# MAT 627, Spring 2025, Stony Brook University

# Topics in Complex Analysis: Quasiconformal Mappings

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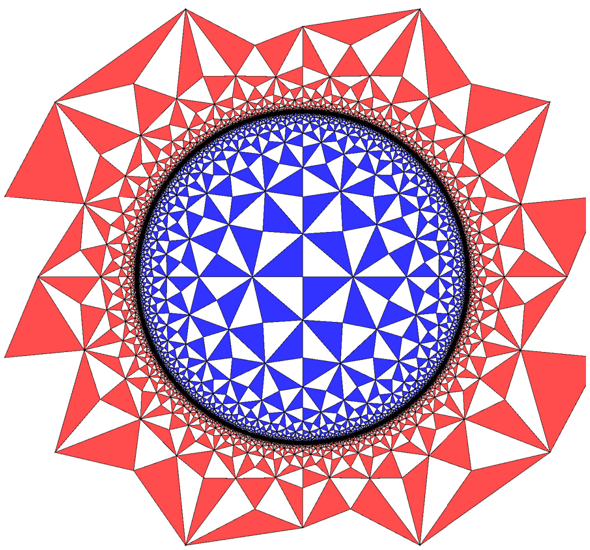
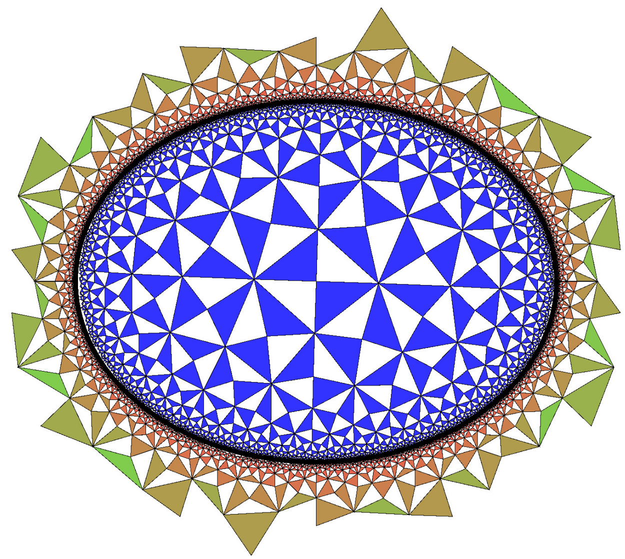




This semester I hope to cover the following topics:

- Review of conformal mappings
- Extremal length and conformal modulus, log capacity, harmonic measure
- Geometric definition quasiconformal mappings
- Basic properties
- Quasisymmetric maps and boundary extension
- Removable sets
- Conformal welding
- Analytic definition of quasiconformal mappings
- The measurable Riemann mapping theorem
- Further topics

# Area Distortion

**Theorem 6.1** (Astala's Therorem)**.** *Suppose $f$ is $K$-quasiconformal of $\mathbb{D}$ to itself with $f(0) = 0$. Then for any measurable $E \subset \mathbb{D}$, we have $|f(E)| \leq C(K) \cdot |E|^{1/K}$.*

**Theorem 6.2** (Astala's Therorem)**.** *Suppose $f$ is $K$-quasiconformal of $\mathbb{D}$ to itself with $f(0) = 0$. Then for any measurable $E \subset \mathbb{D}$, we have $|f(E)| \leq C(K) \cdot |E|^{1/K}$.*

Astala's original proof uses idea from dynamics. Here we will follow a shorter proof due to Alexandre Eremenko and David Hamilton.

Kari Astala

Alex Eremenko


David Hamilton

**Lemma 6.3.** *Let* $\{p_j\}_1^n > 0$ *and* $\{q_j\}_1^n > 0$ *be probability distributions on* $1, \ldots, n$*. Then*

$$\sum_{j=1}^n p_j \log p_j \geq + \sum_{j=1}^n p_j \log q_j.$$

*Proof.* Since $\phi(x) = x \log x$ is convex (compute $\phi''$)

$$\begin{aligned} \sum_{j=1}^n p_j \log p_j - + \sum_{j=1}^n p_j \log q_j &= \sum_{j=1}^n q_j \phi(p_j/q_j) \\ &\geq \phi(\sum_{j=1}^n q_j \cdot p_j/q_j) \\ &= \phi(1) = 0 \quad \square \end{aligned}$$

**Lemma 6.4.** *Let* $\{a_j\}_1^n$ *be positive functions on the unit disk such that* $\log a_j$ *are harmonic and for* $|\lambda| < 1$,

$$\sum_{j=1}^{n} a_j(\lambda) \leq 1. \tag{6.13}$$

*Then for* $|\lambda| < 1$,

$$\sum_{j=1}^{n} a_j(\lambda) \leq \left( \sum_{j=1}^{n} a_j(0) \right)^{(1-|\lambda|)/(1+|\lambda|)} . \tag{6.14}$$

*Proof.* For $\lambda, z \in \mathbb{D}$, define the probability distributions

$$p_j = \frac{a_j(\lambda)}{\sum a_j(\lambda)}, \qquad q_j = \frac{a_j(z)}{\sum a_j(z)}.$$

and for fixed $\lambda$, define

$$H(z) = -\sum_{j=1}^{n} p_j \log a_j(z) + \sum_{j=1}^{n} p_j \log q_j.$$

By our assumption on $a_j$, $H$ is harmonic. By Lemma 6.3 $H(z) \geq 0$. So by Harnack's inequality

$$H(z) \geq \frac{1-|z|}{1+|z|} H(0).$$

By Lemma 6.3

$$\begin{aligned}
H(z) &= -\sum_{j=1}^{n} p_j \log a_j(z) + \sum_{j=1}^{n} p_j \log q_j \\
&= -\sum_{j=1}^{n} p_j \log \sum a_j(z) + p_j \log \frac{a_j(z)}{\sum a_j(z)} + \sum_{j=1}^{n} p_j \log q_j \\
&\geq -\sum_{j=1}^{n} p_j \log \sum a_j(z) \\
&= -\log \sum a_j(z) \sum_{j=1}^{n} p_j \\
&= -\log \sum a_j(z).
\end{aligned}$$

