omega) \leq C \exp(-\pi\lambda(\Gamma)).$$

If $E \subset \partial\Omega$ *is an arc then the two sides are comparable.*

Proof. Let $f : \mathbb{D} \to \Omega$ be conformal. By Koebe's $\frac{1}{4}$-theorem (Theorem 2.11), the disk $D(z, \frac{1}{2})$ in Ω maps to a smooth region K in the unit disk that contains the origin, and ∂K is uniformly bounded away from both the origin and the unit circle.

Thus by Pfluger's theorem applied to the curve family Γ_X connecting K and the compact set $X = f^{-1}(E)$,

$$\frac{1}{\mathrm{cap}(X)} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq \frac{1}{\mathrm{cap}(X)} + C_2(K),$$

for constants C_1, C_2 that are bounded independent of all our choices.

By Lemma 2.28 the right-hand side of

$$1 + \log 4 + \log \frac{1}{|X|} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq 1 + \log 2 + \log \frac{1}{|X|} + C_2(K).$$

holds in general, and the left-hand side also holds if X is an interval.

Multiply by -1 and exponentiate to get

$$\frac{|X|}{2e^{1+\pi+C_2}} \leq \exp(-\pi\lambda(\Gamma_X)) \leq \frac{|X|}{4e^{C_1}}$$

under the same assumptions. Now use $\omega(z, E, \Omega) = \omega(0, X, \mathbb{D}) = |X|/2\pi$ to deduce the result. $\square$

Corollary 2.30 (Ahlfors distortion theorem)**.** *Suppose* Ω *is a Jordan domain,* $z_0 \in \Omega$ *with* $\mathrm{dist}(z_0, \partial\Omega) \geq 1$ *and* $x \in \partial\Omega$*. For each* $0 < t < 1$ *let* $\ell(t)$ *be the length of* $\Omega \cap \{|w - x| = t\}$*. Then there is an absolute* $C < \infty$*, so that*

$$\omega(z_0, D(x, r), \Omega) \leq C \exp(-\pi \int_r^1 \frac{dt}{\ell(t)}).$$

Proof. Let K be the disk of radius $1/2$ around z_0 and let Γ be the family of curves in Ω which connects $D(x,r)\cap\partial\Omega$ to K.

Define a metric ρ by $\rho(z) = 1/\ell(t)$ if $z \in C_t = \{z \in \Omega : |x - z| = t\}$ and $\ell(t)$ is the length of C_t.

Any curve $\gamma \in \Gamma$ has ρ-length at least

$$L = \int_r^{1/2} \frac{dt}{\ell(t)},$$

and

$$A = \iint_\Omega \rho^2 dxdy \geq \int_r^{1/2} \int_{C_r\cap\Omega} \ell(z)^{-2} r dr d\theta = \int \ell(z)^{-1} dr = L.$$

Therefore $\lambda(\Gamma) \geq A/L^2 = 1/L$, and this proves the result. $\square$

Corollary 2.31 (Beurling's estimate)**.** *There is a $C < \infty$ so that if Ω is simply connected, $z \in \Omega$ and $d = \mathrm{dist}(z, \partial\Omega)$ then for any $0 < r < 1$ and any $x \in \partial\Omega$,*

$$\omega(z, D(x, rd), \Omega) \leq C r^{1/2}$$

Proof. Apply Corollary 2.30 at x and use $\theta(t) \leq 2\pi t$ to get

$$\exp\left(-\pi \int_{rd}^{d} \frac{dt}{\theta(t)t}\right) \leq C \exp\left(-\frac{1}{2}\log r\right) \leq C\sqrt{r}.$$

□

Corollary 2.32. *There is an $R < \infty$ so that for any Ω is a Jordan domain and any $z \in \Omega$*

$$\omega(z, \partial\Omega \setminus D(z, R \cdot \mathrm{dist}(z, \partial\Omega), \Omega) \leq 1/2.$$

