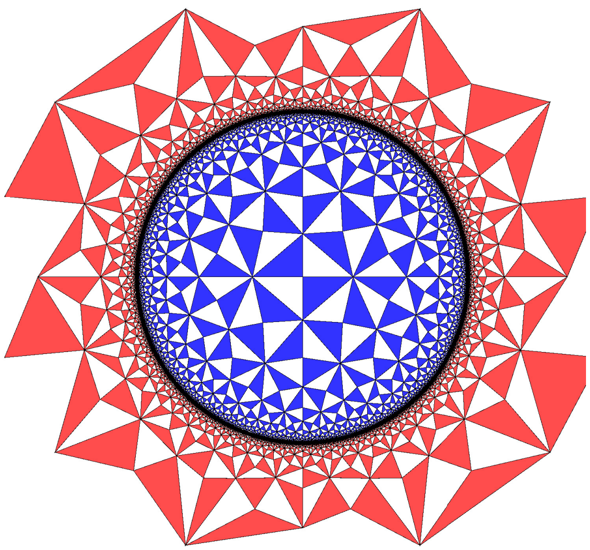
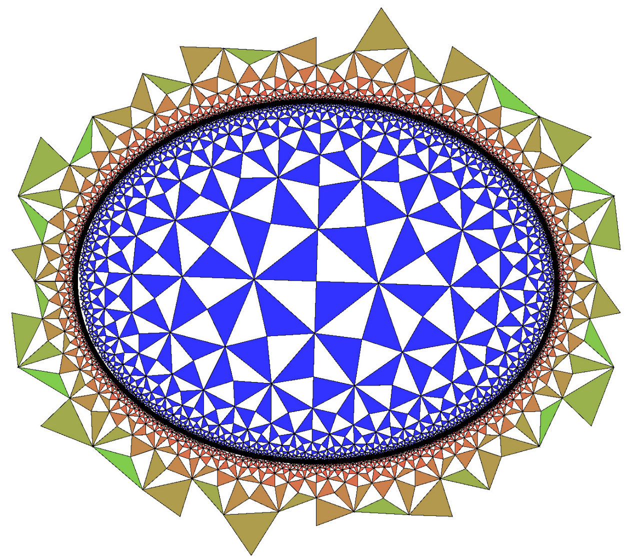
MAT 627, Spring 2025, Stony Brook University

Topics in Complex Analysis: Quasiconformal Mappings

Christopher Bishop

This semester I hope to cover the following topics:

- Review of conformal mappings
- Extremal length and conformal modulus, log capacity, harmonic measure
- Geometric definition quasiconformal mappings
- Basic properties
- Quasisymmetric maps and boundary extension
- Removable sets
- Conformal welding
- Analytic definition of quasiconformal mappings
- The measurable Riemann mapping theorem
- Further topics

Statement of the MRMP

Our goal in this section is to prove:

Theorem 5.1. *[Measurable Riemann Mapping Theorem] Given any measurable function μ on the plane with $\|\mu\|_\infty = k < 1$, there is a $K = (k+1)/(k-1)$ quasiconformal map f with dilatation $\mu_f = \mu$ Lebesgue almost everywhere on $\mathbb{C}$.*

The idea of the proof is fairly simple.

Given a measurable μ find a sequence of "nice" functions $\{\mu_n\}$ with $\mu_n \to \mu$ pointwise and $\sup_{\mathbb{C}} |\mu_n(z)| \leq k = \|\mu\|_\infty < 1$.

For these nice dilatations, we know that there is a corresponding K-QC map f_n with dilatation μ_n, and we may assume these maps are normalized to fix 0 and 1.

By compactness of K-QC maps there is a subsequence that converges uniformly on compact subsets of the plane to a K-QC map f.

Finally, we have to prove f is differentiable almost everywhere, and its dilatation μ_f equals μ almost everywhere.

The last step is the hard one, and requires two deep theorems.

Proposition 5.2. *A K-quasiconformal map f defined on a planar domain Ω is differentiable almost everywhere on Ω. The dilatation $\mu_f = f_{\bar{z}}/f_z$ is well defined and less than $k < 1$ almost everywhere.*

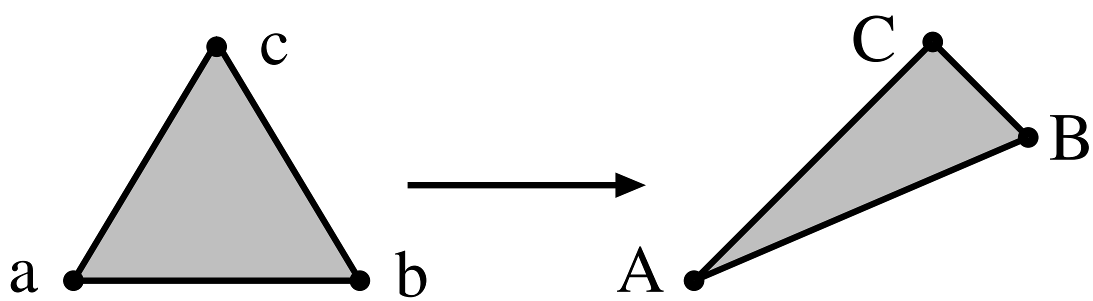
Proposition 5.3. *Suppose $\{f_n\}$, f are all K-quasiconformal maps on the plane with dilatations $\{\mu_n\}$, μ_f respectively, that $f_n \to f$ uniformly on compact sets and that $\mu_n \to \mu$ pointwise almost everywhere. Then $\mu_f = \mu$ almost everywhere.*

On the other hand, finding the "nice" dilatations is relatively easy.

We say that a linear map f is **K-quasiconformal** if $D_f \leq K$. The linear map need not be defined on the whole plane.

Given two triangles T_1, T_2 with vertices a, b, c and A, B, C, there is a unique affine map $T_1 \to T_2$ taking $a \to A$, $b \to B$ and $c \to C$.

The map is orientation preserving if both triangles were labeled in the same orientation.



There is an obvious affine map between these triangles and we can easily compute its quasiconformal constant of this map as follows.