Setting $z = \lambda$,

$$\begin{aligned}
H(\lambda) &\geq \frac{1-|\lambda|}{1+|\lambda|} H(0) \\
&= \frac{1-|\lambda|}{1+|\lambda|} \left( -\sum_{j=1}^{n} p_j \log a_j(0) + \sum_{j=1}^{n} p_j \log q_j \right) \\
&= \frac{1-|\lambda|}{1+|\lambda|} \left( -\sum_{j=1}^{n} p_j \log \sum_k a_k(0) + p_j \log \frac{a_j(0)}{\sum a_l k(0)} + \sum \phi(p_j) \right)
\end{aligned}$$

Now apply Lemma 6.3

$$H(\lambda) \geq \frac{1-|\lambda|}{1+|\lambda|} \left( -\sum_{j=1}^{n} p_j \log \sum_k a_k(0) \right) = \frac{1-|\lambda|}{1+|\lambda|} \left( -\log \sum_k a_k(0) \right)$$

Thus

$$-\log \sum_j a_j(\lambda) \geq H(\lambda) \geq \frac{1-|\lambda|}{1+|\lambda|} \Big( -\log \sum a_j(0) \Big) \, .$$

Switching signs and exponentiating gives

$$\sum_j a_j(\lambda) \leq \left( \log \sum_j a_j(0) \right)^{(1-|\lambda|)/(1+|\lambda|)} \, . \quad \square$$

**Corollary 6.5.** *Let* $a(z,\lambda)$ *is defined on* $E \times \mathbb{D}$ *and assume* $\log a(z,\lambda)$ *is harmonic in* $\lambda$. *positive functions on the unit disk Suppose*

$$\int_E a(z,(\lambda)dxdy \leq 1. \tag{6.15}$$

*Then for* $|\lambda| < 1$,

$$\int_E a(z,\lambda)dxdy \leq \left(\int_E a_j(z,0)dxdy\right)^{(1-|\lambda|)/(1+|\lambda|)}. \tag{6.16}$$

*Proof.* Write the integral as a limit of Riemann sums, apply Lemma 6.3 and take the limit. □

**Lemma 6.6** (Area theorem). *Suppose $f$ is conformal on $\mathbb{D}^* = \{|z| > 1\}$ and $f(z) = z + o(1)$ near infinite. Then $|\mathbb{C} \setminus f(\mathbb{D}^*)| \leq \pi$.*

*Proof.* For $r > 1$, $f(\{|z| = r\}$ is a smooth Jordan curve $\gamma$. Let $A(r)$ be the area of the region $\Omega$ enclosed by this curve. By Green's theorem

$$\begin{aligned} A(r) &= \int_\gamma x dy = - \int_\gamma y dx = \frac{i}{2} \int_\gamma w d\overline{w} \\ &= \frac{i}{2} \int_0^{2\pi} f(z) \overline{f'(z)} \overline{iz dt} \\ &= \frac{1}{2} \int_0^{2\pi} (z + a_1/z + \dots) \overline{(z - a_1/z - \dots)} dt \\ &= \frac{1}{2} \int_0^{2\pi} (1 - |a_1|^2 - 2|a_2|^2 - \dots) dt \leq \pi. \quad \square \end{aligned}$$

**Theorem 6.7.** *Suppose $f$ is $K$-quasiconformal on the plane and is conformal ouside $\mathbb{D}$, and assume $f(z) = z + o(1)$ near infinity. If the dilatation $\mu$ of $f$ is zero on $E \subset \mathbb{D}$, then $|f(E)| \leq \pi^{1-1/K}|E|^{1/K}$.*

*Proof.* Without loss of generality we may assume $\mu$ is smooth. If not, we approximate $\mu$ by smooth dilatations $\{\mu_n\}$ with $\|\mu_n\|_\infty \leq \|\mu\|_\infty$, and use the smooth case to deduce the result for the corresponding maps $f_n$. Since these maps converge uniformly to $f$, by Corollary **??**,

$$|f(E)| \leq \limsup_n |f_n(E)| \leq \pi^{1-1/K}|E|^{1/K}.$$

For $|\lambda| < 1$ define a $K_\lambda$-quasiconformal map $f_\lambda$ with dilatation

$$\mu_\lambda(z) = \lambda \frac{K+1}{K-1} \mu(z),$$

and normzlized so that $f(z) = z + o(1)$ near infinity. Note that $K_\lambda = (1 + |\lambda|)/(1 - |\lambda|)$.

The Jacobian of $f_\lambda$ is

$$J_\lambda(z) = |\partial_z f_\lambda(z)|^2 (1 - |\mu_\lambda(z)|^2).$$

Define

$$a(z, \lambda) = \frac{1}{\pi} J_\lambda(z) = \frac{1}{\pi} |\partial_z f_\lambda(z)|^2.$$

Since, for a fixed $z$, $f_\lambda(z)$ is a nonm-vanishing holomorphic function of $\lambda$, so is its derivative, and hence $a(z, \lambda)$ is harmonic in $\lambda$.

