Dimension of quasicircles

by

STANISLAV SMIRNOV

Université de Genève
Genève, Switzerland

We introduce canonical antisymmetric quasiconformal maps, which minimize the quasiconformality constant among maps sending the unit circle to a given quasicircle. As an application we prove Astala's conjecture that the Hausdorff dimension of a k-quasicircle is at most $1+k^2$.

1. Introduction

A homeomorphism ϕ of planar domains is called *k-quasiconformal*, if it belongs locally to the Sobolev class W_2^1 and its *Beltrami coefficient*

$$\mu_\phi(z) := \frac{\bar{\partial}\phi(z)}{\partial\phi(z)},$$

has bounded L_∞ norm: $\|\phi\| \leqslant k < 1$. Equivalently one can demand that for almost every point z in the domain of definition directional derivatives satisfy

$$\max_\alpha |\partial_\alpha f(z)| \leqslant K \min_\alpha |\partial_\alpha f(z)|,$$

where the constants of quasiconformality are related by

$$k = \frac{K-1}{K+1} \in [0, 1[\quad \text{and} \quad K = \frac{1+k}{1-k} \in [1, \infty[\,.$$

Quasiconformal maps change eccentricities of infinitesimal ellipses at most by a factor of K, and it is common to visualize a quasiconformal map ϕ by considering a measurable field $M(z)$ of infinitesimal ellipses, which is mapped by ϕ to the field of infinitesimal circles. More rigorously, $M(z)$ is an ellipse on the tangent bundle, centered at 0 and

defined up to homothety. Being preimage of a circle under the differential of ϕ at z, in the complex coordinate $v \in T_z \mathbb{C}$ it is given by the equation $|v + \bar{v}\mu_\phi(z)| = \text{constant}$, and so its eccentricity is equal to

$$\frac{\big|1 + |\mu_\phi(z)|\big|}{\big|1 - |\mu_\phi(z)|\big|} \leqslant K.$$

Quasiconformal maps constitute an important generalization of conformal maps, which one obtains for $K = 1$ (or $k = 0$). However, while conformal maps preserve the Hausdorff dimension, quasiconformal maps can alter it. Understanding this phenomenon was a major challenge until the work [1] of Astala, where he obtained sharp estimates for the area and dimension distortion in terms of the quasiconformality constants. In particular, the image of a set of Hausdorff dimension 1 under a k-quasiconformal map can have dimension at most $1 + k$, which can be attained for certain Cantor-type sets (like the Garnett–Ivanov square fractal).

Nevertheless this work left open the question of the dimensional distortion of the subsets of smooth curves. Astala's work [1] implies that a k-quasicircle (that is an image of a circle under a k-quasiconformal map) has the Hausdorff dimension at most $1 + k$, being an image of a 1-dimensional set. However, some years earlier, Becker and Pommerenke have shown in [2] that the Hausdorff dimension of a k-quasicircle cannot exceed $1 + 37k^2$. The Becker–Pommerenke estimate is better than Astala's for very small values of k because of its quadratic behavior, which they have shown to be sharp. Apparently, some properties of a circle were making it harder to increase its dimension by a quasiconformal map. Motivated by the two available estimates, Astala conjectured in [1] that the Hausdorff dimension of k-quasicircles cannot exceed $1 + k^2$.

In this paper we prove Astala's conjecture:

THEOREM 1. *The Hausdorff dimension of a k-quasicircle is at most $1 + k^2$.*

Remark 2. Our proof has some technical simplifications compared to Astala's since we are estimating only the dimension, and do not address the question of the Hausdorff measure.

Remark 3. It easily follows that a k-quasiconformal image of any smooth curve satisfies the same estimate.

The question about the sharpness remains open. In contrast to the general case, where sharpness of Astala's estimates follows relatively easily from the proof, here the question seems to be more difficult and have deep connections. Our methods rely on representing quasicircles by canonical antisymmetric Beltrami coefficients, and are related to difficult questions of quantifying the fine structure of harmonic measures and boundary dimensional distortion of conformal maps.