First use a conformal linear map to send each triangle to one of the form $\{0, 1, a\}$ and $\{0, 1, b\}$. The affine map is then of the form $f(z) \to \alpha z + \beta \bar{z}$ where $\alpha + \beta = 1$ and $\beta = (b - a)/(a - \bar{a})$ and from this we see that

$$K_f = \frac{1 + |\mu_f|}{1 - |\mu_f|},$$

where

$$\mu_f = \frac{f_{\bar{z}}}{f_z} = \frac{\beta}{\alpha} = \frac{b - a}{b - \bar{a}},$$

If the triangle T' is degenerate, or has the opposite orientation as T, we simply give ∞ as our QC bound K.

Triangulate the plane using a triangular grid with elements of size δ_n.

Given a measureable μ on the plane, define μ_n to be the avegage of μ on each triangle of the grid.

Clearly $\|\mu_n\|_\infty \leq \|\mu\|_\infty$ and $\mu_n \to \mu$ (by the Lebsgue differenttiation theorem).

For each triangle T in the grid let T' be the triangle so that affine map between them has constant dilatation $\mu_n|_T$.

Then attach these triangles T' in the same pattern as the T's.

We get a simply connected, non-compact Riemann surface R_n and a QC map $g_n : \mathbb{C} \to R_n$.

By the uniformization theorem R_n is conformally equivalent either the plane or the disk.

Since this surface is QC equivalent to the plane, it must be the plane, i.e., there is a conformal map $f_n \to R_n \to \mathbb{C}$. (Lemma **??**

This gives a quasiconformal map $\phi_n = f_n \circ g_n : \mathbb{C} \to \mathbb{C}$ with dilatation μ. By composing with a conformal linear map, we can assume 0 and 1 are fixed by f_n.

Since the dilatations μ_n have absolute value bounded above by $\|\mu\|_\infty < 1$, there is a subsequence that converges uniformly on compact sets to a quasiconformal map f.

As noted above, we now have to show the hard part: f has a well defined dilatation and this is equal to μ.

The main technical difficulty involves Pompeiu's formula:

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_{\Omega}\frac{f_{\overline{z}}}{z-w}dxdy. \tag{5.13}$$

However, it is not even clear whether this formula makes sense for a quasiconformal map; since f is continuous, the first integral is well defined, but it is not clear whether the second integral is well defined in general; we need to verify that $f_{\overline{z}}$ is defined.

We expect (but have not yet proved) that

$$\begin{aligned}
\operatorname{area}(f(\Omega)) &= \int_\Omega J_f dxdy \\
&= \int_\Omega |f_z|^2 - |f_{\overline{z}}|^2 dxdy \\
&= \int_\Omega |f_z|^2(1-|\mu_f|^2) dxdy,
\end{aligned}$$

which would imply f_z and $f_{\overline{z}}$ are in L^2 locally.

However, $|z-w|^{-1}$ is not in L^2, so we can't be sure that the area integral in the Pompeiu formula is convergent.

However, $|z-w|^{-1}$ it in L^q locally for every $q < 2$, so the integral will be bounded if we can show $f_{\overline{z}} \in L^p$ locally for some $p > 2$.

This is a fundamental result of Bojarski in $\mathbb{C}$ and of Gehring in dimensions ≥ 2 and we will prove it later in this chapter, using the 2-dimensional version of Gehring's proof.

Most of the work consist of showing that for a K-quasiconformal map f, $f_z \in L^p$ for some $p > 2$ that depends only on K.

Some facts from Real Analysis I

Next we recall some facts from real analysis.

Theorem 5.4 (Wiener's Covering Lemma)**.** *Let $\mathcal{B} = \{B_j\}$ be a finite collection of balls in $\mathbb{R}^d$. Then there is a finite, disjoint subcollection $\mathcal{C} \subset \mathcal{B}$ so that*

$$\cup_{B \in \mathcal{B}} B \subset \cup_{B \in \mathcal{C}} 3B.$$

In particular, the Lebesgue measure of the set covered by the subcollection is at least 3^{-d} times the measure covered by the full collection.

Theorem 5.5 (Vitali Covering Lemma)**.** *Suppose $E \subset \mathbb{R}^d$ is a measurable set and $\mathcal{B} = \{B_j\} \subset \mathbb{R}^d$ is a collection of balls so that each point of E is contained in elements of $\mathcal{B}$ of arbitrarily small diameter. Then there is a subcollection $\mathcal{C} \subset \mathcal{B}$ so that $E \setminus \cup_{B \in \mathcal{C}} B$ has zero d-measure.*

Theorem 5.6 (Lebesgue Dominated Convergence theorem). *Suppose* $g \in L^2(\mu)$ *and* $\{f_n\}$ *satisfy* $|f_n| \leq g$ *and* $\lim f_n = f$ *pointwise. Then* $\lim \int f_n d\mu = \int f d\mu$.

Theorem 5.7 (Egorov's Theorem). *Suppose* μ *is a finite positive measure and* $\{f_n\}$ *is a sequence of measurable functions that converge to* f *pointwise almost everywhere on a set* E *with respect to* μ*. Then for every* $\epsilon > 0$ *there is a subset* $F \subset E$ *so that* $\mu(E \setminus F) < \epsilon$ *and* $f_n \to f$ *uniformly on* F.

Lemma 5.8 (The Calderon-Zygmund lemma)**.** *) Suppose Q is a square, $u \in L^1(Q, dxdy)$ and suppose*

$$\alpha > \frac{1}{\text{area}(Q)} \int_Q |u| dxdy.$$

Then there is a countable collection of pairwise disjoint open dyadic subsquares of Q so that

$$\alpha \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} |u| dxdy < 4\alpha, \tag{5.14}$$

$$|u| \leq \alpha \text{ almost everywhere on } Q \setminus \cup_j Q_j, \tag{5.15}$$

$$\sum \text{area}(Q_j) \leq \frac{1}{\alpha} \int_Q |u| dxdy \tag{5.16}$$

Proof. We say a subsquare of Q has property P is the first conclusion above holds and we define a collection of subsquares by iteratively dividing squares that do not have property P into four, equal sized disjoint subsquares, and stopping when property P is achieved.

If the average of u over a square is less than α then average over each of the four subsquares is $< 4\alpha$, so every stopped square has property P.

Any point not in a stopped square is a limit of squares where the average of u is $< \alpha$, so by the Lebesgue differentiation theorem $u \le \alpha$ at almost every such point. Finally,

$$\int_Q |u| dx dy \ge \sum_j \alpha \text{area}(Q_j),$$

which proves the third property. □

For a locally integrable function f, the **Hardy-Littlewood maximal function** of f is defined as

$$\mathcal{H}Lf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r))} |f(y)dy.$$

.