By the area theorem for conformal maps, $f_\lambda(\mathbb{D}) \leq \pi$, so

$$\int_{\mathbb{D}} J_\lambda(z) dxdy \leq \pi.$$

Thus $a(z, \lambda)$ satisfies Corollary 6.5, and hence

$$\frac{1}{\pi} \int_E J_\lambda(z) dxdy \leq (|E|/\pi)^{(1-|\lambda|)/(1+|\lambda|)}.$$

Setting $\lambda = (K-1)/(K+1)$ gives $\mu_\lambda = \mu$ and thus $|f(E)| \leq \pi^{1-1/K}|E|^{1/K}$. $\square$

In what follows, $\Delta$ is a Jordan domain of the form $\Delta = \mathbb{C} \setminus g(\{|z| > 1\})$ where $g$ is conformal on $\mathbb{D}^* = \{|z| > 1\}$ and $g(z) = z + o(1)$ near $\infty$. Often $\Delta = \mathbb{D}$.

We will appy this in the case when $g$ is actually $K$-quasiconformal on the plane and $\Delta = g(\overline{\mathbb{D}})$.

**Theorem 6.8.** *Suppose $f$ is $K$-quasiconformal on the plane and is conformal outside $\Delta$, where $\Delta$ is as above. Assume $f(z) = z + o(1)$ near infinity. If the dilatation $\mu$ of $f$ is zero on $\mathbb{D} \setminus E$, then $|f(E)| \leq K|E|$.*

*Proof.* If suffices to prove this for compact $E$, since for general sets, the area is just the supremum of the areas of all compact subsets.

By Lemma **??**, it suffices to prove this when $f$ is smooth, since we can find smooth approximations to $f$ whose diatiations are supporeted in a neighborhood $U$ of $E$ whose area is a close to $E$ as we wish.

Set $\omega = f_{\overline{z}}$. If $S$ denotes the Beurling transform, then $f_z = 1 + S\omega$ and

$$\omega = \mu(1 + S\mu + S\mu S\mu + \dots).$$

Then

$$\begin{aligned} |f(E)| &= \int_E J_f dxdy = \int_E |f_z|^2 - |f_{\overline{z}}|^2 dxdy \\ &= \int_E |1 + S\omega|^2 - |\omega|^2 dxdy \\ &= \int_E (1 + S\omega)\overline{(1 + S\omega)} - |\omega|^2 dxdy \\ &= \int_E (1 + \mathrm{Re}(S\omega) + |S\omega|^2 - |\omega|^2) dxdy. \end{aligned}$$

Since $S$ is an isometry on $L^2$, and $\omega$ is supported on $E$,

$$\int_E |S\omega|^2 dxdy \le \int_{\mathbb{C}} |S\omega|^2 dxdy = \int_{\mathbb{C}} |\omega|^2 dxdy = \int_E |\omega|^2 dxdy.$$

Thus

$$|f(E)| \le |E| + \int_E \mathrm{Re}(Sw) dxdy.$$

Let $(S\mu)^1 = S\mu$ and inductively define the $k$th iterate $(S\mu)^k = S(\mu(S\mu)^{k-1})$ for $k = 2, \ldots$.

Oserve that by Cauchy-Schwarz and since $S$ is an isometry on $L^2$ , the $k$th iterate satisfies

$$\begin{aligned}\int_E |(S\mu)^k| dxdy &\leq \left(\int_E 1 dxdy\right)^{1/2} \left(\int_{\mathbb{C}} |(S\mu)^k|^2 dxdy\right)^{1/2} \\ &= |E|^{1/2} \left(\int_E |\mu(S\mu)^{k-1}|^2 dxdy\right)^{1/2} \\ &= \|\mu\|_\infty |E|^{1/2} \left(\int_E |(S\mu)^{k-1}|^2 dxdy\right)^{1/2}.\end{aligned}$$

Applying induction we deduce

$$\int_E |(S\mu)^k| dxdy = \|\mu\|_\infty^k |E|^{1/2} \left(\int_E 1 dxdy\right)^{1/2} = \|\mu\|_\infty^k |E|.$$

Since $\|\mu\|_\infty = k = (K-1)/(K+1)$, we Thus

$$|f(E)| \leq |E| + 2|E|(\|\mu\|_\infty + \|\mu\|_\infty^2 + \ldots) = |E|(-1 + \frac{2}{1-k} = K|E|. \quad \square$$

The following result is Astala's theorem with a slightly different normalization.

**Corollary 6.9.** *Suppose $f$ is $K$-quasiconformal on the plane and is conformal ouside $\mathbb{D}$, and assume $f(z) = z + o(1)$ near infinity. If $E \subset \mathbb{D}$, then $|f(E)| \leq K\pi^{1-1/K}|E|^{1/K}$.*

*Proof.* Write $f = h \circ g$ where $g$ is conformal on $E$ and $h$ is conformal off $g(E)$. Then

$$|f(E)| = |h(g(E))| \leq K|g(E)| \leq K\pi^{1-1/K}|E|^{1/K}.$$

□

*Proof of Astala's theorem.* Astala's theorem is for self-maps of the disk, whereas what we have done following Eremenko and Hamilton is for maps that are conformal outside the unit disk.

A $K$-quasiconformal self-map of the disk $f$ can be written as a cposition of two $K$-quasiconformal maps $f = circg$ where $g$ is conformal off the disk, $g(z) = z + o(1)$ near infinity, and $h$ is conformal in $\Omega = g(\mathbb{D})$.

Then $|f(E)| = |h(g(E))|$ and we know $|g(E)| \leq C(K)|E|^{1/K}$, so it is enough to know that $h$ mutiplies the area of $g(E)$ by at most a factor depending only on $K$.

By the compactness properties of $K$-quasiconformal maps, $\Omega$ contains a disk $D(x, 2r)$ of radius $r = r(K)$. Since $h$ is conformal from $\Omega$ to the unit disk, it distorts area by a bounded factor. Thus the image of $g(E) \cap D(x, r)$ has the desired area bound.

