Quasiconformal Mappings

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Contents

Preface

The purpose of these notes is to introduce readers to the basic results about quasiconformal maps of planar domains and their application to various problems of conformal dynamics. They are not exhaustive in any sense, and are not nearly as complete as other sources such as [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]. Rather, they are intended to cover the "bare bones" needed to apply modulus and quasiconformal maps in certain situations arising in conformal. It is my assumption that once the reader has understood the basics, then they can seek out more advanced results elsewhere as the need arises.

We start with a discussion of the basic conformal invariants: extremal length, hyperbolic distance and harmonic measure and how they relate to one another. We then define quasiconformal maps using the geometric definition, i.e., as maps that quasi-preserve extremal length, and deduce the all important compactness properties of K-quasiconformal maps. From this we can deduce a weak version of the measurable Riemann mapping theorem (basically for continuous dilatations) which is still sufficient for many interesting applications in dynamics.

Next we turn to the analytic properties of quasiconformal mappings, eventually leading to the full strength version of the measurable Riemann mapping theorem. Our approach here is a little non-standard, but since there are excellent treatments of this theorem elsewhere, it seemed worthwhile to experiment with something slightly different.

CHAPTER 1

Conformal maps and conformal invariants

Quasiconformal maps are generalizations of conformal maps and a fundamental tool for understanding them are conformal invariants, i.e., numerical values that can be associated to a certain geometric configurations and that remain unchanged (or at least change in predictable ways) under the application of conformal or holomorphic maps. There are three conformal invariants that will be particularly important throughout the notes: extremal length, harmonic measure and hyperbolic distance. Of these, extremal length is the most important because it can be defined in many situations and estimated by direct geometric arguments. The other two are defined on the disk and then transferred to other domains by a conformal map. In this chapter, we introduce extremal length, hyperbolic distance and harmonic measure, and derive a famous estimate for the latter, due to Arne Beurling, using the former.

1. Extremal length

Our first conformal invariant is extremal length. Consider a positive function ρ on a domain Ω. We think of ρ as analogous to $|f'|$ where f is a conformal map on Ω. Just as the image area of a set E can be computed by integrating $\int_E |f'|^2 dxdy$, we can use ρ to define areas by $\int_E \rho^2 dxdy$. Similarly, just as we can define $\ell(f(\gamma)) = \int_\gamma |f'(z)| ds$, we can define the ρ-length of a curve γ by $\int_\gamma \rho ds$. For this to make sense, we need γ to be locally rectifiable (so the arclength measure ds is defined) and it is convenient to assume that ρ is Borel (so that its restriction to any curve γ is also Borel and hence measurable for length measure on γ).

Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω. We say ρ is **admissible** for Γ if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1.$$

In this case we write $\rho \in \mathscr{A}(\Gamma)$. We define the **modulus** of the path family Γ as

$$\mathrm{Mod}(\Gamma) = \inf_\rho \int_M \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ. The **extremal length** of Γ is defined as

$$\lambda(\Gamma) = 1/M(\Gamma).$$

Note that if the path family Γ is contained in a domain Ω, then we need only consider metrics ρ are zero outside Ω. Otherwise, we can define a new (smaller) metric by setting $\rho = 0$ outside Ω; the new metric is still admissible, and a smaller integral than before. Therefore $M(\Gamma)$ can be computed as the infimum over metrics which are only nonzero inside Ω.

Modulus and extremal length satisfy several useful properties that we list as a series of lemmas.

LEMMA 1.1 (Conformal invariance). *If Γ is a family of curves in a domain Ω and f is a one-to-one holomorphic mapping from Ω to Ω' then $M(\Gamma) = M(f(\Gamma))$.*

PROOF. This is just the change of variables formulas

$$\int_\gamma \rho \circ f |f'| ds = \int_{f(\gamma)} \rho ds,$$

$$\int_\Omega (\rho \circ f)^2 |f'|^2 dxdy = \int_{f(\Omega)} \rho dxdy.$$

These imply that if $\rho \in \mathscr{A}(f(\Gamma))$ then $|f'| \cdot \rho \circ f \in \mathscr{A}(f(\Gamma))$, and thus by taking the infimum over such metrics we get $M(f(\Gamma)) \leq M(\Gamma)$ Note that there might be admissible metrics for $f(\Gamma)$ that are not of this form, possibly giving a strictly small modulus. However, by switching the roles of Ω and Ω' and replacing f by f^{-1} we see equality does indeed hold. □

LEMMA 1.2 (Monotonicity). *If Γ_0 and Γ_1 are path families such that every $\gamma \in \Gamma_0$ contains some curve in Γ_1 then $M(\Gamma_0) \leq M(\Gamma_1)$ and $\lambda(\Gamma_0) \geq \lambda(\Gamma_1)$.*

PROOF. The proof is immediate since $\mathscr{A}(\Gamma_0) \supset \mathscr{A}(\Gamma_1)$. □

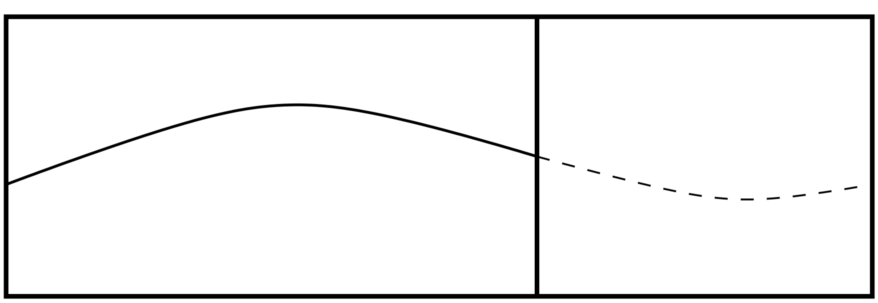


FIGURE 1.1. The Monotone rule: each curve of the first family contains a curve of the second family.

LEMMA 1.3 (Grötsch Principle). *If Γ_0 and Γ_1 are families of curves in disjoint domains then $M(\Gamma_0 \cup \Gamma_1) = M(\Gamma_0) + M(\Gamma_1)$.*

PROOF. Suppose ρ_0 and ρ_1 are admissible for Γ_0 and Γ_1. Take $\rho = \rho_0$ and $\rho = \rho_1$ in their respective domains. Then it is easy to check that ρ is admissible for $\Gamma_0 \cup \Gamma_1$ and, since the domains are disjoint, $\int \rho^2 = \int \rho_1^2 + \int \rho_2^2$. Thus $M(\Gamma_0 \cup \Gamma_1) \le M(\Gamma_0) + M(\Gamma_1)$. By restricting an admissible metric ρ to each domain, a similar argument proves the other direction. $\square$

The Grötsch principle and the monotonicity combine to give

COROLLARY 1.4 (Parallel Rule). *Suppose Γ_0 and Γ_1 are path families in disjoint domains $\Omega_0, \Omega_1 \subset \Omega$ that connect disjoint sets E, F in $\partial\Omega$. If Γ is the path family connecting E and F in Ω, then*

$$M(\Gamma) \ge M(\Gamma_0) + M(\Gamma_1).$$

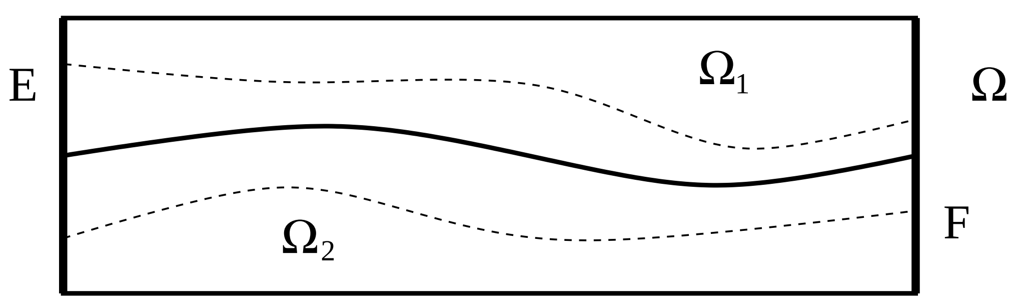


FIGURE 1.2. The Parallel Rule: curves connecting two boundary sets in the whole domain and in two disjoint subdomains.

LEMMA 1.5 (Series Rule). *If Γ_0 and Γ_1 are families of curves in disjoint domains and every curve of $\mathscr{F}$ contains both a curve from both Γ_0 and Γ_1, then $\lambda(\Gamma) \ge \lambda(\Gamma_0) + \lambda(\Gamma_1)$.*

PROOF. If $\rho_j \in \mathscr{A}(\Gamma_j)$ for $j = 0, 1$, then $\rho_t = (1-t)\rho_0 + t\rho_1$ is admissible for Γ. Since the domains are disjoint we may assume $\rho_0\rho_1 = 0$. Integrating ρ^2 then shows

$$M(\Gamma) \le (1-t)^2 M(\Gamma_0) + t^2 M(\Gamma_1),$$

for each t. To find the optimal t set $a = M(\Gamma_1)$, $b = M(\Gamma_0)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

$$\begin{aligned} M(\mathscr{F}) &\leq t^2 a + (1-t^2)b = \frac{b^2 a a^2 b}{(a+b)^2} \\ &= \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a}+\frac{1}{b}} \end{aligned}$$

or

$$\frac{1}{M(\Gamma)} \geq \frac{1}{M(\Gamma_0)} + \frac{1}{M(\Gamma_1)},$$

which, by definition, is the same as

$$\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1). \quad \square$$

Next we actually compute the modulus of some path families. The fundamental example is to compute the modulus of the path family connecting opposite sides of a $a \times b$ rectangle; this serves as the model of almost all modulus estimates. So suppose $R = [0,b] \times [0,a]$ is a b wide and a high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a.

LEMMA 1.6. $\mathrm{Mod}(\Gamma) = a/b$.

PROOF. Then each such curve has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho \, ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dx dy = \frac{1}{b^2} ab = \frac{a}{b}.$$

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$\left(\int fg dx\right)^2 \leq \left(\int f^2 dx\right)\left(\int g^2 dx\right).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible,

$$\int_0^b \rho(x,y) dx \geq 1,$$

and so by Cauchy-Schwarz

$$1 \leq \int_0^b (1 \cdot \rho(x,y)) dx \leq \int_0^b 1^2 dx \cdot \int_0^b \rho^2(x,y) dx.$$

Now integrate with respect to y to get

$$a = \int_0^a 1dy \leq b \int_0^a \int_0^b \rho^2(x,y)dxdy,$$

or

$$\frac{a}{b} \leq \iint_R \rho^2 dxdy,$$

which implies $\mathrm{Mod}(\Gamma) \geq \frac{b}{a}$. Thus $\mathrm{Mod}(\Gamma) = \frac{b}{a}$. □

Another useful computation is the modulus of the family of path connecting the inner and out boundaries of the annulus $A = \{z : r < |z| < R\}$.

LEMMA 1.7. *If $A = \{z : r < |z| < R\}$ then the modulus of the path family connecting the two boundary components is $2\pi/\log\frac{R}{r}$. More generally, if Γ is the family of paths connecting $r\mathbb{T}$ to a set $E \subset R\mathbb{T}$, then $M(\Gamma) \geq |E|/\log\frac{R}{r}$.*

PROOF. By conformal invariance, we can rescale and assume $r = 1$. Suppose ρ is admissible for Γ. Then for each $z \in E \subset \mathbb{T}$,

$$1 \leq (\int_1^R \rho ds)^2 \leq (\int_1^R \frac{ds}{s})(\int_1^R \rho^2 sds) = \log R \int_1^R \rho^2 sds$$

and hence we get

$$\int_0^{2\pi}\int_1^R \rho^2 sdsd\theta \geq \int_E \int_1^R \rho^2 sdsd\theta \geq |E| \int_1^R \rho^2 sds \geq \frac{|E|}{\log R}.$$

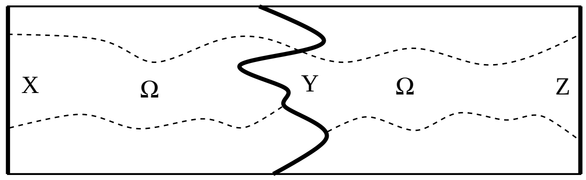
When $E = \mathbb{T}$ we prove the other direction by taking $\rho = (s\log R)^{-1}$. This is an admissible metric and

$$\mathrm{Mod}(\Gamma) \leq \int_0^{2\pi}\int_1^R \rho^2 sdsd\theta = \frac{2\pi}{(\log R)^2}\int_1^R \frac{1}{s}ds = \frac{2\pi}{\log R}. \quad \square$$

Given a Jordan domain Ω and two disjoint closed sets $E, F \subset \partial\Omega$, the **extremal distance** between E and F (in Ω) is the extremal length of the path family in Ω connecting E to F (paths in Ω that have one endpoint in E and one endpoint in F). The series rule is a sort of "reverse triangle inequality" for extremal distance. See Figure 1.3.

Extremal distance can be particularly useful when both E and F are connected. In this case, their complement in $\partial\Omega$ also consists of two arcs, and the extremal distance between these is the reciprocal of the extremal distance between E and F. This holds because of conformal invariance, the fact that it is true for rectangles and an applications of the Riemann mapping theorem (we can always map Ω to a rectangle, so that E and F go to opposite sides (See Exercise 1.1).

Obtaining an upper bound for the modulus of a path family usually involves choosing a metric; every metric gives an upper bound. Giving a



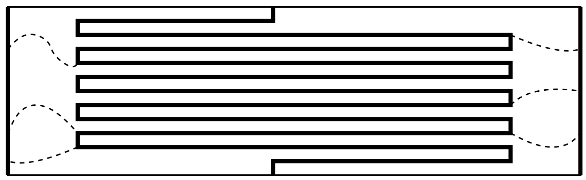


FIGURE 1.3. The series rule says that the extremal distance from X to Z in the rectangle is greater than the sum the extremal distance from X to Y in Ω_1 plus the extremal distance from Y to Z in Ω_2. The bottom figure show a more extreme case where the extremal distance between opposite sides of the rectangle is much larger than either of the other two terms.

lower bound usually involves a Cauchy-Schwarz type argument, which can be harder to do in general cases. However, in the special case of extremal distance between arcs $E, F \subset \partial\Omega$, a lower bound for the modulus can also be computed by giving a upper bound for the reciprocal separating family. Thus estimates of both types can be given by producing metrics (for different families) and this is often the easiest thing to do.

LEMMA 1.8 (Points are removable). *Suppose Q is a quadrilateral with opposite sides E, F znd that Γ is the path family in Q connecting E and F. If $z \in \Omega$, let $\Gamma_0 \subset \Gamma$ be the paths that do not contain z. Then* $\mod(\Gamma_0) = \mod(\Gamma)$.