Here the supremum is over balls centered at x, but it is easy to see that we get some of comparable size we take all ball containing x.

Theorem 5.9 (Hardy-Littlewood maximal theorem)**.** *$\mathcal{H}L$ maps L^1 into weak-L^1, i.e., there is a constant d so that for all $\alpha > 0$*

$$|\{x : \mathcal{H}Lf(x) > \alpha\}| \leq \frac{C}{\alpha} \int |f(x)|dx.$$

Also, $\mathcal{H}L$ is a bounded operator on L^p for $1 < p \leq \infty$, i.e., there is a constant C_p so that $\|\mathcal{H}Lf\|_p \leq C_p \|f\|_p$.

Lemma 5.10. *If $\phi \geq 0$ is a compactly supported, radial, decreasing function with $\|\phi\|_1 = 1$ and f is locally integrable, then $|f * \phi(x)| \leq \mathcal{H}Lf(x)$.*

Theorem 5.11 (Marcinkiewicz interpolation theorem)**.** *Suppose* (X, μ) *and* (Y, ν) *are measure spaces, and suppose* $p_0, q_0, p_1, q_1 \in [1, \infty]$, *such that* $p_0 \leq q_0$, $p_1 \leq q_1$ *and*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for some $0 < t < 1$. *If* T *is a sub-linear map from* $L^{p_0}(\mu) + L^{p_1}(\mu)$ *to the space of measurable functions on* Y *that is weak-type* (p_0, q_0)

In particular, a sublinear operator that maps $L^1(\mu$ *boundedly into weak-*L^1 *and is bounded on* L^∞ *is also bounded from* L^p *to* L^p *for all* $1 < p < \infty$.

For the proof see Theorem 6.28 Folland's book.

Absolute Continuity on Lines

The main type of K-quasiconformal maps used in this text are piecewise C^1 functions that satisfy

$$|f_{\overline{z}}| \leq k|f_z|, \tag{5.17}$$

where $k - (K - 1)/(K + 1)$.

By itself, this equation holding almost everywhere is not enough to guarantee a map is quasiconformal.

For example, suppose $g : [0, 1] \to [0, 1]$ is the usual Cantor singular function.e., a continuous function that increases from 0 to 1 on $[0, 1]$ and is constant on each complementary component $\{I_j\}$ of the Cantor middle-$\frac{1}{3}$ set E. Then the map

$f(x, y) = (x + g(x), y)$, is a homeomorphism of $[0, 1] \times [0, 1]$ to $[0, 2] \times [0, 1]$ that is a translation (hence conformal) on each rectangle $I_j \times [0, 1]$, where I_j is a complementary interval of the Cantor set. Thus $f_{\overline{z}} = 0$ almost everywhere, but there are several way to check that f is not quasiconformal,

It is not conformal because does not preserve the modulus of $[0, 1]^2$.

If I is a covering interval of the Cantor set of length 2^{-n} whose image under g has length 3^{-n}, then the modulus of $I \times [0, 1]$ is changed by a factor of $(3/2)^n$.

A map $f : \mathbb{R} \to \mathbb{C}$ is absolutely continuous if for every compact interval $I \subset \mathbb{R}$ and $\epsilon > 0$ there is a $\delta > 0$ so that $E \subset I$ and $|E| < \delta$ imply $|f(E)| < \epsilon$.

It is a theorem of real analysis that a function is absolutely continuous if it is differentiable almost everywhere, its derivative is locally in L^1, and the fundamental theorem of calculus holds: $f(b) - f(a) = \int_1^b f'(x)dx$.

Theorem 5.12. *If f is quasiconformal, then f is absolutely continuous on almost every line in any given direction.*

Proof. After applying a Euclidean similarity, we may consider horizontal lines in $Q = [0,1]^2$. Define

$$A(y) = \text{area}(f([0,1] \times [0,y])).$$

Then $A(0) = 0$, $A(1) = \text{area}(f(Q)) < \infty$ and A is increasing.

Thus A is continuous except on a countable set and has a finite derivative almost everywhere. Fix a value of y where both this things happen, and we will show that f is absolutely continuous on the horizontal line $L_y = [0,1] \times \{y\}$.

The main idea is that if this failed, then modulus estimates relating length to area will force $A'(y) = \infty$.

Consider the long, narrow rectangle $R = [0,1] \times [y, y + \frac{1}{n}]$ and divide it into $m << n$ disjoint $\frac{1}{m} \times \frac{1}{n}$ sub-rectangles $\{R_j\}$. Let $R'_j = f(r_j)$ and the the "left", "right", and "bottom" edges of R'_j be the images under f of corresponding edges of R_j.

Let b_j be length of $f(L_y \cap \partial R_j)$, i.e., the length of the bottom edge of R'_j. This number might be finite or infinite.

Fix $\epsilon > 0$. In the first case, by taking n large enough, we can insure that any curve in $f(R_j)$ than joins the images of the vertical sides of R_j has length $\geq b_j - \epsilon$. In the second case, we can insure these curves all have length $\geq 1/\epsilon$.

In both case this follows because as $n \to \infty$, any curve in $f(R_j)$ joining the opposite "vertical" sides limits on the bottom edge and hence the liminf of the lengths of such curves is at least the length of the bottom edge of R'_j.

By quasiconformality we know

$$M(R'_j) \geq M(R_j)/K = \frac{m}{Kn},$$

and using the metric $\rho = 1$ on R'_j, shows

$$M(R'_j) \leq \frac{\text{area}(R'_j)}{b_j^2}.$$

Thus by Cauchy-Schwarz,

$$\begin{aligned}
(\sum_{j=1}^{m} b_j)^2 &\leq (\sum_{j=1}^{m} b_j^2 m)(\sum_{j=1}^{m} \frac{1}{m}) \\
&\leq m \sum_{j=1}^{m} \frac{\text{area}(R'_j)}{M(R'_j)} \\
&\leq m \sum_{j=1}^{m} \frac{\text{area}(R'_j)}{m/Kn} \\
&\leq \sum_{j=1}^{m} \text{area}(R'_j) Kn \\
&\leq K \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \\
&\to KA'(y).
\end{aligned}$$

If any of the b_j's is infinite, so is $A'(y)$, so $f(L_y)$ has finite length for our choice of y.

Given a compact set $E \subset L_y$, suppose E is hit by N of the rectangles R_j and that m has been chosen so large that $N/m \leq 2m_1(E)$.