The map $h$ is conformal inside $D(x, r)$ and $g(E) \setminus D(x, r)$ is outside this set, so by inverting and normalizing, we can apply Lemma 6.8 to deduce that the area of $g(E) \setminus D(x, r)$ is also multiplied by at most a bounded factor, depending only on $K$. $\square$

Define $p(K) = \sup\{p : J_f \in L^p_{\text{loc}}(\Omega)\}$ where the supremum is over all $K$-quasiconformal maps $f$ on $\Omega$.

We have seen previously that $p(K) > 1$; Bojarski's Theorem, Theorem **??**.

**Lemma 6.10.** $p(K) \leq K/(K-1)$.

*Proof.* Let $f(z) = z|z|^{(1/K)-1}$ shows that $p(K) \leq K/(K-1)$.

The partials are $O(|z|^{(1/K)-1}$, so $J_f^p$ is of order $|z|^{2p(1-K)/K}$, so to be locally integable, we need $2p(1-K)/K > -2$ or $p < K/(K-1))$. □

**Theorem 6.11.** *For any planar domain* $\Omega$, $p(K) = K/(K-1)$

*Proof.* We only need to prove $p(K) \geq K/(K-1)$.

First consider a $K$-quasiconformal map $f : \mathbb{D} \to \mathbb{D}$ and for $s \geq 0$, set

$$E_s = \{x \in \mathbb{D} : J_f(x) \leq s\}.$$

By Astala's area therorem

$$s|E_s| \leq \int_{E_s} J_f dxdy = |f(E_s)| \leq C(K)|E_s|^{1/K}$$

or, solving for $|E_s|$,

$$|E_s| \leq \left(\frac{C(K)}{s}\right)^{K/(K-1)}.$$

For such a map

$$\int_{\mathbb{D}} J_f^p dxdy \leq \pi + \int_1^\infty |E_s| ds = \pi + M(K) \int_0^\infty s^{p-1} s^{-K/(K-1)} ds.$$

This is finite if $(p-1) - K/(K-1) < -1$ or $p < K/(K-1)$. $\square$

For a general $K$-quasiconformal map on a domnain $\Omega$, choose a compact disk $D$ with $2D \subset \Omega$. Let $\psi$ and $\phi$ be conformal maps of $2D$ and $f(2D)$ respectively to the unit disk. Then the previous argument applies to $g = \phi \circ f \circ \psi^{-1}$.

But by Koebe's theorem the derivative of $\phi$ and $\psi$ are both comparable to constants on $D$ and $f(D)$ and thus $J_f^p$ is integrable on $D$ if and only if $J_g$ is. This proves the result.

**Theorem 6.12.** *Suppose $f : \Omega \to \Omega'$ is $K$-quasiconformal and $E \subset \Omega$ is compact. Then*

$$\dim(f(E)) \leq \frac{2K \dim(E)}{2 + (K-1)\dim(E)}.$$

Astal gives examples showing equality is possible for some Cantor sets.

For $K = (1+k)/(1-k)$ and a line segment $E$, the estimate says

$$\dim(f(E)) \leq \frac{2K}{K+1} = 1 + k^2.$$

Astala conjectured, and Smirnov later proved, that $\dim(f(E)) \leq 1 + k^2$, but Ivrii has show this is not sharp either (at least for small $k$).

The estimate in the theorem can be re-written as

$$\frac{1}{K}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \leq \frac{1}{\dim(f(E))}\frac{1}{2} \leq K\left(\frac{1}{\dim(E)} - \frac{1}{2}\right).$$

**Lemma 6.13.** *Suppose* $0 < t < 1$, $f : \mathbb{D} \to \mathbb{D}$ *is* $K$*-quasiconformal with* $f(0) = 0$, *and* $\{B_j\}$ *are pairwise disjoint balls in* $\mathbb{D}$. *Then if* $tK/(1+t(K-1)) < p \leq 1$,

$$\sum_j |f(B_j)|^p \leq C(K,t,p) \left( \sum_j |B_j|^t \right)^{1/(1+t(K-1)} .$$

Recall that $|B_j|$ denotes the area of $B_j$.

*Proof.* Since

$$\frac{p(1+t(K-1))}{tK} > 1,$$

If $1 < p_0 < K/(K-1)$, then the conjugate exponent $q_0 = p_0/(p_0-1)$ satisfies $K < q_0 < \infty$ or $1 < q_0/K < \infty$..

Since $p(1+t(K-1))/(tK) > 1$, we can choose $p_0$ and $q_0$ so that

$$1 < \frac{q_0}{K} < p\frac{1+t(K-1)}{tK}.$$

Now use Hölder's inequality with exponents $p_o, q_0$ to deduce

$$\begin{aligned} \sum_j |f(B_j)|^p &= \sum_j \left( \int_{B_j} J_f dxdy \right)^p \\ &= \sum_j \left[ \left( \int_{B_j} J_f^{p_0} dxdy \right)^{p/p_0} |B_j|^{p/q_0} \right]. \end{aligned}$$

Next apply Hölder's inequality to the sum with conjugate exponents $p_0/p$ and $p_0/(p_0 - p)$ to get

$$\begin{aligned}\sum_j |f(B_j)|^p &= \sum_j \left(\sum \int_{B_j} J_f^{p_0} dxdy\right)^{p/p_0} \left(\sum_j |B_j|^{(p/q_0)p_0/(p_0-p)}\right)^{p_0-p)/p_0} \\ &= \sum_j \left(\int_{\mathbb{D}} J_f^{p_0} dxdy\right)^{p/p_0} \left(\sum_j |B_j|^{(p/q_0)p_0/(p_0-p)}\right)^{p_0-p)/p_0} .\end{aligned}$$

Since $p_0 < P(K) = K/(K-1)$, the first term is finite.