The strategy of our proof is similar to Astala's [1]: we employ holomorphic motions and thermodynamic formalism of [5] and [6]. The difference is that we establish a "canonical Beltrami representation" for quasicircles, which exhibits an interesting kind of symmetry. This allows us to embed the quasicircle into a symmetric holomorphic motion and use a better version of Harnack's inequality.

We formulate the theorem for a k-quasiline, that is an image of the real line $\mathbb{R}$ under a k-quasiconformal map. One obtains the quasicircle formulation by conjugating by a Möbius map which sends the real line to a circle and hence replaces the complex conjugation $\bar{z}$ by the inversion with respect to the circle in question.

THEOREM 4. (Canonical Beltrami representation) *For a curve Γ the following conditions are equivalent*:
  (i) Γ *is a k-quasiline*;
  (ii) $\Gamma = \psi(\mathbb{R})$ *with* $\|\mu_\psi\| \leqslant 2k/(1+k^2)$ *and* $\mu_\psi(z) = 0$, $z \in \mathbb{C}_+$;
  (iii) $\Gamma = \phi(\mathbb{R})$ *with* $\|\mu_\phi\| \leqslant k$ *and*

$$\mu_\phi(\bar{z}) = -\overline{\mu_\phi(z)}. \tag{1}$$

Remark 5. In the language of ellipses, condition (1) means that the ellipse $M(\bar{z})$ at z is a rotated by $\frac{1}{2}\pi$ complex conjugate of the ellipse $M(z)$.

Remark 6. If

$$\mu_\phi(\bar{z}) = \overline{\mu_\phi(z)}, \tag{2}$$

or equivalently the infinitesimal ellipse $M(z)$ at z is a complex conjugate $\overline{M(\bar{z})}$ of the ellipse at $\bar{z}$, then by symmetry the real line is mapped to the real line (though quasisymmetrically perturbed), so the Beltrami coefficient is "wasted". On the other hand, the theorem above shows that in the case of (1) the Beltrami coefficient is "optimal" among the ones representing the same quasicircle. Note that we represent every quasiconformal map as a superposition of two maps with better quasiconformality constants and satisfying (1) and (2), respectively.

2. Proofs

2.1. Proof of Theorem 4

It is more illustrative to work with ellipse fields rather than Beltrami coefficients. For a (measurable) ellipse field $M(z)$ we will denote by $\|M\|$ the essential supremum of

the eccentricities, which is related to the norm of the corresponding Beltrami coefficient μ by $\|M\|=(1+\|\mu\|)(1-\|\mu\|)$.

(i) $\Rightarrow$ (ii).   Let $N(z)$, $\|N\|\leqslant K:=(1+k)/(1-k)$, be the ellipse field representing a k-quasiconformal map η, which maps $\mathbb{R}$ onto Γ. We will "correct" the map θ by precomposing it with a carefully selected map which preserves the real line. We construct this new quasiconformal map α by means of an ellipse field A:

$$A(z)=\begin{cases} \overline{N(\bar{z})}, & \text{if } z\in\mathbb{C}_-, \\ N(z), & \text{if } z\in\mathbb{C}_+. \end{cases}$$

Then the map $\psi:=\eta\circ\alpha^{-1}$ is the desired one. Indeed, by Remark 6 the map α preserves the real line, so $\psi(\mathbb{R})=\eta(\alpha^{-1}(\mathbb{R}))=\eta(\mathbb{R})=\Gamma$. For z in the upper half-plane both η and α send the ellipse field $N(z)$ to the field of circles, hence the map $\psi=\eta\circ\alpha^{-1}$ preserves the field of circles and is conformal in the upper half-plane. In the lower half-plane both η and α change eccentricities by at most K, so ψ changes eccentricities by at most K^2. Thus $\|\mu_\psi\|\leqslant(K^2-1)/(K^2+1)=2k/(1+k^2)$.