Then repeating the argument above, but only summing over the j's so that the bottom edges of R_j hit E,

$$
\begin{aligned}
(\sum_j b_j)^2 &\le (\sum_j b_j^2 m)(\sum_j \frac{1}{m}) \\
&\le N \sum_j \frac{\text{area}(R'_j)}{M(R'_j)} \\
&\le N \sum_j \frac{\text{area}(R'_j)}{m/Kn} \\
&\le \frac{N}{m} \sum_{j=1}^{m} \text{area}(R'_j) Kn \\
&\le K m_1(E) \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \\
&\to K m_1(E) A'(y).
\end{aligned}
$$

Thus $m_1(E)$ small, implies $\sum b_j$ is small, and hence $f(E)$ has small 1-dimensional measure. Hence f is absolutely continuous on L_y, as desired. $\square$

Basic theorems of real analysis say that if f is absolutely continuous on a line L , then its partial derivative along that lines exists almost everywhere and

$$f((b) - f(a) = \int_a^b f_n ds,$$

where $a, b \in L$ and f_n is the partial in the direction from a to b.

Since we have shown that quasiconformal maps are absolutely continuous on almost every horizontal and almost every vertical line, we see that the partial f_x, f_y exist almost everywhere and hence $f_z, f_{\overline{z}}, \mu_f = f_{\overline{z}}/f_z$ are all well defined almost everywhere.

Next we want to say that at a point w where these all exist, we have

$$f(z) = f(w) + f_z(w)(z - w) + f_{\overline{z}}(w)(\overline{z} - \overline{w}) + o(|z - w|),$$

i.e., f is differentiable at w. However, as explained in most calculus texts, the existence of partial derivatives at at a point does not imply a function is differentiable there (consider $f(x, y) = x^2y/(x^2 + y^2)$ at the origin).

Theorem 5.13. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ and has partials almost everywhere, then it is differentiable almost everywhere.*

Proof. By Egorov's theorem the limits

$$f_x(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

$$f_y(z) = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{h},$$

are uniform and converge to a continuous functions on a compact set $E \subset \Omega$ so that $\text{area}(\Omega \setminus E)$ is as small as we wish.

Almost every point of E is a point of density for the intersection of E with both the vertical and horizontal lines through z_0, so if suffices to proof differentiability at such points.

For simplicity we assume 0 is such a point. The proof follows the usual case in calculus where we assume the partials are continuous, except that here we have to replace continuous on a neighborhood of 0 with continuous on a set E that is measure dense around 0.

Because of the continuity and uniform convergence on E, for any $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f_x(0) - f_x(z)|, |f_y(0) - f_y(z)| < \epsilon,$$

if $z \in E \cap D(0,\delta)$-neighborhood of 0 and

$$|f_x(z) - \frac{f(z+h) - f(z)}{h}|, |f_y(z) - \frac{f(z+ih) - f(z)}{h}| < \epsilon,$$

if $z \in E \cap D(0,\delta)$ and $h \in [-\delta, \delta]$.

Note that

$$\begin{aligned} f(z) - f(0) - xf_x(0) - yf_y(0) &= [f(z) - f(x) - yf_y(0)] + [f(x) - f(0) - xf_x(0)] \\ &\quad +[yf_y(x) - f_y(0)] \\ &= I + II + III. \end{aligned}$$

If $|z| < \delta$ and $x \in E$, then by the inequalities above, $I < \epsilon|y|$, $II < \epsilon|x|$ and $III < \epsilon y$, so the term on the far left is bounded by $3\epsilon|z|$, which proves differentiability if $x \in E$. A similar proof works if $iy \in E$.

Fix $\epsilon > 0$ and choose δ so small that if $0 < x < \delta$, then $E \cap (\frac{x}{1+\epsilon}, x) \neq \emptyset$ (this must be possible since $E \cap \mathbb{R}$ has density 1 at 0) and $E \cap (\frac{iy}{1+\epsilon}, y) \neq \emptyset$.

Thus if $0 < |x|, |y| \leq \delta/(1+\epsilon)$ can find points $x_1, x_2 \in E \cap (\frac{x}{1+\epsilon}, (1+\epsilon)x)$ and $iy_1, iy_2 \in E \cap i(\frac{y}{1+\epsilon}, (1+\epsilon)y)$ and so that $x + iy$ is inside the rectangle $R = (x_1, x_2) \times (y_1, y_2)$.

Since f is a homeomorphism (all we need is that it is continuous and open), $|f|$ takes its maximum on the boundary, so

$$\begin{aligned}
\sup_{z=x+iy\in R} & |f(z) - f(0) - xf_x(0) - yf_y(0)| \\
&\leq \sup_{w=u+iv\in\partial R} |f(zw - f(0) - xf_x(0) - yf_y(0)| \\
&\leq 3\epsilon|w| + \sup_{w=u+iv\in\partial R} |x-u||f_x(0)| + |y-v||f_y(0)| \\
&\leq 3\epsilon(1+\epsilon)|z| + \epsilon|f_x(0)||z| + \epsilon|f_y(0)||z|.
\end{aligned}$$

□

Corollary 5.14. *A K-quasiconformal map f defined on a planar domain Ω is differentiable almost everywhere on Ω.*

Proof. This is immediate from Theorems 5.12 and 5.13. $\square$

Lemma 5.15. . *If f is K-quasiconformal then*

$$\int_Q J_f dxdy \le \text{area}(f(Q)) \le \pi \text{diam}(f(Q))^2,$$

for every square Q.

Proof. We only use the quasiconformal hypothesis to deduce f is differentiable almost everywhere; the result holds for all such maps.

At any point x where f is differentiable we can choose a small square Q_x containing x such that

$$\text{area}(f(Q')) \geq (1-\epsilon) J_f(x) \text{area}(Q'),$$

and by the Lebesgue differentiation theorem, for almost every x we have

$$\int_{Q'} J_f dx dy \leq (1+\epsilon) J_f(x) \text{area}(Q'),$$

for all small enough squares centered at x.