Some artihmeticc shows that $p_0/(p_0 - p) > 1 + t(K - 1)$ so the sum in the second term becomes at most a constant times larger if we replace the first term by the second in the exponent (recall $|B_j| \leq \pi$). Thus

$$\sum_j |f(B_j)|^p = C(K,t,p) \left( \sum_j |B_j|^{(p/q_0)(1+t(K-1)} \right)^{p_0 - p)/p_0} .$$

We choose $q_0$ so that $p/q_0 > t/(1 + t(K - 1)$, so this becomes

$$\sum_j |f(B_j)|^p = C(K,t,p) \left( \sum_j |B_j|^{t)} \right)^{1/(1+t(K-1))} . \quad \square$$

*Proof of Astala's dimension estimate.* Suppse $f : \Omega \to \Omega'$ is $K$-quasiconformal and that $E \subset \Omega$ is compact with Hausdorff dimension strictly less than 2.

Choose $\dim(E)/2 < t < 1$ and cover $E$ by squares with disjoint interiors. Each square contains an inscribed ball of comparable size, giving a collect $\{B_j\}$ of pairwise disjoint balls whose doubles cover $E$.

By Corollary **??** we know $\operatorname{diam}(B_j)^2 \simeq |B_j|$. Thus if $\delta > 2tK/(1 + t(K-1)$, we have

$$\sum_j \operatorname{diam}(f(B_j))^\delta \;=\; C \left( \sum_j |B_j|^{2t)} \right)^{(} 1/(1 + t(K-1)).$$

For any $t > \dim(E)/2$, the sum on the right can be made as small as we wish, by an appropriate choice of covering squares. Thus $\dim(f(E)) \leq \delta$ for any $\delta > 2tK/(1 + t(K-1)$ and thus any

$\delta > \dim(E)K/(1+\dim(E)(K-1)/2 = 2\dim(E)K/(2+\dim(E)(K-1)$. $\square$

**Lemma 6.14.** *If $E \subset \mathbb{D}$ is closed and has zero Hausdorff 1-measure, then any bounded holomorphic map $f$ on $\Omega = \mathbb{D} \setminus E$ extends to be holomorphic on $\mathbb{D}$.*

*Proof.* We can choose $R$ arbitrarily close to 1 so that e circle $C_R = \{|z| = R\}$ does not hit $E$, since otherwise $E$ hits all large enough circles and hence has positive length.

Cover the part of $E$ inside $C_R$ by balls whose total boundary length is less than $\epsilon$. For any $z$ inside $C_R$ but outside the balls, we use the Cauchy integral formula to write $f(z)$ as the Cauchy integral over $C_R$ and a contour $\gamma$ of length at most $\epsilon$ contained in the union of the boundaries of the ball. Since $f$ is bounded, the contribution of $\gamma$ tends to zero with $\epsilon$ and hence $f$ agrees with its Cauchy integral over $C_R$, which defines a holomorphic function on the entire interior of $C_R$. Taking $R \nearrow 1$, shows $f$ extends to be holomorphic on all of $\mathbb{D}$.

□

**Corollary 6.15.** *In planar domains a compact set $E$ with* $\dim(E) < 2/(K+1)$ *is removable for bounded $K$-quasiregular maps.*

Astala contructs sets of any dimension $> 2/(K+1)$ that are not removable.

*Proof.* It suffices to consider maps defined on a disk.

Any $K$-quasiregular map $f$ can be factored as $f = \phi \circ g$ where $\phi$ is holomorphic on $\mathbb{D}$ and $g : \mathbb{D} \to \mathbb{D}$ is $K$-quasiconformal.

If $\dim(E) < 2/(K+1)$, then

$$\dim(g(E)) \leq \frac{2K2/(K+1)}{2 + (K-1)2/(K+1)} = 1.$$

Thus $g(E)$ is removable for $\phi$, i.e., $\phi$ extends to be holomorphic on the whole plane and hence $f$ extends to be quasiregular on the plane. $\square$

**Smirnov's $1 + k^2$ bound**

**Theorem 6.16.** *Suppose $f$ is $K$-quasiconformal on the plane and $K = (1+k)/(1-k)$. Then* $\dim(f(\mathbb{R})) \leq 1 + k^2$.

This is not sharp. Precise bound is not known.

Oleg Ivrii has shown that a better bound is $1 + \Sigma^2 k^2 + O(k^{8/3} - \epsilon)$ where $\Sigma^2$ is a constant less than 1 (by deep work of Hedenmalm).

$\Sigma^2$ is defined as $\sup \sigma^2(S\mu) < 1$ where the supremum is over measurable funtions $\mu$ so that $|\mu| \leq 1$ on $\mathbb{D}$ and is 0 elsewhere, and $\sigma^2$ is the asymptotic variance of a Bloch function

$$\sigma^2(g) = \lim_{R \searrow 1} \frac{1}{2\pi |\log(R-1)|} \int_{|z|=R} |g(z)|^2 |dz|.$$

McMullen showed that if $\mu$ is invariant under a co-compact Fuchsian group then

$$2\frac{d^2}{dt^2}|_{t=0} \dim(\omega^{t\mu}(\mathbb{T})) = \sigma^2(S\mu).$$

We first need the following result.

**Theorem 6.17.** *The following are equivalent:*

*i* $\Gamma$ *is a* $k$*-quasiline.*

*ii* $\Gamma = \psi(\mathbb{R})$ *with* $\|\mu_\psi\|_\infty \leq 2k/(1+k^2)$ *and* $\mu_\psi = 0$ *on* $\mathbb{H} = \{x+iy : y > 0\}$.