PROOF. Since $\Gamma_0 \subset \Gamma$ we have $\mod(\Gamma_0) \leq \mod(\Gamma)$ by monotonicity, to prove the other direction we claim that any metric that is admissible for Γ_0 is also admissible for Γ.

Suppose ρ is not admissible for Γ. Then there is a $\gamma \in \Gamma$ so that $\int_\gamma \rho ds < 1 - \varepsilon$. Choose a small $r > 0$ so $D(z, r) \subset \Omega$ and note that by Cauchy-Schwarz

$$(\int_0^r [\int_0^{2\pi} \rho t d\theta] dt)^2 \leq \pi r^2 \int_{D(z,r)} \rho^2 dxdy = o(r^2).$$

Here we have used the fact that since ρ^2 is integrable on Q, we have $\int_{D(z,r)} \rho^2 dxdy \to 0$ as $r \searrow 0$ (see [?]). Hence

$$\int_0^r [\int_{C_t} \rho ds] dt = \int_0^r \ell_\rho(C_t) dt = o(r),$$

where C_t is the circle of radius t around z. Thus we can find arbitarily small circles centered at z whose ρ-length is less than ε. Then for the path γ chosen above, replace it by a path that follows γ from E to the first time it hits C_t, then follows an arc of C_t, and then follows γ from the last time it hits C_t to to F. This path is in Γ_0 but its ρ-length is at most the ρ-length of γ plus the ρ-length of C_t, and this sum is less than 1. Thus ρ is also not admissible for Γ_0. This proves the claim and the lemma. $\square$

The previous result will be useful in later chapters when we want to prove that quasiconformal map of a punctured disk is actually quasiconformal on the whole disk.

If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let

$$\gamma_u = \gamma \cap \mathbb{H}_u, \quad \gamma_\ell = \gamma \cap \mathbb{H}_l, \quad \gamma_+ = \gamma_u \cup \overline{\gamma_\ell},$$

where $\mathbb{H}_u = \{x+iy : y > 0\}$, $\mathbb{H}_l = \{x+iy : y < 0\}$ denote the upper and lower half-planes. For a path family Γ, define $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma_+ = \{\gamma_+ : \gamma \in \Gamma\}$.

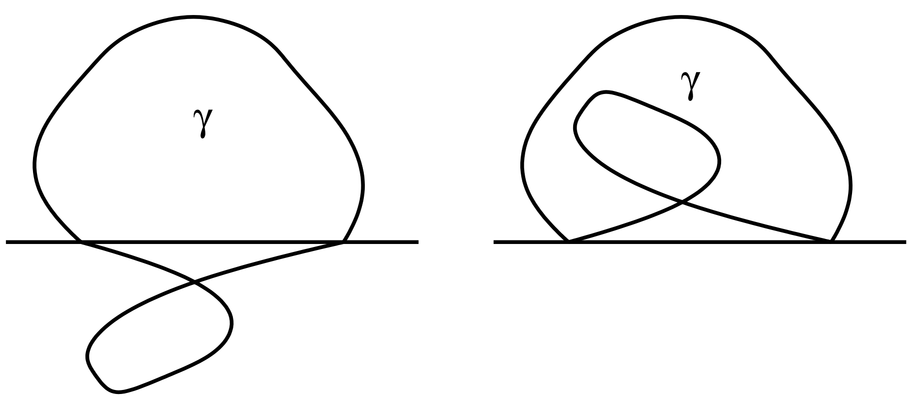


FIGURE 1.4. The curves γ and γ_+

LEMMA 1.9 (Symmetry Rule). *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma_+)$.

PROOF. We start by proving $M(\Gamma) \leq 2M(\Gamma_+)$. Given a metric ρ admissible for γ_+, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$. Then for any $\gamma \in \Gamma$,

$$\begin{aligned}
\int_\gamma \sigma ds &= \int_{\gamma_u} \sigma(z) ds + \int_{\gamma_\ell} \sigma(z) ds \\
&\geq \int_{\gamma_u} \rho(z) ds + \int_{\gamma_\ell} \rho(\bar{z}) ds \\
&= \int_{\gamma_u} \rho(z) ds + \int_{\overline{\gamma_\ell}} \rho(z) ds \\
&\geq \int_{\gamma_+} \rho ds \\
&\geq \inf_{\gamma \in \Gamma} \int_\gamma \rho ds.
\end{aligned}$$

Thus if ρ admissible for Γ_+, then σ is admissible for Γ. Since $\max(a,b)^2 \leq a^2 + b^2$, integrating gives

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z) dxdy + \int \rho^2(\bar{z}) dxdy \leq 2 \int \rho^2(z) dxdy.$$

Taking the infimum over admissible ρ's for Γ_+ makes the right hand side equal to $2M(\Gamma_+)$, proving $\text{Mod}(\Gamma) \leq 2\text{Mod}(\Gamma_+)$.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}_u$ and $\sigma = 0$ if $z \in \mathbb{H}_l$. Then

$$\begin{aligned}
\int_{\gamma_+} \sigma ds &= \int_{\gamma_+} \rho(z) + \rho(\bar{z}) ds \\
&= \int_{\gamma_u} \rho(z) ds + \int_{\gamma_u} \rho(\bar{z}) ds + \int_{\gamma_e ll} \rho(z) + \int_{\gamma_\ell} \rho(\bar{z}) ds \\
&= \int_\gamma \rho(z) ds + \int_{\overline{\gamma}} \rho(\bar{z}) ds \\
&= 2 \inf_\rho \int_\gamma \rho ds.
\end{aligned}$$

Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ_+. Since $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\begin{aligned} M(\Gamma_+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy \\ &= \frac{1}{4}\int_{\mathbb{H}_u} (\rho(z)+\rho(\bar{z}))^2 dxdy \\ &\leq \frac{1}{2}\int_{\mathbb{H}_u} \rho^2(z)dxdy + \int_{\mathbb{H}_u} \rho^2(\bar{z})dxdy \\ &= \frac{1}{2}\int \rho^2 dxdy. \end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. $\square$

LEMMA 1.10. *Let $\mathbb{D}^* = \{z : |z| > 1\}$ and $\Omega_0 = \mathbb{D}^* \setminus [R,\infty)$ for some $R > 1$. Let $\Omega = \mathbb{D}^* \setminus K$, where K is a closed, unbounded, connected set in $\mathbb{D}^*$ which contains the point $\{R\}$. Let Γ_0, Γ denote the path families in these domains with separate the two boundary components. Then $M(\Gamma_0) \leq M(\Gamma)$.*

PROOF. We use the symmetry principle we just proved. The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma_0^+) = \frac{1}{2}M(\Gamma_0)$. The family Γ may not be symmetric, but we can replace it by a larger family that is. Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0. Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so $M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$. Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$.

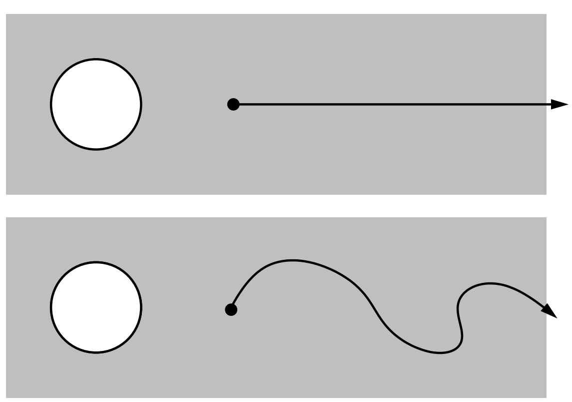


FIGURE 1.5. The topological annulus on top has smaller modulus than any other annulus formed by connecting R to ∞.

Suppose $\gamma \in \Gamma_R$. Since γ has non-zero winding around 0 it must cross both the negative and positive real axes. If it never crossed $(0,R)$ then the winding around 0 and R would be the same, which false, so γ must cross$(0,R)$ as well. Choose points $z_- \in \gamma \cap (-\infty, 0)$ and $z_+ \in \gamma \cap (0,R)$. These points divide γ into two subarcs γ_1 and γ_2. Then $\gamma_+ = (\gamma_1)_+ \cup (\gamma_2)_+$. But if we reflect $(\gamma_2)_+$ into the lower half-plane and join it to $(\gamma_1)_+$ it forms a closed curve γ_0 that is in Γ_0 and $(\gamma_0)_+ = \gamma_+$. Thus $\gamma_+ \in (\Gamma_0)_+$, as desired. □

Let $\Omega_{\varepsilon,R} = \{z : |z| > \varepsilon\} \setminus [R, \infty)$. Note that $\Omega_{1,R}$ is the domain considered in the previous lemma (e.g., see the top of Figure 1.5). We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$

which conformal maps the unit disk to $\mathbb{R}^2 \setminus [\frac{1}{4}, \infty)$ and satisfies $k(0) = 0$, $k'(0) = 1$. Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\varepsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius $\frac{\varepsilon}{4R}(1 \pm O(\frac{\varepsilon}{R}))$. Thus the modulus of $\Omega_{\varepsilon,R}$ is

$$2\pi \log \frac{4R}{\varepsilon} + O(\frac{\varepsilon}{R}). \tag{1.1}$$

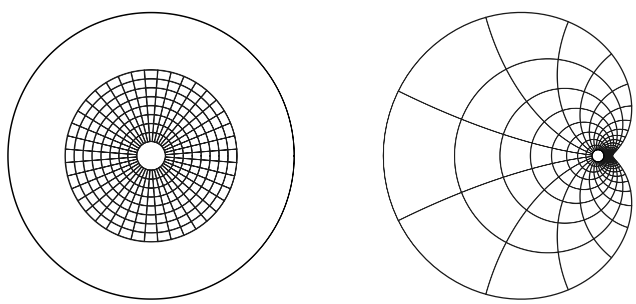


FIGURE 1.6. Plot of the Koebe function.

Next we prove the Koebe $\frac{1}{4}$-theorem for conformal maps. The standard proof of Koebe's $\frac{1}{4}$-theorem uses Green's theorem to estimate the power series coefficients of conformal map (proving the Bieberbach conjecture for the second coefficient). However here we will present a proof, due to Mateljevic [**?**], that uses the symmetry property of extremal length.

THEOREM 1.11 (The Koebe $\frac{1}{4}$ Theorem). *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

PROOF. Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components). Let $R = \mathrm{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\varepsilon,r} = \{z : \varepsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\varepsilon} = M(A_{\varepsilon,1}) = M(f(A_{\varepsilon,1})).$$

Let $\delta = \min_{|z|=\varepsilon} |f(z)|$. Since $f'(0) = 1$, we have $\delta = \varepsilon + O(\varepsilon^2)$. Note that $f(A_{\varepsilon,1}) \subset f(\mathbb{D}) \setminus D(0,\delta)$, so

$$M(f(A_{\varepsilon,1})) \leq M(f(\mathbb{D}) \setminus D(0,\delta)).$$

By Lemma 1.10 and Equation (1.1),

$$M(f(\mathbb{D}) \setminus D(0,\delta)) \leq M(\Omega_{\delta,R}) = 2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\varepsilon},$$

or

$$\log 4R - \log(\varepsilon + O(\varepsilon^2)) + O(\frac{\varepsilon}{R}) \geq -\log \varepsilon,$$

and hence

$$\log 4R \geq -O(\frac{\varepsilon}{R}) + \log(1 + O(\varepsilon)).$$

Taking $\varepsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. $\square$

2. Logarithmic capacity

Logarithmic capacity associates a non-negative number to each Borel subset of the unit circle. Applying a Möbius transformation can change this value, so it is not a conformal invariant, but it will act as an intermediate between extremal and harmonic measure (a conformal invariant that will be defined later).

Suppose μ is a positive, finite Borel measure on $\mathbb{C}$ and define its potential function as

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w), z \in \mathbb{C}.$$

and its energy integral by

$$I(\mu) = \iint \log \frac{2}{|z-w|} d\mu(z) d\mu(w) = \int U_\mu(z) d\mu(z).$$

We put the "2" in the numerator so that the integrand is non-negative when $z, w \in \mathbb{T}$, however, this is a non-standard usage.

Sets of zero logarithmic capacity must be very small, indeed the following computations will show that they must have dimension zero.

LEMMA 2.1. *Suppose $E \subset \overline{\mathbb{D}}$ and φ is a Hausdorff gauge function. Then*

$$\mathscr{H}^{\varphi}_{\infty}(E) \leq C\varphi(\frac{1}{\mathrm{Cap}_{\log}(\mathrm{E})}).$$

PROOF. By Frostman's Lemma (e.g., Lemma 3.1.1 of [**?**]), if E has positive dimension then there is a measure μ supported on E such that $\|\mu\| \geq \mathscr{H}^{\varphi}_{\infty}$ and $\mu(D(x,r)) \leq C\varphi(r)$ for all x and some $C < \infty$. Let

$$U_\mu(z) = \int \log\frac{2}{|z-w|}d\mu(w) = \int_0^2 \log\frac{2}{r}d\mu(z,r) \leq \int_0^{m^*} \log\frac{2}{r}d\varphi(r),$$

where $\varphi(m^*) = m$. Using integration by parts

$$U_\mu(z) \leq m\log\frac{2}{m^*} + \int_0^{m^*} \varphi(r)\frac{dr}{r}$$

□

COROLLARY 2.2. *If E has positive Hausdorff dimension, then it has positive logarithmic capacity.*

PROOF. By Frostman's Lemma (e.g., Lemma 3.1.1 of [**?**]), if E has positive dimension then there is a measure μ supported on E such that $\mu(D(x,r)) \leq Cr^{\alpha}$ for all x and some $C < \infty$ and $\alpha > 0$. By the previous lemma, this implies E has positive capacity. □

LEMMA 2.3. *U_μ is lower semi-continuous, i.e.,*

$$\liminf_{z\to z_0} U_\mu(z) \geq U_\mu(z_0).$$

PROOF. Fatou's lemma. □

Recall that $\mu_n \to \mu$ weak-* if $\int f d\mu_n \to \int f d\mu$ for every continuous function f of compact support.

LEMMA 2.4. *If $\{\mu_n\}$ are positive measures and $\mu_n \to \mu$ weak*, then* $\liminf_n U_{\mu_n}(z) \geq U_\mu(z)$.

PROOF. If we replace $\varphi = \log\frac{2}{|z-w|}$ by the continuous kernel $\varphi_r = \max(r,\varphi)$ in the definition of U to get U^r, then weak convergence implies

$$\lim_n U^r_{\mu_n}(z) \nearrow U^r_\mu(z).$$

Moreover, the convergence is increasing since the measures positive. So for any $\varepsilon > 0$ we can choose N so that $n > N$ implies

$$U^r_{\mu_n}(z) \geq U^r_\mu(z) - \varepsilon.$$

As $r \to \infty$ $U^r \to U$ (by the monotone convergence theorem), so for r large enough and $n > N$ we have

$$U_{\mu_n}(z) \geq U^r_{\mu_n}(z) \geq U_\mu(z) - 2\varepsilon.$$

which proves the result. $\square$

LEMMA 2.5. *If $\mu_n \to \mu$ weak*, then* $\liminf_n I(\mu_n) \geq I(\mu)$.