Combining these two estimates and using the Vitali covering theorem to extract a collection of disjoint squares $\{Q_j\}$ with centers x_j and with these properties that cover almost every point of Q, we get

$$\begin{aligned}\int_Q J_f dxdy &\le \sum_j \int_{Q_j} J_f dxdy \\ &\le (1+\epsilon) J_f(x_j) \text{area}(Q_j) \\ &\le \frac{1+\epsilon}{1-\epsilon} \text{area}(f(Q_j)) \\ &\le \frac{1+\epsilon}{1-\epsilon} \text{area}(f(Q)).\end{aligned}$$

Taking $\epsilon \searrow 0$, gives $\text{area}(f(E)) \ge \int_E J_f dxdy$. The inequality area $\le \pi \text{diam}^2$ is obvious. □

Since $|f_z|^2 \le J_f/(1-k^2)$, we also get

Corollary 5.16. *If* f *is* K*-quasiconformal then*

$$\int_Q |f_z|^2 dxdy \le \frac{\pi}{1-k^2} \mathrm{diam}(f(Q))^2,$$

for every square Q*.*

Lemma 5.17. *If f is K-quasiconformal, then*

$$\frac{(\int_Q |f_z| dxdy)^2}{\text{area}(Q)} \gtrsim \text{diam}(f(Q))^2.$$

with a uniform constant for every square Q.

Proof. The path family connecting opposite sides of a square Q has modulus 1, so the image of this family in $f(Q)$ has modulus between K and $1/K$.

This implies the shortest path in $f(Q)$ connecting the same sides has length $\simeq \text{diam}(f(Q))$, so the integral of $|f_z|+|f_{\overline{z}}|$ along any horizontal segment crossing Q is at least $C\text{diam}(f(Q))$ for some fixed $C > 0$ (depending only on K).

Since $|f_z| \leq |f_z| + |f_{\overline{z}}| \leq 1(1+k)|f_z|$, the same is true for the integral of $|f_z|$.

Integrating over all horizontal segments crossing Q gives

$$\int_Q |f_z| dxdy \gtrsim \operatorname{diam}(Q)\operatorname{diam}(f(Q)).$$

Hence

$$\frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)} \gtrsim \frac{[\operatorname{diam}(Q)\operatorname{diam}(f(Q))]^2}{\operatorname{area}(Q)} \gtrsim \operatorname{diam}(f(Q))^2.$$

$\square$

Lemma 5.18. *If f is K-quasiconformal, then*

$$\int_Q |f_z|^2 dxdy \leq C \frac{(\int_Q |f_z| dxdy)^2}{\text{area}(Q)}$$

Proof. Note that for K-quasiconformal maps, $|\mu_f| \le k = (K-1)/(K+1)$ and

$$|f_z|(1-k^2) \le |f_z|^2(1-|\mu|^2) \le |f_z|^2 - |f_{\overline{z}}|^2 = J_f \le |f_z|^2,$$

so that J_f and $|f_z|^2$ are the same up to a bounded factor. Thus

$$\int_Q |f_z|^2 dxdy \le \int \lesssim \operatorname{diam}(f(Q))^2 \lesssim \frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)}$$

or

$$\int_Q |f_z|^2 dxdy \le C \frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)}$$

for some constant C that depends only on the quasiconformal constant of f (and not on the choice of the square Q). $\square$

Hölder's inequality implies

$$\int_Q |f_z| dxdy \leq \left(\int_Q |f_z|^2 dxdy\right)\left(\int_Q 1 dxdy\right) = \text{area}(Q) \cdot \left(\int_Q |f_z|^2 dxdy\right)$$

The inequality in the lemma goes in the opposite direction, and is called a reverse Hölder inequality. We shall see later that it has profound implications for the behavior of f_z.

Gehring's inequality and Bojarski's theorem

Hölder's inequality says that

$$\int fg d\mu \leq (\int f^p d\mu)^{1/p} (\int g^q d\mu)^{1/q},$$

where $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Applying this to a non-negative function on a square Q we get

$$(\frac{1}{\text{area}(Q)} \int_Q v^p dxdy) \geq (\frac{1}{\text{area}(Q)} \int_Q v dxdy)^p,$$

with equality if and only if v is a.e. constant.

Thus the “reverse Hölder inequality ”

$$(\frac{1}{\text{area}(Q)}\int_Q v^p dxdy) \le (K\frac{1}{\text{area}(Q)}\int_Q vdxdy)^p,$$

can only hold if $K \ge 1$.

If it holds for single Q, this does not say much, except that $v \in L^p \cap L^1$.

However, if it holds (with the same K) for all Q's we can deduce that $v \in L^{p+\epsilon}$ for some $\epsilon > 0$.

This remarkable “self-improvement” estimate is due to Gehring [], although the proof we give follows the presentation in Garnett's book *Bounded Analytic Functions*.

We start with a technical lemma.

Lemma 5.19. *Suppose that* $p > 1$, $v \geq 0$, $E_\lambda = \{z : v(z) > \lambda\}$, *and*

$$\int_{E_\lambda} v^p dxdy \leq A\lambda^{p-1} \int_{E_\lambda} vdxdy,$$

for all $\lambda \geq 1$. *Then there is* $r > p$ *and* $C < \infty$ *so that*

$$(\int_Q v^r dxdy)^{1/r} \leq C(\int_Q v^p dxdy)^{1/p}.$$

Proof. This is basically just arithmetic with distribution functions. Note that it suffices to assume $\text{area}(Q) = 1$ and $\int_Q v^p dxdy = 1$. Then

$$\begin{aligned}
\int_{E_1} v^r dxdy &= \int_{E_1} v^p v^{r-p} dxdy \\
&= (r-p) \int_{E_1} v^p (1 + \int_1^v \lambda^{r-p-1} d\lambda) dxdy \\
&= (r-p) \int_{E_1} v^p + (r-p) \int_1^\infty \lambda^{r-p-1} \int_{E_\lambda} v^p dxdyd\lambda
\end{aligned}$$

Hence,

$$
\begin{aligned}
\int_{E_1} v^r dxdy &\le (r-p)\int_{E_1} v^p + A(r-p)\int_1^\infty \lambda^{r-2}\int_{E_\lambda} v dxdyd\lambda \\
&\le (r-p)\int_{E_1} v^p + A(r-p)\int_{E_1} v(\int_0^v \lambda^{r-2} d\lambda)dxdy \\
&\le (r-p)\int_{E_1} v^p + A\frac{r-p}{r-1}\int_{E_1} v^r dxdy \\
&\le (r-p)\int_{E_1} v^p + \frac{1}{2}\int_{E_1} v^r dxdy
\end{aligned}
$$

where the last inequality holds if r is close enough to p (depending on A and p).