*iii* $\Gamma = \phi(\mathbb{R})$ *with* $\|\mu_\phi\|_\infty \leq k$ *and*

$$\mu_\phi(\overline{x}) = -\overline{\mu_\phi(z)}. \tag{6.17}$$

Condition 3 is similar to the condition

$$\mu_\phi(\overline{z}) = \overline{\mu_\phi(z)}. \tag{6.18}$$

This implies $\phi(\mathbb{R}) = \mathbb{R}$ and $\phi$ is symmetric. Such a map does not raise the dimension of $\mathbb{R}$ at all, and so such a dilatation is "wasted".

The condition in (iii) above gives an ellispse field that is in some sense orthogonal to the symmetric one, indicating it is "optimal" for the given quasicircle.

Recall the connection between ellipse fields and dilatations.

The eccentricity of an ellipse at $z$ is $|\mu(z)|$ and its major axis is in the direction $\arg(\sqrt{\mu(z)})$.

The ambiguity in the sqaure root makes no difference, since the major axis is given by both directions.

A map with dilatation $\mu$ maps the correspodning ellipse field to the “all circles” field, which we will denote by $T$ in what follows.

Condition (6.17) says that the ellipses at $\overline{z}$ is a 90°-degree rotation of the conjugate of the ellipse at $z$.

Condition (6.18) says that the ellipses at $z$ and $\overline{z}$ are conjugates, of each other, i.e., relfections across the real axis.

Since (iii) $\Rightarrow$ (i) is trivial, we need only prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

*Proof of (i) $\Rightarrow$ (ii)* . We follow Smirnov's notation and proof closely, giving the proof in terms of ellipse fields.

Suppose $\|\mu\|_\infty = k$ and let $N(z)$, $\|N\| \le K = (1+k)/(1-k)$, be the ellipse field representing a $k$-quasiconformal map $\eta$, which maps $\mathbb{R}$ onto $\Gamma$.

Define an ellipse field $A$:

$$A(z) = \begin{cases} \overline{N(\bar{z})}, & z \in \mathbb{H}_l \\ N(z), & z \in \mathbb{H} \end{cases} .$$

Let $\alpha$ be the quasiconformal map perserving $\mathbb{R}$ corresponding to the ellipse field $A$. and define $\psi := \eta \circ \alpha^{-1}$.

Note $\psi(\mathbb{R}) = \eta(\alpha^{-1}(\mathbb{R})) = \eta(\mathbb{R}) = \Gamma$.

For $z \in \mathbb{H}$, $\eta$ and $\alpha$ both send the ellipse field $N(z)$ to the field of circles, hence the map $\psi = \eta \circ \alpha^{-1}$ preserves the field of circles and is conformal in the upper half-plane.

In $\mathbb{H}_l$ both $\eta$ and $\alpha$ change eccentrivities by at most $K$, so $\psi$ changes eccentricities by at most $K^2$. Thus

$$\|\mu_\psi\| \leq (K^2 - 1)/(K^2 + 1) = 2k/(1 + k^2).$$

□

*Proof of (ii) $\Rightarrow$ (iii)* . Let $M(z)$ be the ellipse field corresponding to the $2k/(1+k^2)$-quasiconformal map $\psi$, with quasiconstant

$$K' = \frac{1+2k/(1+k^2)}{1-2k/(1+k^2)} = \{\frac{1+k}{1-k}\}^2 = K^2 .$$

Let $\beta$ be a quasiconformal map corresponding to the ellipse field

$$B(z) = \begin{cases} \sqrt{M(z)}, & z \in \mathbb{H}_l \\ \sqrt{\overline{M(\bar{z})}}, & z \in \mathbb{H} \end{cases} .$$

Here $\sqrt{M}$ denotes the ellipse with the same alignment whose eccentricity is the square root of $M$'s eccentricity; the ellipses are not rotated, only their eccentricities change. As before $\beta(\mathbb{R}) = \mathbb{R}$

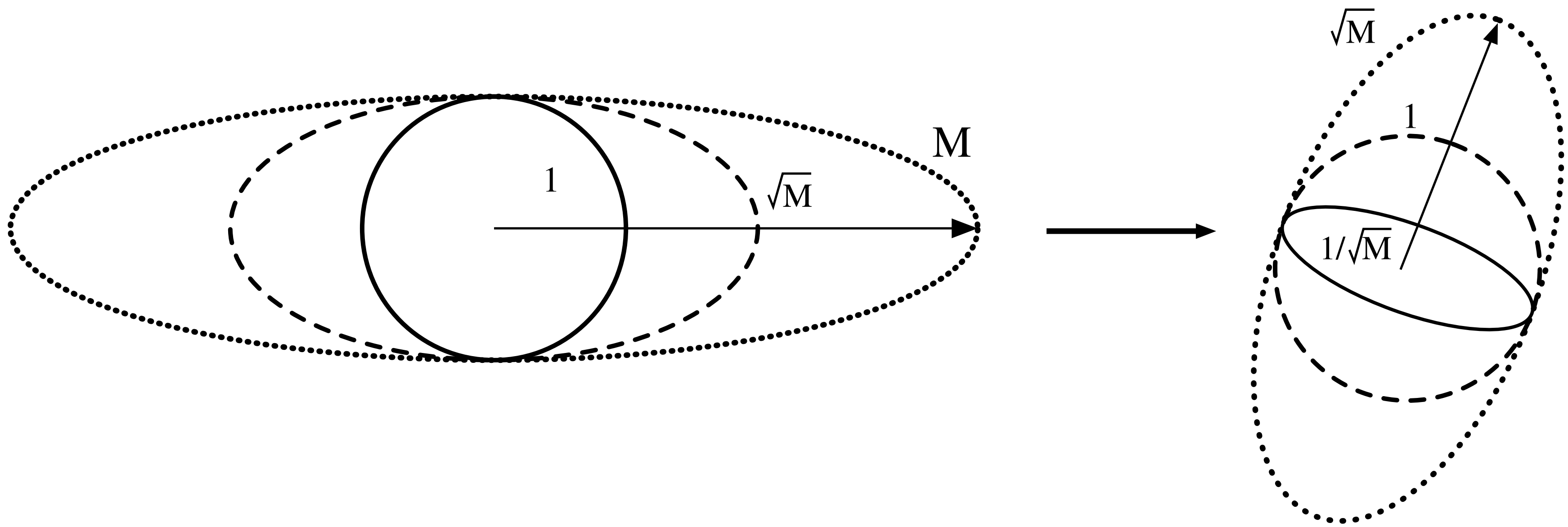
Define $\phi := \psi \circ \beta^{-1}$. Then $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \Gamma$.