PROOF. The proof is almost the same as for the previous lemma, except that we have to know that if $\{\mu_n\}$ converges weak*, then so does the product measure $\mu_n \times \mu_n$. However, weak convergence of $\{\mu_n\}$ implies convergence of integrals of the form

$$\iint f(x)g(y)d\mu_n(x)d\mu_n(y).$$

and Stone-Weierstrass theorem implies that the finite sums of such product functions are dense in all continuous function on the product space. Since weak-* convergent sequences are bounded, the product measures $\mu_n \times \mu_n$ also have uniformly bounded masses, and hence convergence on a dense set of continuous functions of compact support implies convergence on all continuous functions of compact support. This, together with the fact that weak* convergent sequences are bounded ([**?**]), implies that $\mu_n \times \mu_n$ converges weak*. $\square$

Suppose E is Borel and μ is a positive measure that has its closed support inside E. We say μ is admissible for E if $U_\mu \leq 1$ on E and we define the **logarithmic capacity** of E as

$$\text{cap}(E) = \sup\{\|\mu\| : \mu \text{ is admissible for } E\}$$

and we write $\mu \in \mathscr{A}(E)$. We define the **outer capacity** (or exterior capacity) as

$$\text{cap}^*(E) = \inf\{\text{cap}(V) : E \subset V, V \text{open}\}.$$

We say that a set E is **capacitable** if $\text{cap}(E) = \text{cap}^*(E)$.

The logarithmic kernel can be replaced by other functions, e.g., $|z-w|^{-\alpha}$, and there is a different capacity associated to each one. To be precise, we should denote logarithmic capacity as $\text{cap}_{\log}$ or logcap, but to simplify notation we simply use "cap" and will often refer to logarithmic capacity as just "capacity". Since we do not use any other capacities in these notes, this abuse should not cause confusion.

WARNING: The logarithmic capacity that we have defined is **NOT** the same as is used in other texts such as Garnett and Marshall's book [**?**], but is related to what they call the Robin's constant of E, denoted $\gamma(E)$. The exact relationship is $\gamma(E) = \frac{1}{\text{cap}(E)} - \log 2$. Garnett and Marshall [**?**] define the logarithmic capacity of E as $\exp(-\gamma(E))$. The reason for doing this is that the logarithmic kernel $\log \frac{1}{|z-w|}$ takes both positive and negative values in the plane, so the potential functions for general measures and the Robin's constant for general sets need not be non-negative. Exponentiating takes care of this. Since we are only interested in computing the capacity of subsets of the circle, taking the extra "2" in the logarithm gave us a non-negative kernel on the unit circle, and we defined a corresponding capacity in the usual way. Since the kernel is the logarithm, we feel justified in calling the corresponding capacity the logarithmic capacity, despite the divergence with usual usage.

POSSIBLE ALTERNATES : Robin's capacity, conformal capacity, circular capacity.

LEMMA 2.6. *Compact sets are capacitable.*

PROOF. Since $\mathrm{cap}(E) \leq \mathrm{cap}^*(E)$ is obvious, we only have to prove the opposite direction. Set $U_n = \{z : \mathrm{dist}(z, E) < 1/n\}$ and choose a measure μ_n supported in U_n with $\|\mu_n\| \geq \mathrm{cap}(U_n) - 1/n$. Let μ be a weak accumulation point of $\{\mu_n\}$ and note

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w) \leq \int \log \frac{2}{|z-w|} d\mu_n(w) \leq 1$$

so μ is admissible in the definition of $\mathrm{cap}(E)$. Thus

$$\mathrm{cap}(E) \geq \limsup \|\mu_n\| = \lim \mathrm{cap}(U_n) = \lim \mathrm{cap}(U_n) = \mathrm{cap}^*(E).$$

□

It is also true that all Borel sets are capacitable. Indeed, this holds for all analytic sets (i.e., continuous images of complete separable topological spaces). See Appendix B of [**?**].

It is clear from the definitions that logarithmic capacity is monotone

$$E \subset F \quad \Rightarrow \quad \mathrm{cap}(E) \leq \mathrm{cap}(F). \tag{2.1}$$

and satisfies the regularity condition

$$\mathrm{cap}(E) = \sup\{\mathrm{cap}(K) : K \subset E, K \mathrm{compact}\}. \tag{2.2}$$

LEMMA 2.7 (Sub-additive). *For any sets $\{E_n\}$,*

$$\mathrm{cap}(\cup E_n) \leq \sum \mathrm{cap}(E_n). \tag{2.3}$$

PROOF. We can write any $\mu = \sum \mu_n$ as a sum of mutually singular measures so that μ_n gives full mass to E_n. We can then restrict each μ_n to a compact subset K_n of E_n so that $\mu_n(K_n) \geq (1-\varepsilon)\mu(E_n)$. These restrictions are admissible for each E_n and hence

$$\sum \mathrm{cap}(E_n) \geq \sum \mu_n(K_n) \geq (1-\varepsilon) \sum \mu_n(E_n) = (1-\varepsilon)\|\mu\|.$$

Taking $\varepsilon \to 0$ proves the result. □

COROLLARY 2.8. *A countable union of zero capacity sets has zero capacity.*

COROLLARY 2.9. *Outer capacity is also sub-additive.*

PROOF. Given sets $\{E_n\}$ choose open sets $V_n \supset E_n$ so that $\mathrm{cap}(V_n) \leq \mathrm{cap}^*(E_n) + \varepsilon 2^{-n}$. By the sub-additivity of capacity

$$\mathrm{cap}^*(\cup E_n) \leq \mathrm{cap}(\cup V_n) \leq \sum \mathrm{cap}(V_n) \leq \varepsilon + \sum \mathrm{cap}^*(E_n).$$

Taking $\varepsilon \to$ proves the result. $\square$

Although capacity informally "measures" the size of a set, it is not additive, and hence not a measure. See Exercise 6.

LEMMA 2.10. *If μ has a bounded potential, then $\mathrm{Cap}_{\log}(\mathrm{E}) = 0$ implues $\mu(E) = 0$.*

PROOF. If $\mu(E) > 0$ then μ restricted to E also has bounded potential function and proves that E has positive capacity. $\square$

LEMMA 2.11. *If E is compact has positive capacity, then there exists an admissible μ that attains the maximum mass in the definition of capacity and $U_\mu(z) = 1$ everywhere on E, except possible a set of capacity zero.*

PROOF. Let μ_n be a sequence of proability measures on E so that $\|\mu_n\| \to R$ where $R = \inf I(\mu)$ over all proability measures supported on E. This is finite since E has positive capacity. By the Banach-Alogalu theorem there is a weak-* convergent subsequence with limit μ, and by Lemma 2.5,

$$I(\mu) \leq \liminf_n I(\mu_n) = R.$$

We claim that $U_\mu \geq R$ except possibly on a set of zero capacity. Otherwise let $T \subset E$ be a set of positive capacity on which $U_\mu < 1 - \varepsilon$ and let σ be a non-zero, positive measure on T which potential bounded by 1. Define

$$\mu_t = (1-t)\mu + t\sigma.$$

This is a measure on E so that

$$\begin{aligned}
I(\mu_t) &\leq \int \log \frac{1}{|z-w|}((1-t)d\mu + td\sigma)((1-t)d\mu + td\sigma) \\
&\leq (1-t)^2 I(\mu) + 2t \int U_\mu d\sigma + t^2 I(\sigma) \\
&\leq I(\mu) - 2tI(\mu) + 2t \int U_\mu d\sigma + O(t^2) \\
&\leq I(\mu) - 2tI(\mu) + 2t(1-\varepsilon)\|\sigma\| + O(t^2) \\
&< I(\mu),
\end{aligned}$$

if $t > 0$ is small enough. This contradicts minimality of μ, and hence the claim holds.

Next we show that $U_\mu \le 1$ everywhere on the closed support of μ. By the previous step we know $U_\mu \ge 1$ except on capacity zero, hence except on a set of μ-measure zero. If there is a point z in the support of μ such that $U_\mu(z) > 1$, then by lower semi-continuity of potentials, U_μ is $> 1+\varepsilon$ on some neighborhood of z and this neighborhood has positive μ measure (since z is in the support of μ) and thus $I(\mu) = \int U_\mu d\mu > \|\mu\|$, a contradiction.

Finally, let $\sigma = \mu/R$. Then the potential function of σ is bounded by 1 everywhere, so σ is admissible for E and hence $\|\sigma\| \le \mathrm{Cap}_{\log}(\mathrm{E})$. If ν is any other admissible measure for E, then $\nu(\{z \in E : \sigma(z) < 1\}) = 0$ by Lemma 2.10. Hence

$$\|\nu\| = \int 1 d\nu = \int U_\sigma d\nu = \int U_\nu d\sigma \le \int 1 d\sigma = \|\sigma\|,$$

and thus $\|sigma\| \ge \mathrm{Cap}_{\log}(\mathrm{E})$. Thus $\mathrm{Cap}_{\log}(\mathrm{E}) = \|\sigma\| = \|\mu/\mathrm{R}\| = 1/\mathrm{R}$. Hence $R = 1/\mathrm{Cap}_{\log}(\mathrm{E})$ is the Robin's constant of E. Since $\|\sigma\| \ge I(\sigma) = I(\mu)/R^2 = 1/R$

□

The following makes a connection between logarithmic capacity and extremal length. Eventually, this will become a connection between extremal length and harmonic measure.

If $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of K. Let K^* be the reflection of K across $\mathbb{T}$. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals, let Ω be the connected component of $\mathbb{C} \setminus (E \cup K \cup K^*)$ that has E on its boundary. Let $h(z)$ be the harmonic function in Ω with boundary values 0 on K and K^* and boundary value 1 on E. By the usual theory of the Dirichlet problem (e.g. [**?**]), all boundary points are regular (since all boundary components are non-degenerate continua) and hence h extends continuously to the boundary with the correct boundary values. Moreover, h is symmetric with respect to $\mathbb{T}$, and this implies its normal derivative on $\mathbb{T} \setminus E$ is 0. Let $D(h) = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy$. Let Γ_E denote the paths in $\mathbb{D} \setminus K$ that connect K to E.

LEMMA 2.12. *With notation as above, $M(\Gamma_E) = D(h)$.*

PROOF. Clearly $|\nabla h|$ is an admissible metric for Γ_E, so

$$M(\Gamma_E) \le D(h) \equiv \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Thus we need only show the other direction.

Green's theorem states that

$$\iint_\Omega u\Delta v - v\Delta u dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} ds. \tag{2.4}$$

Using this and the fact that $h = 1$ on E, we have

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = -\int_{\mathbb{T}} \frac{\partial h}{\partial n} ds = -\int_E \frac{\partial h}{\partial n} ds = -\int_E h\frac{\partial h}{\partial n} ds.$$

and

$$\begin{aligned} \int_{\partial K} \frac{\partial h}{\partial n} ds &= -\frac{1}{2}\int_E \frac{\partial(h^2)}{\partial n} ds \\ &= \frac{1}{2}\int_{\mathbb{T}\setminus E} \frac{\partial(h^2)}{\partial n} ds + \frac{1}{2}\int_{\partial K} \frac{\partial(h^2)}{\partial n} ds + \frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy. \end{aligned}$$

The first term is zero because h has normal derivative zero on $\mathbb{T}\setminus E$, and hence the same is true for h^2. The second term is zero because h is zero on K and so $\frac{\partial(h^2)}{\partial n} h^2 = 2h\frac{\partial h}{\partial n} = 0$. To evaluate the third term, we use the identity

$$\begin{aligned} \Delta(h^2) &= 2h_x \cdot h_x + 2h \cdot h_{xx} + 2h_y \cdot h_y + 2h \cdot h_{yy} \\ &= 2h\Delta h + 2\nabla h \cdot \nabla h \\ &= 2h \cdot 0 + 2|\nabla h|^2 \\ &= 2|\nabla h|^2, \end{aligned}$$

to deduce

$$\frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy = \int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.$$

Therefore,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = \int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.$$

Thus the tangential derivative of h's harmonic conjugate has integral $D(h)$ around ∂K and therefore $2\pi h/D(h)$ is the real part of a holomorphic function g on $\mathbb{D}\setminus K$. Then $f = \exp(g)$ maps $\mathbb{D}\setminus K$ into the annulus

$$A = \{z : 1 < |z| < \exp(2\pi/D(h))\}$$

with the components of E mapping to arcs of the outer circle and the components of $\mathbb{T}\setminus E$ mapping to radial slits. The path family Γ_E maps to the path family connecting the inner and outer circles without hitting the radial slits, and our earlier computations show the modulus of this family is $D(h)$. $\square$

THEOREM 2.13 (Pfluger's theorem). *If $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of K. Then there are constants C_1, C_2 so that following holds. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals,*

$$\frac{1}{\mathrm{cap}(E)} + C_1 \leq \pi\lambda(\Gamma_E)) \leq \frac{1}{\mathrm{cap}(E)} + C_2,$$

where Γ_E is the path family connecting K to E. The constants C_1, C_2 can be chosen to depend only on $0 < r < R < 1$ if $\partial K \subset \{r \leq |z| \leq R\}$.

PROOF. Using Lemma 2.12, we only have to relate $D(h)$ to the logarithmic capacity of E. Let μ be the equilibrium probability measure for E. We know in general that $U_\mu = \gamma$ where $\gamma = 1/\mathrm{cap}(E)$ almost everywhere on E (since sets of zero capacity have zero measure) and is continuous off E, but since U_μ is harmonic in $\mathbb{D}$ and equals the Poisson integral of its boundary values, we can deduce $U_\mu = \gamma$ everywhere on E. Let $v(z) = \frac{1}{2}(U_\mu(z) + U_\mu(1/\bar{z})$. Then since ∂K has positive distance from 0, there are constants C_1, C_2 so that

$$v + C_1 \leq 0, \qquad v + C_2 \geq 0,$$

on ∂K. Note that $C_1 \geq -\gamma$ by the maximum principle and $C_2 \geq 0$ trivially. Moreover, since μ is a probability measure supported on the unit circle, given $0 < r < R < 1$, U_μ is uniformly bounded on both the annulus $\{r \leq |z| \leq R\}$ and its reflection across the unit circle, since these both have bounded, but positive distance from the unit circle. This proves that C_1, C_2 can be chosen to depend on only these numbers, as claimed in the final statement of the theorem.