Subtracting the last term of the last step from the first step gives

$$\int_{E_1} v^r dxdy \le 2(r-p)\int_{E_1} v^p dxdy.$$

Off E_1 we have $v \le 1$ so $v^r \le v^p$ and hence

$$\int_Q v^r dxdy \le (1+2(r-p))\int_Q v^p dxdy.$$

Because of our normalizations, this proves the lemma. □

Theorem 5.20. *Let $p > 1$. If $v(x) \geq 0$ and $v \in L^p(Q, dxdy)$, and if the "reverse Hölder inequality"*

$$(\frac{1}{\text{area}(Q)} \int_Q v^p dxdy) \leq (K \frac{1}{\text{area}(Q)} \int_Q v dxdy)^p,$$

holds for all subsquares of a square Q_0, then there is an $r > p$ so that

$$(\frac{1}{\text{area}(Q_0)} \int_{Q_0} v^r dxdy)^{1/r} \leq (C(K, p, r) \frac{1}{\text{area}(Q_0)} \int_{Q_0} v dxdy),$$

Proof. We need only verify the hypothesis of Lemma 5.19. Fix λ and set $\beta = 2K\lambda$.

We will split the integral

$$\int_{E_\lambda} v^p dxdy = \int_{E_\lambda \setminus E_\beta} v^p dxdy + \int_{E_\beta} v^p dxdy$$

into two pieces.

The second piece is trivial to bound by the correct estimate because

$$\int_{E_\lambda \setminus E_\beta} v^p dxdy \leq \beta^{p-1} \int_{E_\lambda \setminus E_\beta} vdxdy \leq (2K\lambda)^{p-1} \int_{E_\lambda} vdxdy.$$

To bound the other piece of the integral, we use the Calderon-Zygmund lemma (Lemma 5.8) to find a sequence of disjoint squares $\{Q_j\}$ so that

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy < 2\beta^p,$$

and $v \leq \beta$ almost everywhere off $\cup Q_j$.

Thus $E_\beta \setminus \cup Q_j$ has measure zero and

$$\int_{E_\beta} v^p dxdy \leq \sum_j \int Q_j v^p dxdy \leq 2\beta^p \sum \text{area}(Q_j).$$

We now make use of the reverse Hölder hypothesis to write

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy \leq (\frac{K}{\text{area}(Q_j)} \int_{Q_j} vdx)^p,$$

hence

$$\begin{aligned} \text{area}(Q_j) &\leq \frac{K}{\beta} \int_{Q_j} vdxdy \\ &\leq \frac{K}{\beta} (\int_{Q_j \cap E_\lambda} vdxdy + \lambda \text{area}(Q_j)) \\ &\leq \frac{K}{\beta} \int_{Q_j \cap E_\lambda} vdxdy + \frac{1}{2} \text{area}(Q_j). \end{aligned}$$

Solving for $\text{area}(Q_j)$ gives

$$\begin{aligned}\text{area}(Q_j) &\leq \frac{2K}{\beta}\int_{Q_j} v\,dx\,dy \\ &\leq \frac{1}{\lambda}\int_{Q_j} v\,dx\,dy.\end{aligned}$$

Thus by the defining property of the Q_j's,

$$\begin{aligned}\int_{E_\beta} v^p dxdy &\le \sum_j \int_{Q_j} v^p dxdy \\ &\le 2\beta^p \sum_j \text{area}(Q_j) \\ &\le 2\beta^p \lambda^{-1} \sum_j \int_{Q_j \cap E_\lambda} v dx \\ &\le 2^{p+1} K^p \lambda^{p-1} \int_{E_\lambda} v dx.\end{aligned}$$

Thus the hypothesis of Lemma 5.19 holds with $A = (2K)^{p-1} + 2^{p+1}K^p$, and we deduce that $v \in L^r(Q, dxdy)$ for some $r > p$. $\square$

Theorem 5.21 (Bojarski's Theorem)**.** *If $1 \le K < \infty$, there is a $p > 2$ and $A, B < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $Q \subset \mathbb{C}$ is a square, then*

$$(\frac{1}{\text{area}(Q)} \iint_Q |f_z|^p dxdy)^{1/p} \le A(\frac{1}{\text{area}(Q)} \int_Q |f_z|^2 dxdy)^{1/2} \le B \frac{\text{diam}(f(Q))}{\text{diam}(Q)}$$

Proof. To apply Gehring's inequality to the partial derivatives of quasiconformal maps, we have to show that these partial satisfy a reverse Hölder inequality. What we want is

$$\int_Q |f_z|^2 dxdy \le \frac{C}{\text{area}(Q)} (\int_Q |f_z| dxdy)^2,$$

with a uniform C for all squares in the plane. This was proven in the previous section. $\square$

Lemma 5.22. *If f fixes $0, 1, \infty$, then*

$$\int_Q |L_f(x) - 1|^2 dxdy \leq \epsilon \text{area}(Q),$$

where $L_f = |f_z| + |f_{\overline{z}}|$ and $\epsilon \to 0$ as $\|\mu_f\|_\infty \to 0$.

Proof. Fix a square Q with sides parallel to the axes, let $\ell(Q)$ denote its side length and let S_1, S_2 denote the two vertical sides of S

By Cauchy-Schwarz

$$0 \leq (\frac{1}{\text{area}(Q)} \int_Q |L_f - 1| dxdy)^2 \leq \frac{1}{\text{area}(Q)} \int_Q |L_f - 1|^2 dxdy.$$

Now expand and rearrange

$$\begin{aligned}
&= \frac{1}{\text{area}(Q)} \int_Q (L_f^2 - 2L_f + 1) dxdy \\
&= \frac{1}{\text{area}(Q)} \int_Q (L_f^2 - 1 - 2L_f + 2) dxdy \\
&= \frac{1}{\text{area}(Q)} \int_Q (L_f^2 - 1) dxdy - \frac{2}{\text{area}(Q)} \int_Q (L_f - 1) dxdy
\end{aligned}$$