Let $L(z)$ be the image of the ellipse field $M(z)$ under $\beta$. For $z \in \mathbb{H}$, $\beta$ maps the circle field to $B = \sqrt{M}$ and $\psi$ maps it $M$. So $\phi$ maps $\sqrt{M}$ to $M$. This is the same as the preimage of the circle field under the map $\phi$.

$\beta$ sends the ellipse field $\sqrt{M(z)}$ to the field of circles, $T$, and $\psi$ sends $M$ to $T$. Since $M = \psi^{-1}(T)$ $\psi$ and $\phi = \psi \circ \beta^{-1}$ , $\beta(M) = \phi^{-1}(T)$ is the preimage of the circle field under $\phi$.

For $z$ in the lower hal-plane, $\beta(\sqrt{M}) = T$. so $\|L\| = \sqrt{\|M\|} = \sqrt{K'} = K$.

Note that if a linear map sends an ellipse of eccentricty $\sqrt{M}$ to a circle, then the "parallel" ellipse of eccentricty $M$ is sent to a ellipse of eccenticity $\sqrt{M}$ and a circel also sent to an ellipse of eccentricty $\sqrt{M}$, but with major axis rotated by $\pi/2$.




For $z \in \mathbb{H}$, $M = T$ is the circle field, so $\beta$ sends $M$ to $L = \beta(M) = \beta(T)$ that at $z$ has the same eccentricity as $L(\overline{z})$ but the major axis is conjuagate and rotated by $\pi/2$. In terms of Beltrami coefficients, this is (6.17). □

**Lemma 6.18.** *Let $h$ be a positive harmonic function in the unit disc $\mathbb{D}$, whose partial derivative at the origin vanishes in the direction of some $\lambda \in \mathbb{D}$: $\partial_\lambda h(0) = 0$. Then $h$ satisfies*

$$(1-|\lambda|^2)/(1+|\lambda|^2)h(\lambda) \le h(0) \le (1+|\lambda|^2)/(1-|\lambda|^2)\, h(\lambda)\ .$$

This is a stronger version of Harnack's inequality:

$$(1-|\lambda|)/(1+|\lambda|)h(\lambda) \le h(0) \le (1+|\lambda|)/(1-|\lambda|)\, h(\lambda)\ .$$

that holds for all positive harmonic functions.

*Proof.* By replacing $h$ by $(h(z)+h(z^*))/(2h(0))$, where $^*$ denotes the symmetry with respect to the radial line through $\lambda$, we may assume $h(0) = 1$ and the gradient of $h$ vanishes at the origin.

If $\widetilde{h}$ is the harmonic conjugate of $h$ vanishing at 0, then $h + i\widetilde{h}$ map the disk to the right half-plane, and the function

$$f := \tau \circ (h + i\widetilde{h}) = \frac{z-1}{z+1} \circ (h + i\widetilde{h})$$

maps the disk to itself and satsifies $f(0) = f'(0) = 0$. By the Schwarz lemma, $|f(z)/z| \leq |z|$, hence $|f(z)| \leq |z|^2$.

Thus $h(\{|z| < \lambda\}) \subset |\tau^{-1}(\{|z| < \lambda^2\})$ and a direct calculation shows the latter set lies tween the vertical lines

$$\{x = \tau^{-1}(-\lambda^2)\} = \frac{1+\lambda^2}{1-\lambda^2}$$

and

$$\{x = \tau^{-1}(\lambda^2)\} = \frac{1-\lambda^2}{1+\lambda^2}.$$

This proves the lemma. □

*Proof that quasicircles have dimension* $\leq 1+k^2$. Suppse $\Gamma$ is a $k$-quasiline, i.e., $\Gamma=\phi(\mathbb{R})$ where $\|\mu_\phi\|_\infty \leq k < 1$.

Assume $\mu=\mu_\phi$ satisfies Smirnov's condition (6.17).

Define a holomorphic motion $\phi_\lambda$ with Beltrami coefficients $\mu_\lambda := \mu \cdot \lambda/k$ and which preserve points 0, 1, $\infty$. As usual, $\phi_0=\mathrm{id}$ and $\phi_k=\phi$.

Because of (6.17) holds for $\mu_k$ we can deduce that $\mu_\lambda$ satisfies (6.17) for real $\lambda$ and (6.18) for imaginary $\lambda$.

Moreover, for real values of $\lambda$ one has $\phi_\lambda(z) = \overline{\phi_{-\lambda}(\bar{z})}$.