The following inequalities are easy to check on K, K^* and E,

$$\frac{v(z) + C_1}{\gamma + C_1} \leq h(z) \leq \frac{v(z) + C_2}{\gamma + C_2}.$$

and hence hold on Ω by the maximum principle. Since we have equality on E, we also get

$$\frac{\partial}{\partial n}\left(\frac{v(z) + C_1}{\gamma + C_1}\right) \leq \frac{\partial h}{\partial n} \leq \frac{\partial}{\partial n}\left(\frac{v(z) + C_2}{\gamma + C_2}\right)$$

for $z \in E$. When we integrate over E, the middle term is $-D(h)$ (we computed this above) and by Green's theorem

$$\begin{aligned} -\int_E \frac{\partial}{\partial n} \frac{v(z) + C_1}{\gamma + C_1} ds &= \frac{1}{\gamma + C_1} \int_{\mathbb{D}} \Delta(v) dxdy \\ &= \frac{\pi}{\gamma + C_1} \end{aligned}$$

because v is harmonic except for a $\frac{1}{2}\log\frac{1}{|z|}$ pole at the origin. A similar computation holds for the other term and hence

$$\frac{\pi}{\gamma+C_1} \le D(h) = M(\Gamma_E) \le \frac{\pi}{\gamma+C_2},$$

since $D(h) = \int_E \frac{\partial h}{\partial n} ds$. Hence

$$\gamma + C_1 \le \pi\lambda(\Gamma_E) \le \gamma + C_2.$$

This completes the proof of Pfluger's theorem for finite unions of intervals.

□

Next we prove Pfluger's theorem for all compact subsets of $\mathbb{T}$. First we need a continuity property of extremal length. Recall that an extended real-valued function is lower semi-continuous if all sets of the form $\{f > \alpha\}$ are open.

LEMMA 2.14. *Suppose $E \cap \mathbb{T}$ is compact, $K \subset \mathbb{D}$ is compact, connected and contains the origin, and Γ_E is the path family connecting K and E in $\mathbb{D} \setminus K$. Fix an admissible metric ρ for Γ_E and for each $z \in \mathbb{T}$, define $f(z) = \inf \int_\gamma \rho ds$ where the infimum is over all paths in Γ_E that connect K to z. Then f is lower semi-continuous.*

PROOF. Suppose $z_0 \in \mathbb{T}$ and use Cauchy-Schwarz to get

$$\begin{aligned}
\int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r} \rho ds\right)^2 dr &\le \int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r} \rho^2 ds\right) dr \left(\int_{|z-z_0|=r} 1 ds\right) dr \\
&\le \int_{2^{-n-1}}^{2^{-n}} r \int_0^{2\pi} \rho^2 r d\theta dr \\
&\le \pi 2^{-n} \int_{2^{-n-1}<|z-z_0|<2^{-n}} \rho^2 dxdy \\
&= o(2^{-n}).
\end{aligned}$$

Therefore we can choose circular cross-cuts $\{\gamma_n\} \subset \{z : 2^{-n-1} < |z-z_0| < 2^{-n}\}$ of $\mathbb{D}$ centered at z_0 and with ρ-length ε_n tending to 0. By taking s subsequence we may assume $\sum \varepsilon_n < \infty$. Now choose $z_n \to z_0$ with

$$f(z_n) \to \alpha \equiv \liminf_{z \to z_0} f(z).$$

We want to show that there is a path connecting K to z_0 whose ρ-length is as close to α as we wish. Passing to a subsequence we may assume z_n is separated from K by δ_n. Let c_n be the infimum of ρ-lengths of paths connecting γ_n and γ_{n+1}. By considering a path connecting K to z_n, we see that $\sum_1^n c_k \le f(z_n)$, for all n and hence $\sum_1^\infty c_n \le \alpha$.

Next choose $\varepsilon > 0$ and choose n so that we can connect K to z_n (and hence to γ_n) by a path of ρ-length less than $\alpha + \varepsilon$. We can then connect γ_n

to z_0 by a infinite concatenation of arcs of γ_k, $k > n$ and paths connecting γ_k to γ_{k+1} that have total length $\sum_n^\infty (\varepsilon_n + c_n) = o(1)$. Thus K can be connected to z_0 by a path of ρ-length as close to α as we wish. $\square$

COROLLARY 2.15. *Suppose $E \subset \mathbb{T}$ is compact and $\varepsilon > 0$. Then there is a finite collection of closed intervals F so that $E \subset F$ and*

$$\lambda(\Gamma_E) \le \lambda(\Gamma_F) + \varepsilon,$$

where the path families are defined as above.

PROOF. Choose an admissible ρ so that $\int \rho^2 dxdy \le M(\Gamma_E) + \varepsilon$. Set

$$r = \left(\frac{M(\Gamma_E)+\varepsilon}{M(\Gamma_E)+2\varepsilon}\right)^{1/2}$$

By Lemma 2.14 $V = \{z \in \mathbb{T} : f(z) > r\}$ is open, and therefore we can choose a set F of the desired form inside V. Then ρ/r is admissible for Γ_F, so

$$M(\Gamma_F) \le \int \left(\frac{\rho}{r}\right)^2 dxdy = \frac{M(\Gamma_E)+2\varepsilon}{M(\Gamma_E)+\varepsilon} \int \rho^2 dxdy \le M(\Gamma_E) + 2\varepsilon.$$

Thus an inequality in the opposite direction holds for extremal length. $\square$

COROLLARY 2.16. *Pfluger's theorem holds for all compact sets in $\mathbb{T}$.*

PROOF. Suppose E is compact. Using Corollary 2.15 and Lemma 2.6 we can choose nested sets $E_n \searrow E$ that are finite unions of closed intervals and satisfy

$$\lambda(\mathscr{F}_{E_n}) \to \lambda(\mathscr{F}_E),$$

and

$$\operatorname{cap}(E_n) \to \operatorname{cap}(E).$$

Thus the inequalities in Pfluger's theorem extend to E. $\square$

3. Hyperbolic distance

We start on the disk, and then extend to simply connected domains via the Riemann mapping theorem and to general planar domains via the uniformization theorem.

The **hyperbolic metric** on $\mathbb{D}$ is given by $d\rho(z) = |dz|/(1-|z|^2)$. This means that the hyperbolic length of a rectifiable curve γ in $\mathbb{D}$ is defined as

$$\ell_\rho(\gamma) = \int_\gamma \frac{|dz|}{1-|z|^2}, \tag{3.1}$$

and the hyperbolic distance between two points $z, w \in \mathbb{D}$ is the infimum of the lengths of paths connecting them (we shall see shortly that there is an explicit formula for this distance in terms of z and w). In many sources, there is a “2” in the numerator of (3.1), but we follow [**?**], where the definition is as given in (3.1). For most applications this makes no difference, but the reader is warned that some of our formulas may differ by a factor of 2 from the analogous formulas in some papers and books.

We define the **hyperbolic gradient** of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ as

$$D_H^H f(z) = |f'(z)| \frac{1-|z|^2}{1-|f(z)|^2}.$$

More generally, given a map f between metric spaces (X,d) and (Y,ρ) we define the gradient at a point z as

$$D_d^\rho f(z) = \limsup_{x \to z} \frac{\rho(f(z), f(x))}{d(x,z)}.$$

The use of the word “gradient” is not quite correct; a gradient is usually a vector indicating both the direction and magnitude of the greatest change in a function. We use the term in a sense more like the term “upper gradient” that occurs in metric measure theory to denote a function $\rho \geq 0$ that satisfies

$$|f(b) - f(a)| \leq \int_\gamma \rho \, ds,$$

for any curve γ connecting a and b. I hope that the slight abuse of the term will not be confusing.

In these notes, the most common metrics we will use are the usual Euclidean metric on $\mathbb{C}$, the spherical metric

$$\frac{ds}{1+|z|^2},$$

on the Riemann Sphere, S^2 and the hyperbolic metric on the disk or on some other hyperbolic planar domain. To simplify notation, we use E, S and H to denote whether we are taking a gradient with respect to Euclidean, spherical or hyperbolic metrics. For example if $f : U \to V$, the symbol $D_H^H f$ means that we are taking a gradient from the hyperbolic metric on U to the hyperbolic metric on V (assuming the domains are clear from context; otherwise we write D_U^V or $D_{\rho_U}^{\rho_v}$ if we need to be very precise.)

In this notation, the spherical derivative of a function, usually denoted

$$f^\#(z) = \frac{|f'(z)|}{1+|f(z)|^2},$$

is written $D_E^S f(z)$ since it is a limit of quotients where the numerator is measured in the spherical metric and the denominator is measured in the Euclidean metric. Similarly D_H^S denotes a gradient measuring expansion from a hyperbolic to the spherical metric. This particular gradient is important in the theory of normal families (e.g., see Montel’s theorem in [**?**]).

Another variation we will use is $D^E_{\mathbb{D}} f$. If this is bounded on the disk, then f is a Lipschitz function from the hyperbolic metric on the disk to the Euclidean metric on the plane. Such functions are called Bloch functions.

A **linear fractional transformation** is a map of the form

$$z \to \frac{a+bx}{c+dz},$$

where $a,b,c,d \in \mathbb{C}$. These exactly the 1-to-1, holomorphic maps of the Riemann sphere to itself. Such maps are also called **Möbius transformations**.

LEMMA 3.1. *Möbius transformations of $\mathbb{D}$ to itself are isometries of the hyperbolic metric.*

PROOF. When f is a Möbius transformation of the disk we have

$$f(z) = \frac{z-a}{1-\bar{a}z}, \qquad f'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

Thus

$$\begin{aligned}
D^H_H f(z) &= \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|f(z)|^2} = \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|\frac{z-a}{1-\bar{a}z}|^2} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2-|z-a|^2} = \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z)(1-a\bar{z})-(z-a)(\bar{z}-\bar{a})} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z-a\bar{z}+|az|^2)-(|z|^2-a\bar{z}-z\bar{a}+|a|^2)} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1+|az|^2-|z|^2-|a|^2)} = 1.
\end{aligned}$$

Note that

$$\ell_\rho(f(\gamma)) \le \int_\gamma D^H_H f(z)\frac{|dz|}{1-|z|^2}.$$

Thus Möbius transformations multiply hyperbolic length by at most one. Since the inverse also has this property, we see that Möbius transformation preserve hyperbolic length. □

The segment $(-1,1)$ is clearly a geodesic for the hyperbolic metric and since isometries take geodesics to geodesics, we see that geodesics for the hyperbolic metric are circles orthogonal to the boundary.

On the disk it is convenient to define the pseudo-hyperbolic metric

$$T(z,w) = |\frac{z-w}{1-\bar{w}z}|.$$

The hyperbolic metric between two points can then be expressed as

$$\rho(w,z) = \frac{1}{2}\log\frac{1+T(w,z)}{1-T(w,z)}. \tag{3.2}$$

On the upper half-plane the corresponding function is

$$T(z,w) = |\frac{z-w}{w-\bar{z}}|,$$

and ρ is related as before.

LEMMA 3.2 (Schwarz's Lemma). *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ then $|f'(0)| \le 1$ with equality iff f is a rotation. Moreover, $|f(z)| \le |z|$ for all $|z| < 1$, with equality for $z \neq 0$ iff f is a rotation.*

PROOF. Define $g(z) = f(z)/z$ for $z \neq 0$ and $g(0) = f'(0)$. This is a holomorphic function since if $f(z) = \sum a_n z^n$ then $a_0 = 0$ and so $g(z) = \sum a_n z^{n-1}$ has a convergent power series expansion. Since $\max_{|z|=r}|g(z)| \le \frac{1}{r}\max_{|z|=r}|f| \le \frac{1}{r}$. By the maximum principle $|g| \le \frac{1}{r}$ on $\{|z| < r\}$. Taking $r \nearrow 1$ shows $|g| \le 1$ on $\mathbb{D}$ and equality anywhere implies g is constant. Thus $|f(z)| \le |z|$ and $|f'(0)| = |g(0)| \le 1$ and equality implies f is a rotation. $\square$

In terms of the hyperbolic metric this says that

$$\rho(f(0), f(z)) = \rho(0, f(z)) \le \mathbb{H}_r(0,z),$$

which shows the hyperbolic distance from 0 to any point is non-increasing. For an arbitrary holomorphic self-map of the disk f and any point $w \in \mathbb{D}$ we can always choose Möbius transformations τ, σ so that $\tau(0) = w$ and $\sigma(f(w)) = 0$, so that $\sigma \circ f \circ \tau(0) = 0$. Since Möbius transformations are hyperbolic isometries, this shows

COROLLARY 3.3. *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic then $\rho(f(w), f(z)) \le \rho(w,z)$.*

LEMMA 3.4. *If $\{f_n\}$ are holomorphic functions on a domain Ω that converge uniformly on compact sets to f and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.*

PROOF. We may assume $\{z_n\}$ are contained in some disk $D \subset \Omega$ around z. Let $E = \{z_n\}_1^\infty \cup \{z\}$. This is a compact set so it has a positive distance d from $\partial\Omega$. The points within distance $d/2$ of E form a compact set F on which the functions $\{f_n\}$ are uniformly bounded on E, say by M. By the Cauchy estimate the derivatives are bounded by a constant M' on E (e.g., see [**?**]). Thus

$$|f(z) - f_n(z_n) \le |f(z) - f_n(z)| + |f_n(z) - f_n(z_n)| \le |f(z) - f_n(z)| + M'|z - z_n|,$$

and both terms on the right tend to zero by hypothesis. $\square$

A planar domain Ω is called **hyperbolic** if $\mathbb{C} \setminus \Omega$ has at least two points.

THEOREM 3.5. *Every hyperbolic plane domain Ω is holomorphically covered by $\mathbb{D}$ (i.e., there is a locally 1-to-1, holomorphic covering map from $\mathbb{D}$ to Ω).*

We will prove this in three steps: bounded domains, simply connected domains and finally the general case.

UNIFORMIZATION FOR BOUNDED DOMAINS. If Ω is bounded, then by a translation and rescaling, we may assume $\Omega \subset \mathbb{D}$ and $0 \in \Omega$. We will define a sequence of domains $\{\Omega_n\}$ with $\Omega_0 = \Omega$ and covering maps $p_n : \Omega_n \to \Omega_{n-1}$ such that $p(0) = 0$. We will show that Ω_n contains hyperbolic disks centered at 0 of arbitrarily large radius and that the covering map $q_n = p_1 \circ \cdots \circ p_n : \Omega_n \to \Omega_0 = \Omega$ converges uniformly on compacta to a covering map $q : \mathbb{D} \to \Omega$.

If $\Omega_0 = \mathbb{D}$ we are done, since the identity map will work. In general assume that we have $q_n : \Omega_n \to \Omega_0$ and that there is a point $w \in \mathbb{D} \setminus \Omega_n$. Let τ and σ be Möbius transformations of the disk to itself so that $\tau(w) = 0$, choose a square root α of $\tau(0)$ and choose σ so $\sigma(\alpha) = 0$. Then $p_{n+1}(z) = \sigma(\sqrt{\tau(z)})$ and let Ω_{n+1} be the component of $U = p_{n+1}^{-1}(\Omega_n)$ that contains the origin (the set U will have one or two components; two if w is in a connected component of $\mathbb{D} \setminus \Omega_n$ that is compact in $\mathbb{D}$, and one otherwise). Since σ and τ are hyperbolic isometries and $\sqrt{z}$ expands the hyperbolic metric, we see that Ω_{n+1} contains a larger hyperbolic ball around 0 than Ω_n did.