(ii) $\Rightarrow$ (iii).   Let $M(z)$, $\|M\|\leqslant K'$, be the ellipse field representing the $2k/(1+k^2)$-quasiconformal map ψ, with

$$K'=\frac{1+2k/(1+k^2)}{1-2k/(1+k^2)}=\left(\frac{1+k}{1-k}\right)^2=K^2.$$

Define a new quasiconformal map β by means of a new ellipse field B:

$$B(z)=\begin{cases} \sqrt{M(z)}, & \text{if } z\in\mathbb{C}_-, \\ \sqrt{\overline{M(\bar{z})}}, & \text{if } z\in\mathbb{C}_+. \end{cases}$$

Here $\sqrt{M}$ denotes the ellipse with the same alignment whose eccentricity is the square root of M's eccentricity (note that we do not rotate the ellipses, just change their eccentricities). Then the map $\phi:=\psi\circ\beta^{-1}$ is the desired one. As before, by Remark 6 the map β preserves the real line, so $\phi(\mathbb{R})=\psi(\mathbb{R})=\Gamma$.

Let $L(z)$ be the image of the field $M(z)$ under the map β, or equivalently the preimage of the circle field under the map ϕ. For z in the lower half-plane, β sends the ellipse field $\sqrt{M(z)}$ to the field of circles, hence we conclude that there $\|L\|\leqslant K=\sqrt{K'}$. Using "symmetric" construction of β we conclude that for z in the upper half-plane, it sends $M(z)$, which is just a circle field there, to $iL(z)$, i.e. the rotation of $L(z)$ by $\frac{1}{2}\pi$. Rephrasing in terms of Beltrami coefficients, we arrive at (1).

(iii) $\Rightarrow$ (i).   This is trivial since ϕ is already k-quasiconformal.

2.2. The symmetric Harnack inequality

LEMMA 7. (Symmetric Harnack inequality) *Let h be a positive harmonic function in the unit disk $\mathbb{D}$, whose partial derivative at the origin vanishes in the direction of some $\lambda \in \mathbb{D}$: $\partial_\lambda h(0) = 0$. Then h satisfies the following stronger version of Harnack's inequality*:

$$\frac{1-|\lambda|^2}{1+|\lambda|^2} h(\lambda) \leqslant h(0) \leqslant \frac{1+|\lambda|^2}{1-|\lambda|^2} h(\lambda).$$

Remark 8. Unlike the usual Harnack's inequality, the one-sided estimate is not valid for subharmonic functions. Therefore the combination of Becker–Pommerenke's and Astala's results does not automatically give the $1+k^2$ estimate.

Proof. Without loss of generality, we may assume that $h(0)=1$ and the gradient of h vanishes at the origin. Otherwise we consider the function $(h(z)+h(z^*))/2h(0)$, where $*$ denotes the symmetry with respect to the line through the origin and λ.

Then the function
$$f := \frac{z-1}{z+1} \circ (h+i\tilde{h})$$

is analytic from the unit disk $\mathbb{D}$ to itself (here $\tilde{h}$ denotes the harmonic conjugate). Moreover, f preserves the origin and $f'(0)=0$, since $\nabla h(0)=0$. Thus we can apply the Schwarz lemma to the function $f(z)/z$, concluding that $|f(\lambda)| \leqslant |\lambda|^2$. This implies that

$$h(\lambda) \leqslant \frac{1+|\lambda|^2}{1-|\lambda|^2} \quad \text{and} \quad h(\lambda) \geqslant \frac{1-|\lambda|^2}{1+|\lambda|^2}. \qquad \square$$

2.3. Proof of Theorem 1

Take a k-quasiline Γ and its representation by a k-quasiconformal map ϕ with Beltrami coefficient μ satisfying (1). Include ϕ into the standard holomorphic motion of maps ϕ_λ, $\lambda \in \mathbb{D}$, whose Beltrami coefficients are given by $\mu_\lambda := \mu\lambda/k$ and which preserve the points 0, 1 and ∞. As usual, $\phi_0 = \mathrm{id}$ and $\phi_k = \phi$. Because of the particular symmetric behavior (1) of μ_k, we conclude that μ_λ satisfies (1) for real λ and (2) for imaginary λ. There is additional symmetry: for real values of λ one has $\phi_\lambda(z) = \overline{\phi_{-\lambda}(\bar{z})}$, which follows from the same property for μ_λ.

We proceed as in [1] with some simplifications in estimating the distortion.