Now use $(Lf)^2 = (|f_z| + |f_{\overline{z}}|)^2 \leq K(|f_z| - |f_{\overline{z}}|)(|f_z| + |f_{\overline{z}}|) = KJ_f$ to get

$$
\begin{aligned}
&\leq \frac{1}{\text{area}(Q)} \int_Q (KJ_f - 1) - \frac{2}{\text{area}(Q)} \int_Q (L_f - 1) dxdy \\
&\leq \frac{1}{\text{area}(Q)} \int_Q (KJ_f - +J_f - J_f - 1) - \frac{2}{\text{area}(Q)} \int_Q (L_f - 1) dxdy \\
&= \frac{1}{\text{area}(Q)} \int_Q (K-1) J_f dxdy + \frac{1}{\text{area}(Q)} \int_Q (J_f - 1) dxdy \\
&\quad - \frac{2}{\text{area}(Q)} \int_Q (L_f - 1) dxdy
\end{aligned}
$$

We claim each terms tends to zero with $\|\mu\|_\infty$.

$$\frac{1}{\text{area}(Q)}\int_Q (K-1)J_f dxdy = O(\|\mu\|_\infty)\int J_f dxdy == O(\|\mu\|_\infty)\frac{1}{\text{area}(Q)}\int J_f dxdy ==$$

Since f tends to the identity on Q,

$$\frac{1}{\text{area}(Q)}\int_Q (J_f-1)dxdy = \frac{1}{\text{area}(Q)}\int_Q J_f dxdy - \frac{1}{\text{area}(Q)}\int_Q 1 dxdy \text{area}(f(Q)) - \text{area}(Q)$$

Finally, the integral of $|f_z| + |f_{\overline{z}}|$ over a horizonal segment in Q gives an upper bound for the length of the image curve, and this must be at least the distance between the two vertical sides of Q. Thus

$$\frac{2}{\mathrm{area}(Q)} \int_Q (L_f - 1) dxdy = 2\left(\frac{1}{\mathrm{area}(Q))} \int_Q L_f dxdy - 1)\right) \geq 2\left(\frac{\mathrm{dist}(S_1, S_2)}{\mathrm{area}(Q))} \int_Q L_f dxdy$$

since, as before, f tends uniformly to the identity on Q. Because of the negative sign in front of the third term in our sum of integrals, this proves the result. $\square$

Corollary 5.23. *If f fixes $0, 1, \infty$, then there is a $p > 2$, so that*
$$\int_Q |L_f(x) - 1|^p dxdy \to 0,$$
where $L_f = |f_z| + |f_{\overline{z}}|$ as $\|\mu_f\|_\infty \to 0$.

Proof. We know there is a $q = 2 + 2\epsilon > 2$ so that $L^f \in L^q(Q)$ with a bound depending only on p and Q. Taking $p = (q+2)/2 = 2 + \epsilon$, we can use Hölder's inequality to write
$$\|L_f - 1\|_p \leq \|L_f\|_2^{1+1/(1\epsilon)} \cdot \|L_f - 1\|_q^{2+\epsilon}.$$
The L^2 norm on the right tends to zero by Lemma **??** and the L^q is uniformly bounded by Bojarski's theorem, if q is close enough to 2. Thus the product tends to zero. □

This will be important later when we want to show the map $\mu \to f_\mu$ is continuous from the unit ball of L^∞ to Hölder continuous functions.

Corollary 5.24. *If Ω has a piecewise C^1 boundary and f is quasiconformal on Ω, then*

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_{\Omega}\frac{f_{\overline{z}}}{z-w}dxdy. \tag{5.18}$$

Proof. Smooth and take a limit using the L^p boundedness of the the Hardy-Littlewood maximal theorem and the Lebesgue dominated convergence theorem. $\square$

Corollary 5.25. *If f is quasiconformal, then f maps sets of zero area to zero area and*

$$\mathrm{area}(f(E)) = \int_E J_f dxdy.$$

Proof. Since $\nu(E) = \mathrm{area}(f(E))$ and $\nu(E) = \int_E J_f dxdy$ are both non-negative Borel measures, it suffices to show that they are equal for some convenient basis of sets, say squares with sides parallel to the coordinate axes. Let Q be such a square.

We have already proved the "$\geq$" direction in Lemma 5.15.

To prove the other direction, we use the fact that $J_f \in L^p(Q, dxdy)$ for some $p > 1$. Define a smoothed version f_n of f by convolving f with a smooth, non-negative bump function φ_n of total mass 1 and support in $D(0, \frac{1}{n})$.

Since f is continuous on $\mathbb{C}$, $f_n \to f$ uniformly on Q. Since convolution is linear, the partials of f_n are the partials of f convolved with φ_n and therefore the supremum over n of these partials is bounded by the Hardy-Littlewood maximal function of f_z, i.e.,

$$\sup_n |(f_n)_z(x)| \leq \mathcal{H}L(f_z)(x),$$

and similarly for $f_{\overline{z}}$.

Since the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p < \infty$, and $f_z, f_{\overline{z}} \in L^p$ for some $p > 1$, we see that $\{((f_n)_z)\}$, $\{((f_n)_{\overline{z}})\}$ are dominated by an L^p function and hence by an L^2 function on Q (since $L^p \subset L^2$ on bounded sets).

Thus the sequence of Jacobians $\{J_{f_n}\}$ is dominated by an L^1 function on Q, so by the Lebesgue dominated convergence theorem,

$$\int_Q J_{f_n} dxdy \to \int_Q J_f dxdy.$$

Moreover, since f_n is smooth

$$\int_Q J_{f_n} dxdy \geq \text{area}(f_n(Q)),$$

(equality may not hold since we don't known f_n is 1-to-1, and the integral computes area with multiplicity) and since $f_n \to f$ uniformly, $f_n(Q)$ eventually contains any compact subset of $f(Q)$ and hence

$$\limsup_n \text{area}(f_n(Q)) \geq \text{area}(f(Q)).$$

Thus $\text{area}(f(Q)) \leq \int_Q J_f dxdy$, as desired. $\square$

Later we will need the following result, which was proven as part of the preceeding argument.

Corollary 5.26. *Suppose f is K-quasiconformal. Then f can be approximated, uniformly on compact sets, by smooth K-quasiconformal maps $\{f_n\}$ whose dilatations $\{\mu_n\}$ converge pointwise to the dilatation μ of f, and such that for any measurable set E, $|f_n(E)| \to |f(E)|$.*

Lemma 5.27. *Suppose* $\{g_n\} \in L^p(R, dxdy)$ *for some* $p > 2$ *and*

$$\lim_n \iint_R \frac{g_n(z)}{z - w} dxdy = 0$$

for all $w \in R$*. Then* $\lim_n \iint_R g_n dxdy = 0$*.*

Proof. Fix rectangles $R'' \subset R' \subset R$, each compactly contained in the interior of the next.