More precisely, suppose $\operatorname{dist}(\partial\Omega_n, 0) < r < 1$ for all n. Since $f(z) = z^2$ maps the disk to itself, it strictly contracts the hyperbolic metric; a more explicit computation shows

$$D_H^H f(z) = |2z| \frac{1 - |z|^2}{1 - |z|^4} = \frac{2|z|}{1 + |z|^2} < 1.$$

Thus $g(z) = \sqrt{z}$ is locally an expansion of the hyperbolic metric, at least on a subdomain $W \subset \mathbb{D}$ where it has a well defined branch. For $z \neq 0$,

$$D_H^H g(z) = \left|\frac{1}{2\sqrt{z}}\right| \frac{1 - |z|^2}{1 - |z|} \geq \frac{1 + |z|}{2\sqrt{z}}. \tag{3.3}$$

Then (3.3) says that

$$D_H^H p_n(0) = D_H^H \sqrt{z}(\tau(0)) > \frac{1 + r}{2\sqrt{r}} > 1,$$

since $|\tau(0)| = |w| < r$. Hence $D_H^H q_n(0)$ increases by this much at every step. But $D_H^H q_n(0) \leq 1$, which is a contradiction. Thus $d_n \to 1$.

Thus $\{q_n\}$ is a sequence of uniformly bounded holomorphic functions on the disk. By Montel's theorem, there a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $q : \mathbb{D} \to \Omega$. It is non-constant since it has non-zero gradient at the origin; moreover, by Hurwitz's theorem (see [**?**]), q' never vanishes on $\mathbb{D}$ since it is the locally uniform limit of the sequence $\{q_n'\}$, and these functions never vanish since they are all derivatives of locally univalent covering maps. Next we show that q is a covering map $\mathbb{D} \to \Omega$.

Fix $a \in \Omega$ and let $d = \mathrm{dist}(a, \partial\Omega)$. Since Ω is bounded, this is finite. Let $D = D(a,d) \subset \Omega$. Since q_n is a covering map, every branch of q_n^{-1} is 1-to-1 holomorphic map of D into $\mathbb{D}$ and hence each q_n is a contraction from the hyperbolic metric on D to the hyperbolic metric on $\mathbb{D}$. Thus every preimage of $\frac{1}{2}D$ has uniformly bounded hyperbolic diameter.

Now fix a point $b \in q^{-1}(a)$. Since $q_n(b) \to q(b) = a$, $q_n(b) \in \frac{1}{2}D$ for n large enough, so there is branch of q_n^{-1} that contains b. Since these branches are uniformly bounded holomorphic functions, by Montel's theorem we can pass to a subsequence so that they converge to a holomorphic function g from $\frac{1}{2}D$ into $\mathbb{D}$. Moreover,

$$q(g(z)) = \lim_n q_n(q_n^{-1}(z)) = z,$$

by Lemma 3.4. $\square$

This proves the existence of a covering map for bounded domains Ω. If Ω is bounded and simply connected, then we have proved the Riemann mapping theorem for Ω. For unbounded simply connected domains we use the following argument.

RIEMANN MAPPING THEOREM. It suffices to show any simply connected planar domain, except for the plane itself, can be conformally mapped to a bounded domain. If the domain Ω is bounded, there is nothing to do. If Ω. omits a disk $D(x,r)$ then the map $z \to 1/(z-x)$ conformal maps Ω to a bounded domain. Otherwise, translate the domain so that 0 is on the boundary and consider a continuous branch of $\sqrt{z}$. The image is a 1-1, holomorphic image of Ω, but does not contain both a point and its negative. Since the image contains some open ball, it also omits an open ball and hence can be mapped to a bounded domain by the previous case. $\square$

The final step is to deduce the uniformization theorem for all hyperbolic plane domains (we have only proved it for bounded domains so far). It suffices to show that any hyperbolic plane domain has a covering map from some bounded domain W, for then we can compose the covering maps $\mathbb{D} \to W$ and $W \to \Omega$. We can reduce to the following special case:

THEOREM 3.6. *There is a holomorphic covering map from $\mathbb{D}$ to $\mathbb{C}^{**} = \mathbb{C} \setminus \{0,1\}$*

PROOF. Let

$$\Omega = \{z = x+iy : y > 0, 0 < x < 1, |z - \frac{1}{2}| > \frac{1}{2}\} \subset \mathbb{H}_u.$$

This is simply connected and hence can be conformally mapped to $\mathbb{H}_u$ with $0, 1, \infty$ each fixed. We can then use Schwarz reflection to extend the map across the sides of Ω. Every such reflection of Ω stays in $\mathbb{H}_u$ maps to either the lower or upper half-planes. Continuing this forever gives a covering map from a simply connected subdomain U of $\mathbb{H}_u$ to W. Since U is simply connected and not the whole plane (it is a subset of $\mathbb{H}_u$) it is conformally equivalent to $\mathbb{D}$ and hence a covering $q : \mathbb{D} \to W$ exists. (Actually $U = \mathbb{H}_u$, but we do not need this stronger result. See Exercise 6.) $\square$

UNIFORMIZATION OF GENERAL PLANAR DOMAINS. Let $q : \mathbb{D} \to \mathbb{C}^{**} = \mathbb{C} \setminus \{0,1\}$. be a covering map of the twice punctured plane. If $\{a,b\} \in \mathbb{C} \setminus \Omega$ then $h(z) = bq(z) + a$ is a covering map from $U = h^{-1}(\Omega) \subset \mathbb{D}$ to Ω. Any connected component of U shows that Ω has a covering from a bounded plane domain, finishing the proof. $\square$

We can now define a hyperbolic metric ρ on any hyperbolic domain using the covering map $p : \mathbb{D} \to \Omega$. The function ρ should be defined so that p is locally an isometry, i.e.,

$$\begin{aligned} 1 &= D_{\mathbb{D}}^{\Omega} p(w) \\ &= D_{\mathbb{D}}^{E} \mathrm{Id}(w) \cdot D_{E}^{E} p(w) \cdot D_{E}^{\rho_\Omega} \mathrm{Id}(p(w)) \\ &= \frac{1}{\rho_{\mathbb{D}}(w)} \cdot |p'(w)| \cdot \rho_\Omega(z) \end{aligned}$$

and so we take

$$\rho_\Omega(z) = \frac{|p'(w)|}{1 - |w|^2} = |p'(w)| \rho_{\mathbb{D}}(w)$$

where $p(w) = z$. Different choices of p and w give the same value for $\rho_\Omega(z)$ since they differ by an isometry of $\mathbb{D}$. Thus every hyperbolic planar domain has a hyperbolic metric.

We want to give some useful estimates for ρ_Ω in terms of more geometric quantities, such as the quasi-hyperbolic metric, defined as

$$\widetilde{\rho}_\Omega(z) ds = \frac{ds}{\mathrm{dist}(z, \partial\Omega)}.$$

For simply connected domains, ρ and $\widetilde{\rho}$ are boundedly equivalent; for more general domains this can fail, but some useful estimates are still available.

The first observation is that if $f : U \to V$ is conformal and $\rho_U(z)ds$ and $\rho_V(z)ds$ are the densities of the hyperbolic metrics on U and V then

$$\rho_V(f(z)) = \rho_U(z)/|f'(z)|.$$

Applying this to the map $\tau(z) = (z+1)/(z-1)$ that maps the right half-plane $\mathbb{H}_r = \{x+iy : x > 0\}$ to the unit disk $\mathbb{D}$, we see that the hyperbolic density for the half-plane is

$$\rho_{\mathbb{H}_r}(z) = |\tau'(z)|\rho_{\mathbb{D}}(\tau(z)) = \frac{2}{|z-1|^2}\frac{1}{1-|\tau(z)|^2} = \frac{1}{2x} = \frac{1}{2\,\mathrm{dist}(z, \partial\mathbb{H}_r)}.$$

Thus the hyperbolic density on a half-plane is approximately the same as the quasi-hyperbolic metric. Using Koebe's theorem (Lemma 1.11) we can deduce that that this is true for any simply connected domain.

LEMMA 3.7. *For simply connected domains, the hyperbolic and quasi-hyperbolic metrics are bi-Lipschitz equivalent, i.e.,*

$$d\rho_\Omega \le d\widetilde{\rho}_\Omega \le 4d\rho_\Omega. \tag{3.4}$$

PROOF. Using Koebe's theorem,

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \le \rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\mathrm{dist}(f(z), \partial\Omega} = \frac{1}{\mathrm{dist}(f(z), \partial\Omega} = \widetilde{\rho}(f(z)),$$

which is one half of the result. The other half is similar:

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \ge \frac{1}{4}\rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\mathrm{dist}(f(z), \partial\Omega)} = \frac{1}{4}\widetilde{\rho}(f(z)).$$

□

COROLLARY 3.8. *If $f : \Omega \to \Omega'$ is conformal, then*

$$\frac{\mathrm{dist}(f(z), \partial\Omega')}{4\,\mathrm{dist}(z, \partial\Omega)} \le |f'(z)| \le \frac{4\,\mathrm{dist}(f(z), \partial\Omega')}{\mathrm{dist}(z, \partial\Omega)}.$$

PROOF. Write $f = g \circ h^{-1}$ where $g : \mathbb{D} \to \Omega'$ and $h : \mathbb{D} \to \Omega$ and use the chain rule and Koebe's theorem. □

The following is immediate from Schwarz's lemma.

COROLLARY 3.9. *If $U \subset V$ are both hyperbolic, then $\rho_U \ge \rho_V$.*

PROOF. If $\Pi_U : \mathbb{D} \to U$ and $\Pi_V : \mathbb{D} \to V$ are the covering maps then the inclusion map $U \to V$ can be lifted to conformal map $\mathbb{D} \to \Pi_V^{-1}(U) \subset \mathbb{D}$. Applying Schwarz's lemma to this map (and using the fact that the projections are local isometries) gives the result. □

LEMMA 3.10. *If f is conformal on the disk, and $\varphi = \log f'$, then $|\varphi'(z)| \le 6/(1-|z|^2)$ for all $z \in \mathbb{D}$.*

PROOF. Define
$$F(z) = \frac{f(\tau(z)) - f(w)}{(1-|w|^2)f'(w)},$$
where
$$\tau(z) = \frac{z+w}{1-\bar{w}z}.$$
Then F is conformal, $F(0)=0$ and $F'(0)=1$, so Lemma **??** says that $|F''(0)| \leq 4$. A computation shows
$$F''(0) = \frac{f''(z)}{f'(z)}(1-|z|^2) + (-2\bar{z}),$$
and $\varphi' = (\log f')' = f''/f'$, so
$$|\varphi'|(1-|z|^2) \leq |F''(0)| + |2z| \leq 4+2 = 6.$$
□

Another way to state the lemma is that $D_H^E \varphi \leq 6$. In other words, φ is a Lipschitz holomorphic function from the disk with its hyperbolic metric to the plane with its Euclidean metric. The set of such functions is called the Bloch class and is a Banach space with the norm
$$\|\varphi\|_{\mathscr{B}} = |\varphi(0)| + \sup_{|z|<1} |\varphi'(z)|(1-|z|^2).$$
In a later chapter, we shall see that Lemma 6 leads to an intimate connection between conformal maps and martingales that allows various results from probability theory about the latter to be directly to the former, e.g., Makarov's law of the iterated logarithm.

4. Boundary continuity

The boundary of a simply connected domain need not be a Jordan curve, nor even locally connected, and such examples arise naturally in complex dynamics as the Fatou components of various polynomials and entire functions. However, this makes little difference to the study of harmonic measure. In this section we show that, from the point view of harmonic measure, it is always enough to consider regions with locally connected boundaries.

LEMMA 4.1. *Suppose Q is a quadrilateral with opposite pairs of sides E,F and C,D. Assume*

(1) *E and F can be connected in Q by a curve σ of diameter $\leq \varepsilon$,*
(2) *any curve connecting C and D in Q has diameter at least* 1.

Then the modulus of the path family connecting E and F in Q is larger than $M(\varepsilon)$ where $M(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

PROOF. Define a metric on Q by $\rho(z) = \frac{1}{2}|z-a|^{-1}/\log(1/2\varepsilon)$ for $\varepsilon < |z-a| < 1/2$. Any curve γ connecting C and D must cross σ and since γ has diameter ≥ 1 it must leave the annulus where ρ is non-zero. This shows that the modulus of the path family in Q separating E and F is small, hence the modulus of the family connecting them is large. □

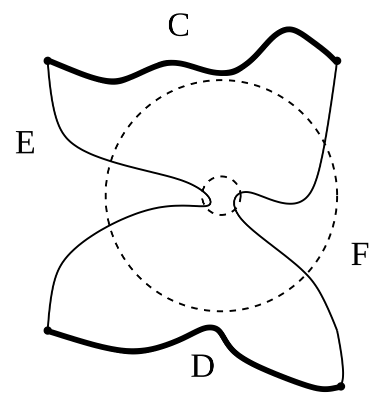


FIGURE 4.1. Proof of Lemma 4.1.

The following fundamental fact says that hyperbolic geodesics are almost the same as Euclidean geodesics.