To this effect, fix a radius $\varrho \in \left(\frac{1}{2}, 1\right)$, and consider for the time being only λ inside the slightly smaller disk $\varrho\mathbb{D}$, so the maps ϕ_λ have uniform quasiconformality constants. Note

that by the results of Gehring (see e.g. [7, Theorem 18.1]) the maps ϕ_λ are uniformly quasisymmetric, so there is a constant $C = C(\varrho)$ such that

$$|z-x| \leqslant |y-x| \quad \Longrightarrow \quad |\phi_\lambda(z) - \phi_\lambda(x)| \leqslant C|\phi_\lambda(y) - \phi_\lambda(x)|, \tag{3}$$

$$C|z-x| \leqslant |y-x| \quad \Longrightarrow \quad 2|\phi_\lambda(z) - \phi_\lambda(x)| \leqslant |\phi_\lambda(y) - \phi_\lambda(x)|. \tag{4}$$

It is enough to estimate the dimension of the image of the interval $[0,1]$. Cover it by n intervals $[a_j, b_j]$ of length $1/n$, and denote by $B_j(\lambda)$ the ball centered at $\phi_\lambda(a_j)$ whose boundary circle passes through $\phi_\lambda(b_j)$. Note that its "complex radius"

$$r_j(\lambda) := \phi_\lambda(b_j) - \phi_\lambda(a_j)$$

is a holomorphic function of λ.

The image of the interval $[0,1]$ is covered by the images of small intervals, each having diameter at most $C|r_j(\lambda)|$ by (3). So, to estimate the Hausdorff dimension, we have to estimate the sums

$$\sum_{j=1}^{n} \operatorname{diam}(\phi_\lambda[a_j, b_j])^p \leqslant C^p \sum_{j=1}^{n} |r_j(\lambda)|^p. \tag{5}$$

We will estimate the logarithm of the right-hand side. The logarithm is a concave function, so applying the Jensen inequality to the collection of points $\{|r_j(\lambda)|^p/\nu_j\}_{j=1}^n$ with weights $\{\nu_j\}_{j=1}^n$ summing to one, we obtain that

$$\log \sum_{j=1}^{n} |r_j(\lambda)|^p = \log \sum_{j=1}^{n} \nu_j \frac{|r_j(\lambda)|^p}{\nu_j} \geqslant \sum_{j=1}^{n} \nu_j \log \frac{|r_j(\lambda)|^p}{\nu_j} = I_\nu - p\Lambda_\nu(\lambda), \tag{6}$$

where $I_\nu := -\sum_{j=1}^{n} \nu_j \log \nu_j$ is the "entropy" and

$$\Lambda_\nu(\lambda) := -\sum_{j=1}^{n} \nu_j \log |r_j(\lambda)|$$

is the "Lyapunov exponent" of the probability distribution $\{\nu_j\}_{j=1}^n$. Note that the Lyapunov exponent is a harmonic function of λ, since $r_j(\lambda)$ are holomorphic.

The equality in (6) is attained if ν_j are chosen proportional to $|r_j(\lambda)|^p$, and we deduce the *variational principle* from Astala's work [1]:

$$\log \sum_{j=1}^{n} |r_j(\lambda)|^p = \sup_\nu (I_\nu - p\Lambda_\nu(\lambda)), \tag{7}$$

where the supremum is taken over all probability distributions ν.

Fix some distribution ν and consider the function

$$H(\lambda):=2\Lambda_\nu(\lambda)-I_\nu+3\log C.$$

It is harmonic in λ (since Λ_ν is) and is an even function on the real line (because of the symmetry of our motion $r_j(\lambda)=\overline{r_j(-\lambda)}$ for $\lambda\in\mathbb{R}$). By (4), the balls $B_j(\lambda)$ cover every point at most C times, while, by (3), their union is contained in a ball of radius C. Hence we can write

$$\sum_{j=1}^n |r_j(\lambda)|^2 \leqslant C^3,$$

deducing by the variational principle (7) that

$$I_\nu - 2\Lambda_\nu(\lambda) \leqslant \log C^3,$$

and concluding that H is non-negative in the disk $\varrho\mathbb{D}$. Finally,

$$I_\nu - \Lambda_\nu(0) \leqslant \log \sum_{j=1}^n |r_j(0)| = \log 1 = 0,$$

and therefore

$$H(0) = 2\Lambda_\nu(0) - I_\nu + 3\log C \geqslant I_\nu + 3\log C.$$

We can now apply Lemma 7 (in the smaller disk $\varrho\mathbb{D}$) to obtain that