Proof. Rescale so $z = 1$ and $\mathrm{dist}(z, \partial\Omega) = 1$. Then apply $w \to 1/w$ which fixes z and maps $\partial\Omega \setminus D(z, R)$ into $D(0, 1/R - 1)$. Then Lemma 2.31 implies the result holds if $R \geq 4C^2 + 1$ (and C is as in Lemma 2.31). □

Corollary 2.33. *For any Jordan domain and any* $\epsilon > 0$*,*

$$\omega(z, \partial\Omega \cap D(z, (1+\epsilon)\mathrm{dist}(z, \partial\Omega)), \Omega) > C\epsilon,$$

for some fixed $C > 0$*.*

Proof. Renormalize so $z = 0$ and 1 is a closest point of $\partial\Omega$ to z. By Corollary 2.32, the set $E = \partial\Omega \cap D(0, 1+\epsilon)$ has harmonic measure at least $1/2$ from the point $1-\epsilon/R$. Since $\omega(z, E, \Omega)$ is a positive, harmonic function on $\mathbb{D}$, Harnack's inequality says it is larger than $C\epsilon/R$ at the origin. □

This is a weak version of the Beurling projection theorem which says that the sharp lower bound is given by the slit disk $D(0, 1 + \epsilon) \setminus [1, 1 + \epsilon)$.

The harmonic measure of the slit in this case can be computed as an explicit function of ϵ because this domain can be mapped to the disk by sequence of elementary functions.

Theorem 2.34. *Suppose Ω is a Jordan domain and $E \subset \partial\Omega$ has zero $\frac{1}{2}$-Hausdorff measure. Then E has zero harmonic measure in Ω.*

Proof. Since dilations do not change dimension or harmonic measure, we can rescale so that Ω contains a unit disk centered at some point z. It suffices to show E has harmonic measure zero with respect to z.

The hypothesis means that for any $\epsilon > 0$, the set E can be covered by open disks $\{D(x_j, r_j)\}$ that satisfy $\sum_j r_j^{1/2} \leq \epsilon$. By Beurling's estimate, this implies

$$\omega(z, E, \Omega) \leq \sum_j \omega(z, D_j, \Omega) \leq O(\sum_j r_j^{1/2}) = O(\epsilon).$$

□

This result was not improved until Lennart Carleson showed in a tour de force that the $\frac{1}{2}$ could be replaced by some $\alpha > \frac{1}{2}$ in

That result was not improved until Makarov showed it holds for all $\alpha < 1$.

Even though we have not defined harmonic measure for multiply connected domains, it is clear that no analog is possible in that case: if the boundary of Ω is a Cantor set of dimension α, then it must have full harmonic measure, even if α is small.

A famous result of Peter Jones and Tom Wolff says that harmonic measures gives full mass to a set of dimension at most 1 for any planar domain.

One might think that this holds for domain in $\mathbb{R}^n$ with bound $n-1$, but Wolff found a counterexample (Wolff snowflakes).

Currently active area of research in higher dimensions.

We recall a result from real analysis.

Theorem 2.35 (Vitali Covering Lemma). *Suppose $E \subset \mathbb{R}^d$ is a measurable set and $\mathcal{B} = \{B_j\} \subset \mathbb{R}^d$ is a collection of balls so that each point of E is contained in elements of $\mathcal{B}$ of arbitrarily small diameter. Then there is a subcollection $\mathcal{C} \subset \mathcal{B}$ so that $E \setminus \cup_{B \in \mathcal{C}} B$ has zero d-measure.*

For a proof see Folland's textbook.

Corollary 2.36. *If Ω is Jordan domain, then harmonic measure is singular to area measure.*

Proof. By the Lebesgue density theorem, at Lebesgue almost every point z of a set E of positive area, all small enough disks satisfy

$$\text{area}(E \cap D(z,r)) \geq (1-\epsilon)\text{area}(D(z,r)), \text{ for all } .$$

In particular we must have $\theta(t) \leq \frac{\epsilon}{t}$ (angle measure of $\Omega \cap \{|w-z| = t\}$) on a set z of measure at least $r/4$ in $[r/2, r]$.