THEOREM 4.2 (Gehring-Hayman inequality). *There is an absolute constant $C < \infty$ to that the following holds. Suppose $\Omega \subset \mathbb{C}$ is hyperbolic and simply connected. Given two points in Ω, let γ be the hyperbolic geodesic connecting these two points and let σ be any other curve in Ω connecting them. Then $\ell(\gamma) \leq C\ell(\sigma)$.*

PROOF. Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that γ is the image of $I = [0,r] \subset \mathbb{D}$ for some $0 < r < 1$. Without loss of generality we may assume $r = r_N 1 - 2^{-N}$ for some N. Let

$$Q_n = \{z \in \mathbb{D} : 2^{-n-1} < |z-1| < 2^{-n}\},$$

and let

$$\gamma_n = \{z \in \mathbb{D} : |z-1| = 2^{-n}\},$$
$$z_n = \gamma_n \cap [0,1).$$

Let $Q'_n \subset Q_n$ be the sub-quadrilateral of points with $|\arg(1-z)| < \pi/6$. Each of these has bounded hyperbolic diameter and hence by Koebe's theorem its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart. In particular, this means that any curve in $f(Q_n)$ separating $f(\gamma_n)$ and $f(\gamma_{n+1})$ must cross $f(Q'_n)$ and hence has diameter $\gtrsim d_n$. Since Q_n has bounded modulus, so does $f(Q_n)$ and so Lemma 4.1 says that the shortest curve in $f(Q_n)$ connecting γ_n and γ_{n+1} has length $\ell_n \simeq d_n$. Thus any curve γ in Q connecting γ_n and γ_{n+1} has length at least ℓ_n, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\sigma)). \quad \square$$

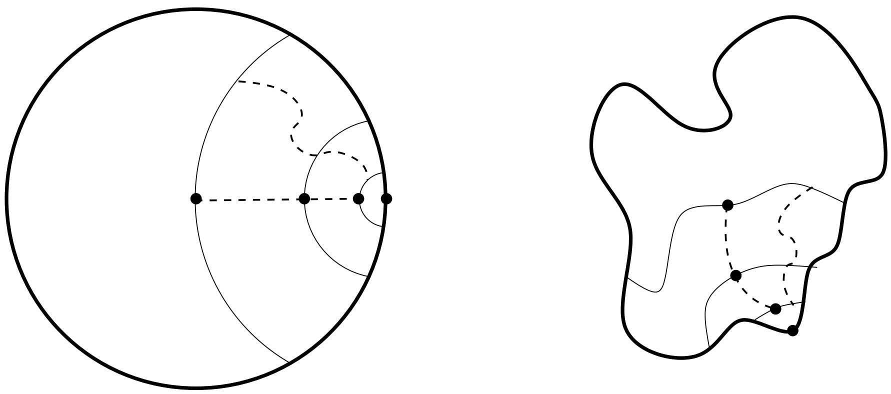


FIGURE 4.2. Proof of the Gehring-Hayman inequality.

If $f : \mathbb{D} \to \Omega$ is conformal define

$$a(r) = \text{area}(\Omega \setminus f(r \cdot \mathbb{D}).$$

If Ω has finite area (e.g., if it is bounded), then clearly $a(r) \searrow 0$ as $r \nearrow 1$.

LEMMA 4.3. *There is a $C < \infty$ so that the following holds. Suppose $f : \mathbb{D} \to \Omega$ and $\frac{1}{2} \leq r < 1$. Let $E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(rx)| \geq \delta$ for some $r < s < 1\}$. Then the extremal length of the path family $\mathscr{P}$ connecting $D(0,r)$ to E is bounded below by $\delta^2/Ca(r)$.*

PROOF. Let $z = f(sx)$ and suppose $w \in f(D(0,r))$. By the Gehring-Hayman estimate, the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them. But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe distortion theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ. Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then ρ is admissible for $f(\mathscr{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$. Thus $\lambda(\mathscr{P}) \geq \frac{\delta^2}{C^2 a(r)}$. $\square$

LEMMA 4.4. *Suppose $f : \mathbb{D} \to \Omega$ is conformal, and for $R \geq 1$,*

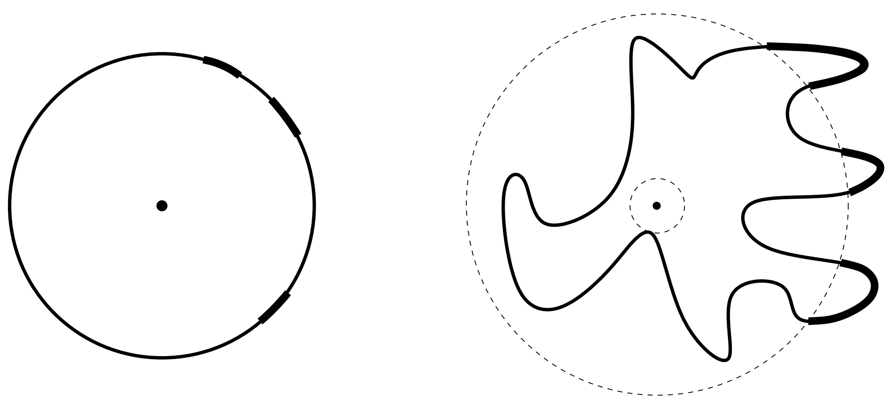
$$E_R = \{x \in \mathbb{T} : |f(x) - f(0)| \geq R \operatorname{dist}(f(0), \partial\Omega)\}.$$

Then E_R has capacity $O(1/\log R)$ if R is large enough.

PROOF. Assume $f(0)=0$ and $\operatorname{dist}(0,\partial\Omega)=1$ and let $\rho(z)=|z|^{-1}/\log R$ for $z\in\Omega\cap\{1<|z|<R\}$. Then ρ is admissible for the path family Γ connecting $D(0,1/2)$ to $\partial\Omega\setminus D(0,R)$ and $\iint\rho^2dxdy\le 2\pi/\log R$. By definition $M(\Gamma)\le 2\pi/\log R$ and $\lambda(\Gamma)\ge(\log R)/2\pi$. By the Koebe distortion theorem $f^{-1}(D(0,1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω. By Pfluger's theorem (Theorem 2.13),

$$\cap(E_r)\le\frac{2}{-2C_2+\log R},$$

which proves the result. $\square$



COROLLARY 4.5. *If $f:\mathbb{D}\to\Omega$ is conformal, then f has radial limits except on a set of zero capacity (and hence has finite radial limits a.e. on $\mathbb{T}$).*

PROOF. Let $E_{r,\delta}\subset\mathbb{T}$ be the set of $x\in\mathbb{T}$ so that $\operatorname{diam}(f(rx,x))>\delta$, and let $E_\delta=\cap_{0<r<1}E_{r,\delta}$. If f does not have a radial limit at $x\in\mathbb{T}$, then $x\in E_\delta$ for some $\delta>0$, and this has zero capacity by Lemma 4.3. Taking the union over a sequence of δ's tending to zero proves the result. The set where f has a radial limit ∞ has zero capacity by Lemma 4.4, so we deduce f has finite radial limits except on zero capacity. $\square$

Combining the last two results proves

COROLLARY 4.6. *Given $\varepsilon>0$ there is a $C<\infty$ so that the following holds. If $f:\mathbb{D}\to\Omega$ is conformal, $z\in\mathbb{D}$ and $I\subset\mathbb{T}$ is an arc that satisfies $|I|\ge\varepsilon(1-|z|)$ and $\operatorname{dist}(z,I)\le\frac{1}{\varepsilon}(1-|z|)$, then I contains a point w where f has a radial limit and $|f(w)-f(z)|\le C\operatorname{dist}(f(z),\partial\Omega)$.*

We can now prove:

THEOREM 4.7 (Carathéodory). *Suppose that $f : \mathbb{D} \to \Omega$ is conformal, and that $\partial\Omega$ is compact and locally path connected (for every $\varepsilon > 0$ there is a $\delta > 0$ so that any two points of $\partial\Omega$ that are within distance δ of each other can be connected by a path in $\partial\Omega$ of diameter at most ε). Then f extends continuously to the boundary of $\mathbb{D}$.*

PROOF. Suppose $\eta > 0$ is small. Since $\partial\Omega$ is compact $\Omega \setminus f(\{|z| < 1 - \frac{1}{n}\})$ has finite area that tends to zero as $n \nearrow \infty$. Thus if n is sufficiently large, this region contains no disk of radius η.

Choose $\{z_j\}$ to be n equally spaced points on the unit circle and using Lemma **??** choose interlaced points $\{w_j\}$ so that f has a radial limit $f(w_j)$ at w_j and this limit satisfies $|f(w_j) - f(rw_j)| \leq C\eta$ where $r = 1 - 1/n$. Then

$$\begin{aligned} |f(w_j) - f(w_{j+1})| &\leq |f(w_j) - f(rw_j)| \\ &\qquad + |f(rw_j) - f(rw_{j+1})| \\ &\qquad\qquad + |f(rw_{j+1}) - f(w_{j+1})| \\ &\leq C\delta, \end{aligned}$$

where the center term is bounded by Koebe's theorem and the other two by definition.

Fix $\varepsilon > 0$ and choose $\delta > 0$ as in the definition of locally connected. Thus if η is so small that $C\eta < \delta$, then the shorter arc of $\partial\Omega$ with endpoints $f(w_j)$ and $f(w_{j+1})$ can be connected in $\partial\Omega$ by a curve of diameter at most ε. Thus the image under f of the Carleson square with base I_j (the arc between w_j and w_{j+1}) has diameter at most $C\eta + \varepsilon$. This implies f has a continuous extension to the boundary. $\square$

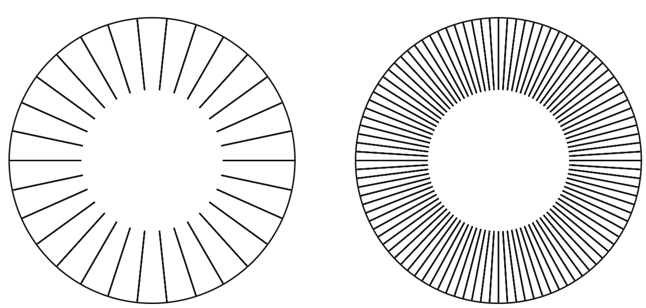


FIGURE 4.3. Radius two disks with 30 and 100 radial slits removed respectively. As the number of slits increases to infinity, the conformal maps onto these regions converge on compact sets to the identity, but each have modulus 2 on a set of fixed, positive logarithmic capacity.

Uniform convergence on compact subsets of $\mathbb{D}$ does not imply uniform convergence on the boundary (see Figure 4.5). However, it is true that the conformal boundary values will converge if the image domains have some parameterizations that converge. In other words, if a sequence of simply connected domains have boundaries with continuous parameterizations that converge uniformly to the continuous parameterization of the limiting domain, then we also get uniform convergence for the conformal parameterizations of the boundaries. (This is analogous to Carathédory's theorem: if a domain boundary has any continuous parameterization, then the conformal parameterization is also continuous.) See Theorem 4.9.

It is an inconvenient fact is that conformal maps do not have to extend continuously to the boundary. We noted above however, that radial do exist almost everywhere. Another convenient substitute for full continuity says that every conformal map is continuous on a subdomain of $\mathbb{D}$ whose boundary hits "most of" $\partial\mathbb{D}$. The precise statement requires a new definition.

Given a compact set $E \subset \mathbb{T}$ we will now define the associated "sawtooth" region W_E Suppose $\{I_n\}$ are the connected components of $\mathbb{T} \setminus E$ and for each n let $\gamma_n(\theta)$ be the circular arc in $\mathbb{D}$ with the same endpoints as I_n and which makes angle θ with I_n (so $\gamma_n(0) = I_n$ and $\gamma_n(\pi/2)$ is the hyperbolic geodesic with the same endpoints as I_n). Let $C_n(\theta)$ be the region bounded by I_n and $\gamma_n(\theta)$, and let $W_E(\theta) = \mathbb{D} \setminus \cup_n C_n(\theta)$. Let $W_E = W_E(\pi/8)$ (and let $W_E^* \subset \overline{\mathbb{D}}^c$ be its reflection across $\mathbb{T}$).

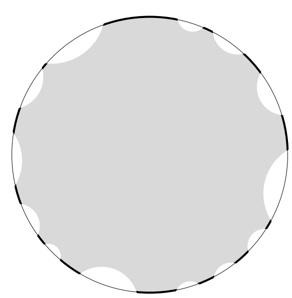


FIGURE 4.4. The sawtooth domain W_E

If $f : \mathbb{D} \to \Omega$ and $0 < r < 1$, then define

$$d_f(r) \quad = \quad \sup\{|f(z) - f(w)| : |z| = |w| = r \text{ and } |z - w| \le 1 - r\}. \tag{4.1}$$

If $\partial\Omega$ is bounded in the plane, then it is easy to see this goes to zero as $r \nearrow 1$, since otherwise any neighborhood of $\partial\Omega$ would contain infinitely many disjoint disks of a fixed, positive size by Koebe's theorem (Theorem 1.11).

LEMMA 4.8. *Suppose $f:\mathbb{D}\to\Omega\subset S^2$ is conformal. Then for any $\varepsilon>0$ there is a compact set $X\subset\mathbb{T}$ with $\operatorname{cap}(\mathbb{T}\setminus X)<\varepsilon$ such that f is continuous on $\overline{W_X}$.*

PROOF. By applying a square root and a Möbius transformation, we may assume that $\partial\Omega$ is bounded in the plane. Given $r<1$ let

$$E(\delta,r)=\{x\in\mathbb{T}:|f(sx)-f(tx)|>\varepsilon \text{ for some } r<s<t<1\}$$

and note that by Pfluger's theorem (Theorem 2.13) and Lemma 4.3

$$\operatorname{cap}(E(\delta,r))\le\exp(-\pi\varepsilon^2/Ca(r)),$$

where $a(r)=\operatorname{area}(f(\mathbb{D})\setminus f(r\cdot\mathbb{D}))$, as before. Moreover, this set is open since f is continuous at the points sx and tx. Fix $\varepsilon>0$, take $\varepsilon_n=2^{-n}$, and choose r_n so close to 1 that $\operatorname{cap}(E_n)\equiv\operatorname{cap}(E(\varepsilon_n,r_n))\le\varepsilon 2^{-n}$. If we define $X=\mathbb{T}\setminus\cup_{n>1}E_n$, then X is closed and $\mathbb{T}\setminus X$ has capacity $\le\varepsilon$ by subadditivity.

To show f is continuous at every $x\in\overline{W_X}$, we want to show that $|x-y|$ small implies $|f(x)-f(y)|$ is small. We only have to consider points $x\in\partial W_X\cap\mathbb{T}$. First suppose $y\in\partial W_X\cap\mathbb{T}$. Choose the maximal n so that $s=|x-y|\le 1-r_n$. Then $x,y\notin E_n$, so

$$|f(x)-f(y)|\le|f(x)-f(sx)|+|f(sx)-f(sy)|+|f(sy)-f(y)|.$$

The first and last terms on the right are $\le\varepsilon_{n-1}$ by the definition of X. The middle term is at most $d_f(1-s)$ (defined in (4.1), which tends to 0 as $s\to 0$. Thus $|f(x)-f(y)|$ is small if $|x-y|$ is.