Fix $\rho \in (1/2, 1)$, and consider $\lambda$ inside the slightly smaller disk $\rho\mathbb{D}$. Within this region, the maps $\phi_\lambda$ are uniformly quasisymmetric, so there is a constant $C = C_\rho$ such that

$$|z - x| \ \le\ |y - x| \ \Rightarrow\ |\phi_\lambda(z) - \phi_\lambda(x)| \le C_\rho \cdot |\phi_\lambda(y) - \phi_\lambda(x)|, \tag{6.19}$$

(6.20)

$$C_\rho \cdot |z - x| \ \le\ |y - x| \ \Rightarrow\ 2|\phi_\lambda(z) - \phi_\lambda(x)| \le |\phi_\lambda(y) - \phi_\lambda(x)|. \tag{6.21}$$

It suffices bound $\dim(\phi([0,1])$.

Cover $[0,1]$ by $n$ intervals $I_j = [a_j, b_j]$ of length $1/n$, and let $B_j(\lambda)$ be the ball centered at $\phi_\lambda(a_j)$ whose boundary circle passes through $\phi_\lambda(b_j)$.

Note that its "complex radius" $r_j(\lambda) := \phi_\lambda(b_j) - \phi_\lambda(a_j)$, is a holomorphic function of $\lambda$.

$\phi([0,1])$ is covered by the images of the $I_j$, and $\operatorname{diam}(I_j) \le C_\rho |r_j(\lambda)|$ by (6.19). To estimate the $\dim(\phi([0,1])$ we have to bound the sum

$$\sum_j \operatorname{diam}(\phi_\lambda(I_j))^p \le C^p \sum_j |r_j(\lambda)|^p. \tag{6.22}$$

We will estimate the logarithm of the right-hand side.

Since the logarithm is concave, if $\{\nu_j\}$ is a proability vector, then Jensen's inequality applied to $\{|r_j(\lambda)|^p/\nu_j\}$ gives

$$\log \sum_j |r_j(\lambda)|^p = \log \sum_j \nu_j \frac{|r_j(\lambda)|^p}{\nu_j} \tag{6.23}$$

$$\geq \sum_j \nu_j \log \left( \frac{|r_j(\lambda)|^p}{\nu_j} \right) \tag{6.24}$$

$$= I_\nu - p\Lambda_\nu(\lambda) \ , \tag{6.25}$$

where $I_\nu := -\sum_j \nu_j \log \nu_j$ is the "entropy" and $\Lambda_\nu(\lambda) := -\sum_j \nu_j \log |r_j(\lambda)|$ is the "Lyapunov exponent" of the probability distribution $\{\nu_j\}$.

Note $\Lambda_\nu(\lambda)$ is a harmonic function of $\lambda$, since $r_j(\lambda)$ are holomorphic.

Since log is strictly concave, equality is acheived if and only if all the mass is concentrated at one point, i.e., $|r_j(\lambda)|/\nu_j$ is independent of $j$.

This means $|r_j(\lambda)|$ is proportional to $\nu_j$.

Thus

$$\log \sum_j |r_j(\lambda)|^p = \sup_\nu \{I_\nu - p\Lambda_\nu(\lambda)\} \ , \tag{6.26}$$

where the supremum is taken over all probability distributions $\nu$.

Fix some $\nu = \{\mu_j\}$ and define

$$H(\lambda) := 2\Lambda_\nu(\lambda) - I_\nu + 3\log C_\rho.$$

Then $H$ is harmonic in $\lambda$ (since $\Lambda_\nu$ is) and is an even function on the real line (because of the symmetry of our motion $r_j(\lambda) = \overline{r_j(-\lambda)}$ for $\lambda \in \mathbb{R}$).

By (6.21) the balls $B_j(\lambda)$ cover every point at most $C_\rho$ times.

By (6.19) their union is contained in a ball of radius $C_\rho$.

Hence $\sum_j |r_j(\lambda)|^2 \le C_\rho^3$, and so by (6.26) we have $I_\nu - 2\Lambda_\nu(\lambda) \le \log C_\rho^3$. Therefore $H \ge 0$ on $\rho\mathbb{D}$ and thus

$$I_\nu - \Lambda_\nu(0) \le \log \sum |r_j(0)| = \log 1 = 0.$$

Thus

$$H(0) = 2\Lambda_\nu(0) - I_\nu + 3\log C_\rho \ge I_\nu + 3\log C_\rho.$$

Apply Lemma 6.18 (stronger Harnack inequality) in the disk $\rho\mathbb{D}$ to obtain

$$\begin{aligned} 2\Lambda_\nu(k) - I_\nu + 3\log C_\rho = H(k) &= H(k) \geq \frac{1-k^2\rho^{-2}}{1+k^2\rho^{-2}} H(0) \\ &\geq \frac{1-k^2\rho^{-2}}{1+k^2\rho^{-2}} \{I_\nu + 3\log C_\rho\}, \end{aligned}$$

which implies

$$\frac{2}{1+k^2\rho^{-2}} I_\nu - 2\Lambda_\nu(k) \leq \frac{2k^2\rho^{-2}}{1+k^2\rho^{-2}} 3\log C_\rho,$$

which can be rewritten as

$$I_\nu - \{1 + k^2\rho^{-2}\Lambda_\nu(k)\} \leq 3\frac{k^2}{\rho^2} \log C_\rho.$$

The last equation holds for all distributions $\nu$, so by the variational principle (6.26)

$$\log \sum_j |r_j(k)|^p = \sup_\nu \{I_\nu - p\Lambda_\nu(k)\} \leq k^2\rho^{-2} 3 \log C_\rho \leq 12 \log C_\rho, \tag{6.27}$$

where we set $p := 1 + k^2\rho^{-2}$.

Sending $n$ to infinity, (6.22) and (6.27) imply that the $p$-dimensional Hausdorff measure of $\phi[0,1]$ is bounded by $C_\rho^{14}$, and hence $\dim(\phi([0,1]) \leq p = 1 + k^2\rho^{-2}$.

Let $\rho \nearrow 1$ to obtain the desired estimate. $\square$