Using the Cauchy integral formula for the constant function 1 on the curve $\partial R'$ we see that we can uniformly approximate the constant function 1 on R'' by a finite sum $s(z) = \sum \frac{a_k}{z - w_k}$ with $w_k \in \partial R'$ and $\sum |a_k|$ is uniformly bounded.

Then

$$\begin{aligned}\iint_R g_n(z)dxdy &= \iint_R g_n(z)s(z)dxdy + \iint_R g_n(z)(1-s(z))dxdy \\ &= o(1) + \iint_{R''} g_n(z)(1-s(z))dxdy + \iint_{R\setminus R''} g_n(z)(1-s(z))dxdy.\end{aligned}$$

For a fixed n, the first integral can be made as close to zero as we wish by taking s close to 1 on R''.

The second integral can be made small by taking area$(R \setminus R'') \to 0$; this implies the L^p norm of g_n on $R \setminus R''$ tends to zero (hence so does its L^1 norm) whereas the L^q norm of s remains uniformly bounded (it is a convex combination of L^q functions with bounded norm).

Thus we can make $\iint_R g_n dxdy$ as small a we wish if n is large, proving the lemma. $\square$

Lemma 5.28. *If $\{g_n\}$ are K-quasiconformal maps that converge uniformly on compact sets to a quasiconformal map g, then for any rectangle R.*

$$\iint_R [(g_n)_z - g_z] dxdy \to 0,$$

$$\iint_R [(g_n)_{\overline{z}} - g_{\overline{z}}] dxdy \to 0.$$

and $(g_n)_z \to g_z$ and $(g_n)_{\overline{z}} \to g_{\overline{z}}$ weakly.

Proof. First consider the $\overline{z}$-derivative. Let $h_n = (g_n)_{\overline{z}} - g_{\overline{z}}$.

By the Pompeiu formula and the fact that $g_n \to g$ uniformly on R, we deduce that

$$\lim_{n\to\infty} \iint_R \frac{h_n(z)}{z-w} dxdy = 0$$

for any $w \in R$. That $\iint_R h_n dxdy \to 0$, follows from Lemma **??**.

To prove weak conference, take any continuous f of compact support and uniformly approximate it to within ϵ by a function $\tilde{f}$ that is constant on finite union of rectangles. Then

$$\iint f h_n dxdy = \iint (f - \tilde{f}) h_n dxdy + \iint \tilde{f} h_n dxdy.$$

The first integral is bounded by $\epsilon \iint |h_n| dxdy$, which is small since $\|h_n\|_1 \leq C\|h_n\|_p$ is uniformly bounded on a large ball containing the support of both f and $\tilde{f}$.

The second integral tends to zero since is a finite linear combination of integrals of h_n over rectangles.

The result for z-derivatives follows from the same proof applied to the complex conjugates of g and $\{g_n\}$, using the fact that $(\bar{f})_{\bar{z}} = \overline{f_z}$. $\square$

Convergence of maps implies convergence of dilatations

Theorem 5.29. *Suppose $\{f_n\}$, f are all K-quasiconformal maps on the plane with dilatations $\{\mu_n\}$, μ_f respectively, that $f_n \to f$ uniformly on compact sets and that $\mu_n \to \mu$ pointwise almost everywhere. Then $\mu_f = \mu$ almost everywhere.*

Proof. We restrict attention to some domain Ω with compact closure. We know that $f_{\bar{z}} = \mu_f f_z$ almost everywhere and we know that f_z is non-zero almost everywhere, so it suffices to show that for almost every w,

$$f_{\bar{z}}(w) - \mu(w) f_z(w) = 0.$$

To prove this it suffices to show that the integral of $f_{\bar{z}}(w) - \mu(w) f_z(w)$ over any rectangle R is zero (this is an application of the Lebesgue differentiation theorem: at almost every point an integrable function is the limit of its averages over rectangles shrinking down to that point).

We re-write this function as

$$
\begin{aligned}
f_{\bar{z}}(w) - \mu(w) f_z(w) &= [f_{\bar{z}}(w) - (f_n)_{\bar{z}}(w)] \\
&\quad + [(f_n)_{\bar{z}}(w) - \mu_n (f_n)_z(w)] \\
&\quad + [\mu_n(w)(f_n)_z(w) - \mu(w)(f_n)_z(w)] \\
&\quad + [\mu(w)(f_n)_z(w) - \mu(w) f_z] \\
&= I + II + III + IV.
\end{aligned}
$$

Term II equals zero almost everywhere, so we need only show that the other three terms tend to zero as n tends to ∞.

Case I: This is Lemma 5.28.

Case III: We use Cauchy-Schwarz to show the integral of the third term is bounded by

$$\left(\iint_R (\mu-\mu_n)^2 dxdy\right)^{1/2}\left(\iint_R |(f_n)_x|^2 dxdy\right)^{1/2},$$

The first integrand tends to zero pointwise and is bounded above by 2 almost everywhere, so the integrals tend to zero by the Lebesgue dominated convergence theorem. On the other hand

$$\left(\iint_R |(f_n)_x|^2 dxdy\right)^{1/2} \simeq \operatorname{diam}(f_n(R)),$$

by Lemma 5.16, and since $\{f_n\}$ converges uniformly on compact sets, this remains bounded.

Thus the integral of III is bounded above by a term tending to zero times a term that is uniformly bounded, and hence it tends to zero.

Case IV: The same lemma as in Case I, but applied to $f_z = (\bar{f})_{\bar{z}}$, and using the fact that $(\bar{f})_{\bar{z}} = \overline{(f_z)}$, shows that for every rectangle R, we have

$$\iint_R (f_z - (f_n)_z)dxdy \to 0.$$

Now approximate μ in the $L^q(R, dxdy)$ norm by a function ν that is constant on a finite collection of disjoint squares (such functions are dense in L^q) and we deduce

$$\int_n \iint_R \mu((f_z-(f_n)_z)dxdy = \lim_n \iint_R (\mu-\nu)((f_z-(f_n)_z)dxdy \leq \lim_n \|\mu-_\nu\|_q \|(f_z-(f_n)_z)\|$$

The first term is as small as we wish and the second is uniformly bounded, so the product is as small as we wish. Thus the limit must be zero, as desired. $\square$

This completes the proof of the measurable Riemann mapping theorem in the general case.