$$2\Lambda_\nu(k) - I_\nu + 3\log C = H(k) \geqslant \frac{1-k^2/\varrho^2}{1+k^2/\varrho^2} H(0) \geqslant \frac{1-k^2/\varrho^2}{1+k^2/\varrho^2}(I_\nu + 3\log C),$$

and deduce that

$$\frac{2}{1+k^2/\varrho^2} I_\nu - 2\Lambda_\nu(k) \leqslant \frac{2k^2/\varrho^2}{1+k^2/\varrho^2} 3\log C,$$

which can be rewritten as

$$I_\nu - \left(1+\frac{k^2}{\varrho^2}\right)\Lambda_\nu(k) \leqslant 3\frac{k^2}{\varrho^2}\log C.$$

Since the latter holds for all distributions ν, we conclude by the variational principle (7) that

$$\log \sum_{j=1}^n |r_j(k)|^p = \sup_\nu (I_\nu - p\Lambda_\nu(k)) \leqslant 3\frac{k^2}{\varrho^2}\log C \leqslant 12\log C, \tag{8}$$

where we set $p:=1+k^2/\varrho^2$.

Letting n tend to infinity, we conclude by (5) and (8) that the p-dimensional Hausdorff measure of the quasiline $\phi[0,1]$ is bounded by C^{14}, and hence its dimension is at most $p=1+k^2/\varrho^2$. Letting now ϱ tend to 1, we obtain the desired estimate.

3. Quasisymmetric distortion

For images of subsets of the circle, the methods above cannot be applied to the full extent (since symmetrizing Beltrami preserves the image of the circle but not of particular subsets).

But if the circle is preserved (e.g. we are dealing with maps which restrict to quasisymmetric maps of the unit circle), we can work with the same symmetrized holomorphic motions, only we are exploring a direction orthogonal to the previous one. We say that a mapping of the unit circle is k-*quasisymmetric* if it can be extended to a k-quasiconformal map of the complex plane. Note that this quasisymmetry constant is different from the usual ϱ.

In this case, if we start with a subset of the circle of dimension 1, its dimension can never be bigger than 1 when λ takes real values, so pressure-type quantities have zero gradient, and the same symmetric Harnack inequality gives that the k-quasisymmetric image has dimension at least $1-k^2$ (which is the "dual" statement to the one that in the orthogonal direction we get at most $1+k^2$). This and related results are discussed by Prause in [3].

If we start with a subset of the circle of dimension $a<1$, we can use that its dimension will be at most 1 for real λ, and improve Harnack's inequality using this global information. The corresponding estimates are more complicated, and are the subject of a separate paper [4].

References

[1] ASTALA, K., Area distortion of quasiconformal mappings. *Acta Math.*, 173 (1994), 37–60.
[2] BECKER, J. & POMMERENKE, C., On the Hausdorff dimension of quasicircles. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 12 (1987), 329–333.
[3] PRAUSE, I., A remark on quasiconformal dimension distortion on the line. *Ann. Acad. Sci. Fenn. Math.*, 32 (2007), 341–352.
[4] PRAUSE, I. & SMIRNOV, S., Quasisymmetric distortion spectrum. Preprint, 2009. `arXiv:0910.4723v1 [math.CV]`.
[5] RUELLE, D., *Thermodynamic Formalism.* Encyclopedia of Mathematics and its Applications, 5. Addison-Wesley, Reading, MA, 1978.
[6] — Repellers for real analytic maps. *Ergodic Theory Dynamical Systems*, 2 (1982), 99–107.
[7] VÄISÄLÄ, J., *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Mathematics, 229. Springer, Berlin–Heidelberg, 1971.

Stanislav Smirnov
Section de Mathématiques
Université de Genève
2-4 rue du Lièvre, Case postale 64
CH-1211 Genève 4
Switzerland
stanislav.smirnov@unige.ch