Thus by the Ahlfors distortion theorem

$$\omega(D(z, r_0 2^{-n}) \leq C \exp\left(-\pi \int_{2^{-n} r_0}^{r_0} \frac{dt}{\epsilon t}\right) \leq C 2^{-\pi n/\epsilon}.$$

This is much less than $(2^{-n} r_0)$ if n is large. Thus almost every point of $\partial\Omega$ can be covered by arbitrarily small disks so that $\omega(D(z_j, r_j)) = o(r_j^2)$.

Use Vitali's theorem to take a disjoint cover of a set of full harmonic measure, and we deduce that harmonic measure gives full mass to set of zero area. $\square$

Corollary 2.37. *There is an $\epsilon > 0$ so that harmonic measure on a planar Jordan domain always gives full measure to a set of Hausdorff dimension at most $2 - \epsilon$.*

Proof. Fix a large integer b and consider the b-adic squares in the plane. Take one such square Q that intersects $\partial\Omega$ and consider its b^2 children squares.

We claim that if b is large enough, then at least one of them has harmonic measure that is less than $(2b^2)^{-1}$ times the harmonic measure of Q.

If there is a subsquare that misses $\partial\Omega$, then its harmonic measure is zero, and the claim is true. Therefore we may assume every subsquare hits $\partial\Omega$.

Suppose Q has side length 1 and define a finite sequence of squares S_k, concentric with Q and with side lengths $\frac{1}{b}, \frac{3}{b}, \frac{6}{b}, \dots, 1$. If $z \in \partial S_k$, then $\mathrm{dist}(z, \partial\Omega) \le \sqrt{2}/b$ and $\mathrm{dist}(z, S_{k-1}) > 3/b$, so by Corollary 2.33 ,

$$\max_{z \in \partial S_k} \omega(z, \partial\Omega \cap S_{k-1}, \Omega \setminus S_{k-1}) < 1 - \delta,$$

for some uniform $\delta > 0$ (independent of k and b).

By the maximum principle and induction,

$$\omega(S_1) \le (1-\delta)^{b/3},$$

and this is less than $1/(2b^2)$ if b is large enough. This prove the claim, that ω deviates from the uniform distribution on the sub-squares by a fixed amount.

The rest is standard.

The deviation from uniformity implies that the entropy

$$h(\mu) = -\sum_{k=1}^{b^2} \omega(Q_j) \log_b \omega(Q_j),$$

is strictly less than 2, the maximum that occurs when every square has equal measure.

By the strong law of large numbers and Billingsley's lemma, ω has dimension strictly less than 2, with a bound that depends on b, but not on Ω. $\square$

Theorem 2.38 (Strong Law of Large Numbers). *Let $(X, d\nu)$ be a probability space and $\{f_n\}$, $n = 1, 2 \ldots$ a sequence of orthogonal functions in $L^2(X, d\nu)$. Suppose $E(f_n^2) = \int |f_n|^2 \, d\nu \leq 1$, for all n. Then*

$$\frac{1}{n} S_n = \frac{1}{n} \sum_{k=1}^{n} f_k \to 0,$$

a.e. (with respect to ν) as $n \to \infty$.

Lemma 2.39 (Billingsley's Lemma). *Let $A \subset [0, 1]$ be Borel and let μ be a finite Borel measure on $[0, 1]$. Suppose $\mu(A) > 0$. If*

$$\alpha_1 \leq \liminf_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \leq \beta_1, \tag{2.11}$$

for all $x \in A$, then $\alpha_1 \leq \dim(A) \leq \beta_1$.

Both these are proven in Chapter 1 of the text by Bishop and Peres.

Jean Bourgain proved this holds for general domains in higher dimensions, with a δ that depends only on the dimension. We shall see later that the bound $\dim(\omega) \leq 1$ holds in the plane.

Some small gaps in his proof were noticed and filled by Badger and Genschaw in Lower bounds on Bourgain's constant for harmonic measure. In $\mathbb{R}^3$, they show that harmonic measure has dimension at most

$$2.99999999999999$$

It is conjectured that the upper bound is $2.5 = (n-1) + (n-1)/(n-1)$.