Now suppose $x\in\partial W_X\cap\mathbb{T}$, $y\in\partial W_X\setminus\mathbb{T}$. From the definition of W_X it is easy to see there is a point $w\in\partial W_X\cap\mathbb{T}$ such that $|w-y|\le 2(1-|y|)\le 2|x-y|$. For the point w we know by the argument above that $|f(x)-f(w)|$ is small. On the other hand,

$$|f(y)-f(w)|\le|f(y)-f(|y|w)|+|f(|y|w)-f(w)|.$$

The first term is bounded by $Cd_f(|y|)$ and the second is small since $w\notin E_n$. Thus $|f(x)-f(y)|$ is small depending only on $|x-y|$. Hence f is continuous on $\overline{W_X}$. □

However, even though uniform convergence on compacta of a sequence of conformal maps does not generally imply convergence on all of $\partial\mathbb{D}$, it is true that the conformal boundary values will converge if the image domains have some parameterizations that converge. In other words, if a sequence of simply connected domains have boundaries with continuous parameterizations that converge uniformly to the continuous parameterization of the

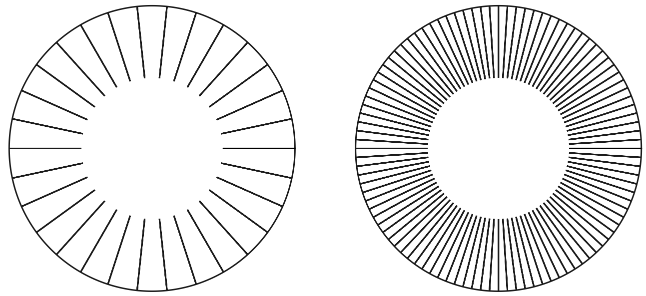


FIGURE 4.5. Radius two disks with 30 and 100 radial slits removed respectively. As the number of slits increases to infinity, the conformal maps onto these regions converge on compact sets to the identity, but each have modulus 2 on a set of fixed, positive logarithmic capacity.

limiting domain, then we also get uniform convergence for the conformal parameterizations of the boundaries. (This is analogous to Carathéodory's theorem: if a domain boundary has any continuous parameterization, then the conformal parameterization is also continuous.)

LEMMA 4.9. *Suppose $\{f_n\}$ are conformal maps of $\mathbb{D} \to \Omega_n$ that converge uniformly on compact subsets of $\mathbb{D}$ to a conformal map $f : \mathbb{D} \to \Omega$. Suppose that the boundary of each Ω_n is the homeomorphic image $\partial\Omega_n = \sigma_n(\mathbb{T})$ and that $\{\sigma_n\}$ converges uniformly on $\mathbb{T}$ to a homeomorphism $\sigma : \mathbb{T} \to \partial\Omega$. Then $f_n \to f$ uniformly on the $\overline{\mathbb{D}}$.*

PROOF. Fix $\varepsilon > 0$ and choose n so large that if we divide $\mathbb{T}$ into n equal sized intervals $\{J_j\}_1^n$, then σ maps each of them to a set I_j of diameter at most $\varepsilon/2$. Let $I_j^k = f_k(J_j)$. Because $\sigma_k \to \sigma$ uniformly, the sets I_j all have diameter at most ε, if k is large enough.

Next choose $\eta > 0$ so small that if $k, m > 1/\eta$ and $\sigma_m(J_j)$ and $\sigma_k(J_i)$ contain points at most distance $C\eta$ apart, then J_i and J_k are the same or adjacent to each other. We can do this because of the uniform convergence and the fact that σ is 1-to-1. By passing to the limit the same property holds if we replace σ_m by σ.

Next choose m so large that $f(\mathbb{D}) \setminus f(\{|z| < 1 - \frac{1}{m}\})$ is contained in an η-neighborhood of $\partial\Omega$. Choose m points $\{z_j\}$ equally spaced on the circle $|z| = 1 - \frac{1}{m}$,and let $K_j^m \subset \mathbb{T}$ be the arc centered at $z_j/|z_j|$ of length $4\pi/m$. Fix a small number $\delta > 0$ (δ will be determined below, depending only on η).By Lemma 4.4 choose a point $w_j \in K_j^m$ so that $|w_j - z_j| \leq 2/m$ and

$$|f(w_j) - f(w_j(1 - \frac{1}{m}))| \leq C\delta.$$

Similarly, choose points $w_j^k \in K_j^m$ so that

$$|f_k(w_j^k) - f_k(z_j)| \leq 2C\delta.$$

This is possible since $f_k \to f$ uniformly on the compact set $\{|z| \leq 1 - \frac{1}{m}\}$ and thus$\partial f_k(\mathbb{D})$ is contained in an2δ-neighborhood of $\partial\Omega$ for k large enough and $\partial\Omega_k$ is contained in a δ-neighborhood of $\partial\Omega$ because of the uniform convergence of the parameterizations.

By taking m larger, if necessary, we can also arrange that each I_j contains at least one of the points $f(z_m/|z_m|)$. Thus each $f(K_j^m)$ is mapped into the union of at most 2 of the I_j and hence its image has diameter at most 2ε. Also, the points $f(w_p^k)$ and $f(w_{p+1}^k)$ are at most $C\delta$ apart, so belong to the same or adjacent sets I_j. Thus $f_k(K_p)$ is a union of at most 4 such adjacent sets and hence has diameter $O(\varepsilon)$.

For each w_p^k there is an arc J_j so that $f_k(w_p^k) \subset \sigma_k(J_j)$. Similarly, there is an arc J_i so that $f(w_p) \in I_i = \sigma(J_i)$. Since $f_k \to f$ uniformly on the finite set $\{z_n\}$, we have, for k sufficiently large

$$\begin{aligned} |f_k(w_n^k) - f(w_n)| &\leq |f_k(w_n^k) - f_k(z_n)| \\ &\quad +|f_k(z_n) - f(z_n)| \\ &\quad\quad +|f(z_n) - f(w_n)| \\ &\leq (2C+1+C)\delta. \end{aligned}$$

This is less than η if δ is small enough. Since I_i and I_j each have diameter at most ε,there union has diameter $< 2\varepsilon$ and the union of the intervals adjacent to these is at most 4ε. Similarly for I_i^k and J_j^k. Thus $f_k(K_p)$ and $f(K_p)$ are contained in $O(\varepsilon)$-neighborhoods of each other. Thus $f_k \to f$ uniformly on $\mathbb{T}$. By the maximum principle, this implies uniform convergence on the closed disk, as desired. □

COROLLARY 4.10. *If $\{f_n\}$ are homeomorphisms that converge uniformly to a homeomorphism f then $M(f_n(Q)) \to M(f(Q))$*

PROOF. ??? □

The convergence of modulus need not occur if the quadrilaterals merely converge in the Hausdorff metric. See Figure 1.23

FIGURE SHOWING QUADS CONVERGING IN HAUSDORFF METRIC BUT MODULI NOT CONVERGING

Even without the convergence of parameterizations, uniform convergence on compact sets implies convergence of a subsequence on on “most” of the boundary. See [**?**].

(*reminder -Cite Lundberg and David Hamilton*)

5. Harmonic measure

Suppose Ω is a planar domain bounded by a Jordan curve, $z \in \Omega$ and $E \subset \partial\Omega$ is Borel. Suppose $f : \mathbb{D}\Omega$ is conformal and $f(0) = z$ (by the Riemann mapping theorem there is always such a map). By Carathéodory's theorem, f extends continuously (even homeomorphically) to the boundary, so $f^{-1}(E) \subset \mathbb{T}$ is also Borel. We define "the harmonic measure of the set E for the domain Ω, with respect to the point z" as

$$\omega(z, E, \Omega) = |E|/2\pi,$$

where $|E|$ denotes the Lebesgue 1-dimensional measure of E. This depends on the choice of the Riemann map f, but any two maps, both sending 0 to z, will differ only by a pre-composition with a rotation. Thus the two possible pre-images of E differ by a rotation and hence have the same Lebesgue measure. If we fix E and Ω, then $\omega(z, E, \Omega)$ is a harmonic function of z (Exercise 1.12), giving rise the name "harmonic measure". Since we always have $0 \leq \omega(z, E, \Omega) \leq 1$, we can deduce that if E has harmonic measure with respect to one point z in Ω then it has zero harmonic measure with respect to all points (Exercise 1.13).

If $\partial\Omega$ is merely locally connected, then Carathéodory's theorem still implies that the Riemann map f has a continuous extension to the boundary, so the same definition of harmonic measure works. We can define harmonic measure for general simply connected domains, by taking an increasing union of domains with locally connected boundaries as given by Lemma 4.8, but we will postpone this discussion until later, as we will postpone the discussion of harmonic measure on multiply connected domains (defined via covering maps). For the moment, Jordan domains and locally connected sets will provide sufficiently many interesting examples.

We want estimate harmonic measure in terms of extremal length. We have already seen how to relate extremal length to logarithmic capacity, and the following relates the latter to harmonic measure:

LEMMA 5.1. *For any compact $E \subset \mathbb{T}$,*

$$\text{cap}(E) \geq \frac{1}{1 + \log 2 + \pi + \log \frac{1}{|E|}}.$$

If $E \subset \mathbb{T}$ has positive Lebesgue measure, then it has positive capacity. In particular, if $E \subset \mathbb{T}$ is an arc, then

$$\text{cap}(E) \leq \frac{1}{\log 4 + \log \frac{1}{|E|}}.$$

For arcs of small measure, the two bounds are comparable.

PROOF. If μ is Lebesgue measure restricted to E, then clearly the corresponding potential function is less than potential function of an arc I of the same measure evaluated at the center x of that arc. Since $\frac{2}{\pi}t \leq |x-y| \leq t$ if the arclength between $x,y \in \mathbb{T}$ is t, this value is at most

$$\int_I \log \frac{2}{|x-y|} dy \leq 2\int_0^{|E|/2} \log \frac{\pi}{t} dt = |E| \log \frac{2}{|E|} + (1+\pi)|E|$$

If we normalize the measure to have mass one, then we get

$$U_\mu \leq \log \frac{2}{|E|} + 1 + \pi = \log \frac{1}{|E|} + 1 + \log 2 + \pi.$$

If E is an arc, then the center x of the arc is at most distance $|E|/2$ from any other point of the arc, and so

$$U_\mu(x) \geq \log \frac{2}{|E|/2} = \log \frac{4}{|E|} = \log \frac{1}{|E|} + \log 4,$$

for any probability measure supported on E. This gives the desired estimate. $\square$

The following is the fundamental estimate for harmonic measure, from which all other estimates flow (at least, all the ones that we will use).

THEOREM 5.2. *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with* $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ *and $E \subset \partial\Omega$. Let Γ be the family of curves in Ω which connects $D(z_0, 1/2)$ to E. Then*

$$\omega(z_0, E, \Omega) \leq C \exp(-\pi\lambda(\Gamma)).$$

If $E \subset \partial\Omega$ is an arc then the two sides are comparable.

PROOF. Let $f : \mathbb{D} \to \Omega$ be conformal. By Koebe's $\frac{1}{4}$-theorem (Theorem 1.11), the disk $D(z, \frac{1}{2})$ in Ω maps to a smooth region K in the unit disk that contains the origin, and ∂K is uniformly bounded away from both the origin and the unit circle. Thus by Pfluger's theorem applied to the curve family Γ_X connecting K and the compact set $X = f^{-1}(E)$,

$$\frac{1}{\operatorname{cap}(X)} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq \frac{1}{\operatorname{cap}(X)} + C_2(K),$$

for constants C_1, C_2 that are bounded independent of all our choices.

By Lemma 5.1 the right-hand side of

$$1 + \log 4 + \log \frac{1}{|X|} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq 1 + \log 2 + \log \frac{1}{|X|} + C_2(K).$$

holds in general, and the left-hand side also holds if X is an interval. Multiply by -1 and exponentiate to get

$$\frac{|X|}{2e^{1+\pi+C_2}} \leq \exp(-\pi\lambda(\Gamma_X)) \leq \frac{|X|}{4e^{C_1}}$$

under the same assumptions. Now use $\omega(z,E,\Omega) = \omega(0,X,\mathbb{D}) = |X|/2\pi$ to deduce the result. $\square$

One of the most famous and most useful applications of this result is

COROLLARY 5.3 (Ahlfors distortion theorem). *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with $\operatorname{dist}(z_0,\partial\Omega) \geq 1$ and $x \in \partial\Omega$. For each $0 < t < 1$ let $\ell(t)$ be the length of $\Omega \cap \{|w-x| = t\}$. Then there is an absolute $C < \infty$, so that*

$$\omega(z_0, D(x,r), \Omega) \leq C\exp(-\pi\int_r^1 \frac{dt}{\ell(t)}).$$

PROOF. Let K be the disk of radius $1/2$ around z_0 and let Γ be the family of curves in Ω which connects $D(x,r)\cap\partial\Omega$ to K. Define a metric ρ by $\rho(z) = 1/\ell(t)$ if $z \in C_t = \{z \in \Omega : |x-z| = t\}$ and $\ell(t)$ is the length of C_t. Any curve $\gamma \in \Gamma$ has ρ-length at least

$$L = \int_r^{1/2} \frac{dt}{\ell(t)},$$

and

$$A = \iint_\Omega \rho^2 dxdy \geq \int_r^{1/2}\int_{C_r\cap\Omega} \ell(z)^{-2} r dr d\theta = \int \ell(z)^{-1} dr = L.$$

Therefore

$$\lambda(\Gamma) \geq A/L^2 = 1/L,$$

and this proves the result. $\square$

COROLLARY 5.4 (Beurling's estimate). *There is a $C < \infty$ so that if Ω is simply connected, $z \in \Omega$ and $d = \operatorname{dist}(z,\partial\Omega)$ then for any $0 < r < 1$ and any $x \in \partial\Omega$,*

$$\omega(z, D(x,rd), \Omega) \leq Cr^{1/2}$$

PROOF. Apply Corollary 5.3 at x and use $\theta(t) \leq 2\pi t$ to get

$$\exp(-\pi\int_{rd}^d \frac{dt}{\theta(t)t}) \leq C\exp(-\frac{1}{2}\log r) \leq C\sqrt{r}.$$

$\square$

COROLLARY 5.5. *There is an $R < \infty$ so that for any Ω is a Jordan domain and any $z \in \Omega$*

$$\omega(z, \partial\Omega \setminus D(z, R\,\mathrm{dist}(z, \partial\Omega), \Omega) \leq 1/2.$$

PROOF. Rescale so $z = 1$ and $\mathrm{dist}(z, \partial\Omega) = 1$. Then apply $w \to 1/w$ which fixes z and maps $\partial\Omega \setminus D(z, R)$ into $D(0, 1/R - 1)$. Then Lemma 5.4 implies the result if $R \geq 4C^2 + 1$ (C is as in Lemma 5.4). $\square$

COROLLARY 5.6. *For any Jordan domain and any $\varepsilon > 0$,*

$$\omega(z, \partial\Omega \cap D(z, (1+\varepsilon)\,\mathrm{dist}(z, \partial\Omega)), \Omega) > C\varepsilon,$$

for some fixed $C > 0$.

PROOF. Renormalize so $z = 0$ and 1 is a closest point of $\partial\Omega$ to z. By Corollary 5.5, the set $E = \partial\Omega \cap D(0, 1+\varepsilon)$ has harmonic measure at least $1/2$ from the point $1 - \varepsilon/R$. Since $\omega(z, E, \Omega)$ is a positive, harmonic function on $\mathbb{D}$, Harnack's inequality says it is larger than $C\varepsilon/R$ at the origin. $\square$

This is a weak version of the Beurling projection theorem which says that the sharp lower bound is given by the slit disk $D(0, 1+\varepsilon) \setminus [1, 1+\varepsilon)$. The harmonic measure of the slit in this case can be computed as an explicit function of ε because this domain can be mapped to the disk by sequence of elementary functions.

6. Exercises

EXERCISE 1.1. If Ω is a Jordan domain and $E, F \subset \partial\Omega$ are disjoint closed subarcs, then there is a conformal map of Ω to some rectangle so that E and F map to opposite sides.

EXERCISE 1.2. If Ω is a topological annulus bounded by two Jordan curves, show that it can be conformally mapped to a round annulus.

EXERCISE 1.3. Let $E \subset \mathbb{C}$ be a closed set and z a point not in E. Compute the modulus of the path family connecting E to $\{z\}$.

EXERCISE 1.4. Let $E_n \subset \mathbb{T}$ be defined by $\{z : \mathrm{Re}(z^n) > 0\}$. Show that $\mathrm{Cap_{log}(E_n)} \to \mathrm{Cap_{log}}(\mathbb{T})$ as $n \to \infty$. Since $\mathbb{T} \setminus E_n$ clearly has the same capacity as E_n, this implies capacity is not additive.

EXERCISE 1.5. Show that the linear fractional transformations that map $\mathbb{D}$ 1-to-1, onto itself are exactly those of the form $z \to \lambda(z-a)/(1-\bar{a}z)$ where $|a| < 1$ and $|\lambda| = 1$.

EXERCISE 1.6. Show a hyperbolic ball in the disk is also a Euclidean ball, but the hyperbolic and Euclidean centers are different (unless they are both the origin). Compute the Euclidean center and radius of a hyperbolic ball of radius r centered at z in $\mathbb{D}$.

EXERCISE 1.7. Show that the only isometries of the hyperbolic disk are Möbius transformations and their reflections across $\mathbb{R}$.

EXERCISE 1.8. Show that the domain U constructed in the proof of Theorem 3.6 is equal to $\mathbb{H}_u$.

EXERCISE 1.9. If $\{f_n\}$ are holomorphic functions on a domain Ω that converge uniformly on compact sets to f and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.

EXERCISE 1.10. Suppose E is compact and supports a positive measure μ so that $\mu(D(x,r)) \leq \varphi(r)$, where

$$\sum_{n=0}^{\infty} n\varphi(2^{-n}) < \infty,$$

Then E has positive capacity.

EXERCISE 1.11. If $E \subset \mathbb{T}$ is compact and has positive Hausdorff dimension, then it has positive capacity.

EXERCISE 1.12. Suppose Ω is a planar Jordan domain and $E \subset \partial\Omega$ is Borel. Prove that $\omega(z,E,\Omega)$ is a harmonic function of z.

EXERCISE 1.13. Suppose Ω is a planar Jordan domain and $E \subset \partial\Omega$ is Borel. Show that if $\omega(z,E,\Omega) = 0$ for some $z \in \Omega$, then it is zero on all of Ω.

EXERCISE 1.14. If $\{p_k\}_{k=1}^n$ are non-negative numbers and $\sum_{k=1}^n p_k = 1$, show that $h = -\sum_{k=1}^n p_k \log p_k$ is maximized uniquely when $p_k = 1/n$ for all k.

EXERCISE 1.15. Suppose $g(z) = \frac{1}{z} + b_0 + b_1 z + \dots$ is univalent in $\mathbb{D}$. Then $\sum_{n=0}^{\infty} n|b_n|^2 \leq 1$. In particular, $|b_1| \leq 1$. This is the area theorem.

EXERCISE 1.16. Use the area theorem to prove that if $\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent on the unit disk with $\varphi'(0) = 1$, then $|a_2| \leq 2$. This is the case $n = 2$ of the Bieberbach conjecture (later to become deBrange's theorem [], []).

EXERCISE 1.17. Use the previous exercise to give a second proof of the Koebe $\frac{1}{4}$-theorem.

EXERCISE 1.18. If f is conformal on the disk, and $\varphi = \log f'$, then $|\varphi'(z)| \leq 6/(1-|z|^2)$ for all $z \in \mathbb{D}$.

SOLUTION. Define

$$F(z) = \frac{f(\tau(z)) - f(w)}{(1-|w|^2)f'(w)},$$

where

$$\tau(z) = \frac{z+w}{1-\bar{w}z}.$$

Then F is conformal, $F(0) = 0$ and $F'(0) = 1$, so Lemma **??** says that $|F''(0)| \leq 4$. A computation shows

$$F''(0) = \frac{f''(z)}{f'(z)}(1-|z|^2) + (-2\bar{z}),$$

and $\varphi' = (\log f')' = f''/f'$, so

$$|\varphi'|(1-|z|^2) \leq |F''(0)| + |2z| \leq 4+2 = 6.$$

EXERCISE 1.19. If φ is conformal on $\mathbb{D}$ then

$$\frac{1-|z|}{(1+|z|)^3} \leq |\varphi'(z)| \frac{1+|z|}{(1-|z|)^3}.$$

This is the distortion theorem. See e.g., Theorem I.4.5 of [**?**].

EXERCISE 1.20. If φ is conformal on $\mathbb{D}$ then

$$\frac{|z|}{(1+|z|)^2} \leq |\varphi(z)| \frac{|z|}{(1-|z|)^2}.$$

This is the growth theorem. See e.g., Theorem I.4.5 of [**?**].

EXERCISE 1.21.

EXERCISE 1.22.

EXERCISE 1.23.

Solutions (eventually move to end of book)

SOLUTION. 1.1 First map Ω to the disk by the Riemann mapping theorem. Then use a Möbius transformation to arrange for the images of E and F to be arcs centered at ± 1 and symmetric with respect to the real line. Then the Schwarz-Christoffel formula gives a map to the desired rectangle.

SOLUTION. Use uniformization theorem to get covering by disk. Then use Riemann map to get covering by vertical strip with deck transformations being vertical translations. Then use exponential map to send strip to annulus and collapsing orbits to single points.

SOLUTION. Take an annulus around the point that is disjoint from E, but has modulus close to zero, and use monotonicity.

SOLUTION. The logarithmic capacity of the circle is $1/\log 2$. Compute the potential of Lebesgue measure restricted to E_n and show that it is bounded by $1/2\log 2 + o(1)$ Therefore approximately twice this measure is still admissible, which means the capacity of E_n is close to the capacity of the circle, if n is large..

SOLUTION.

SOLUTION. 1.9 We may assume $\{z_n\}$ are contained in some disk $D \subset \Omega$ around z. Let $E = \{z_n\} \cup \{z\}$. This is a compact set so it has a positive distance d from $\partial\Omega$. The points within distance $d/2$ of E form a compact set F on which the functions $\{f_n\}$ are uniformly bounded on E, say by M. By the Cauchy estimate the derivatives are bounded by a constant M' on E. Thus

$$|f(z) - f_n(z_n) \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_n)| \leq |f(z) - f_n(z)| + M'|z - z_n|,$$

and both terms on the right tend to zero by hypothesis.

SOLUTION. 1.10 The condition easily implies U_μ is bounded, hence $\text{supp}(\mu)$ has positive capacity.

SOLUTION. 1.11 This follows from Frostman's theorem (Theorem **??**) since if $\dim(E) > 0$ then E supports a measure that satisfies $\mu(D(x,r)) = O(r^\varepsilon)$ for some $\varepsilon > 0$ and $\sum_n 2^{-\varepsilon n} < \infty$.

SOLUTION. 1.12 Show that $\omega(z, E, \mathbb{D})$ must agree with the Poisson integral of the indicator function of E (the function that is 1 on E and 0 off E). This holds because the derivative of a Möbius transformation of the disk to itself has absolute value equal to the Poisson kernel when restricted to the unit circle.

SOLUTION. 1.13 By the maximum principle, a harmonic function that attains a minimum or maximum is constant.

SOLUTION. 1.15 For $0 < r < 1$ let $D_r = \mathbb{C} \setminus g(D(0,r))$. If $z = g(w)$ and $w = e^{i\theta}$ then $dw = iwd\theta$, so by (**??**),

$$\text{area}(D_r) = \iint_{D_r} dxdy = \frac{1}{2i}\int_{\partial D_r} \bar{z}dz = \frac{-1}{2i}\int_{\partial D(0,r)} \bar{g}(w)g'(w)dw.$$

To evaluate the right hand side note that

$$\begin{aligned} g(z) &= \frac{1}{z} + b_0 + b_1 z + \ldots, \\ g'(z) &= \frac{1}{z^2} + 0 + b_1 + 2b_2 z + \ldots, \end{aligned}$$

so that

$$\begin{aligned} \int_{|w|=r} \bar{g}(w)g'(w)dw &= i\int \bar{g}(w)g'(w)wd\theta \\ &= i\int (\frac{1}{\bar{w}} + \bar{b}_0 + \bar{b}_1\bar{w} + \ldots)(-\frac{1}{w} + b_1 w + 2b_2 w + \ldots)d\theta \\ &= 2\pi i(-\frac{1}{r^2} + |b_1|^2 r^2 + 2|b_2|r^4 + \ldots) \end{aligned}$$

Thus,

$$0 \le \text{area}(D_r) = \pi(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}).$$

Taking $r \to 1$ gives the result.

SOLUTION. 1.16 Let $F(z) = z\sqrt{f(z^2)/z^2}$. Then the quantity inside the square root is even and doesn't vanish in $\mathbb{D}$, so F is odd, univalent and

$$F(z) = z + \frac{a_2}{2}z + \ldots.$$

Thus

$$g(z) = \frac{1}{F(z)} = \frac{1}{z} - \frac{a_2}{2}z + \ldots,$$

is univalent and satisfies Theorem **??**, so $|a_2| \le 2$.

SOLUTION. 1.17 By pre-composing with a Möbius transformation and post-composing by a linear map, we may assume $z = 0$, $f(0) = 0$ and $f'(0) = 1$. Then the right hand inequality is just Schwarz's lemma applied to f^{-1}. To prove the left hand inequality, suppose f never equals w in $\mathbb{D}$. Then

$$\begin{aligned} g(z) &= \frac{wf(z)}{w - f(z)} \\ &= w(z + a_2z^2 + \ldots)\frac{1}{w}[(1 + \frac{1}{w}(z + a_2z^2 + \ldots) + \frac{1}{w^2}(z + a_2z^2 + \ldots)^2 + \ldots)] \\ &= z + (a_2 + \frac{1}{w})z^2 + \ldots, \end{aligned}$$

is univalent with $f(0) = 0$ and $f'(0) = 1$. Applying Corollary 1.16 to f and g gives

$$\frac{1}{|w|} \le |a_2| + |a_2 + \frac{1}{w}| \le 2 + 2 = 4.$$

Thus the omitted point w lies outside $D(0, 1/4)$, as desired.

SOLUTION.

SOLUTION. 1.19 Fix a point $w \in \mathbb{D}$ and write the Koebe transform of f,

$$F(z) = \frac{f(\tau(z)) - f(w)}{(1 - |w|^2)f'(w)},$$

where

$$\tau(z) = \frac{z + w}{1 - \bar{w}z}.$$

This is univalent, so by Corollary 1.16, $|a_2(w)| \leq 2$. Differentiation and setting $z = 0$ shows

$$F'(z) = \frac{f'(\tau(z))\tau'(z)}{(1-|w|^2)f'(w)},$$
$$F''(z) = \frac{f''(\tau(z))\tau'(z)^2 + f'(\tau(z))\tau''(z)}{(1-|w|^2)f'(w)},$$
$$\tau'(0) = 1-|w|^2, \tau''(0) = -2(1-|w|^2),$$
$$F''(0) = \frac{f''(w)}{f(w)}(1-|w|^2) - 2\bar{w}.$$

This implies that the coefficient of z^2 (as a function of w) in the power series of F is

$$a_2(w) = \frac{1}{2}((1-|w|^2)\frac{f''(w)}{f'(w)} - 2\bar{w}).$$

Using $|a_2| \leq 2$ and multiplying by $w/(1-|w|^2)$, we get

$$|\frac{wf''(w)}{f'(w)} - \frac{2|w|^2}{1-|w|^2}| \leq \frac{4|w|}{1-|w|^2}.$$

Thus

$$\frac{2|w|^2 - 4|w|}{1-|w|^2} \leq \frac{wf''(w)}{f'(w)} \leq \frac{4|w| + 2|w|^2}{1-|w|^2}.$$

Now divide by $|w|$ and use partial fractions,

$$\frac{-1}{1-|w|} + \frac{-3}{1+|w|} \leq \frac{1}{|w|}\frac{wf''(w)}{f'(w)} \leq \frac{3}{1-|w|} + \frac{1}{1+|w|}$$

$$\begin{aligned} \frac{\partial}{\partial r}\log|f'(re^{i\theta})| &= \frac{\partial}{\partial r}\mathrm{Re}\log f'(z) \\ &= \mathrm{Re}\frac{z}{|z|}\frac{\partial}{\partial z}\log f'(z) \\ &= \frac{1}{|z|}\mathrm{Re}(\frac{zf''(z)}{f'(z)}) \end{aligned}$$

Since $w = re^{i\theta}$ and $f'(0) = 1$, we can integrate to get

$$\log(1-r) - 3\log(1+r) \leq \log|f'(re^{i\theta})| \leq -3\log(1-r) + \log(1+r).$$

Exponentiating gives the result.

SOLUTION. 1.23

CHAPTER 2

Geometric properties of quasiconformal maps

In this chapter we define quasiconformal maps and deduce a variety of properties including compactness. In a later chapter we will study analytic properties, such as differentiability almost everywhere.

1. Distortion of smooth maps

Conformal maps preserves angles; quasiconformal maps can distort angles, but only in a controlled way. To make this distinction more precise we must have a way to measure angle distortion and we start with a discussion of linear maps.

Consider the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax+by, cx+dy).$$

Let M^T denote the transpose of the real matrix M, i.e., its reflection over the main diagonal. Then

$$M^T \cdot M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} \equiv \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is positive and symmetric and hence has two positive eigenvalues λ_1, λ_2, assuming M in non-degenerate. The square roots $s_1 = \sqrt{\lambda_1}$, $s_2 = \sqrt{\lambda_2}$ are the singular values of A (without loss of generality we assume $s_1 \geq s_2$). Then

$$M = U \cdot \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \cdot V,$$

where U, V are rotations. Thus M maps the unit circle to an ellipse whose major and minor axes have length s_1 and s_2. Thus M preserves angles iff it maps the unit circle to a circle iff $s_1 = s_2$. Otherwise M distorts angles and we let $D = s_1/s_2$ denote the dilatation of the linear map M. This is the eccentricity of the image ellipse and is ≥ 1, with equality iff M conformal.

The inverse of a linear map with singular values $\{s_1, s_2\}$ has singular values $\{\frac{1}{s_2}, \frac{1}{s_1}\}$ and hence dilatation $D = (1/s_2)/(1/s_1) = s_1/s_2$. Thus the dilatation of a linear map and its inverse are the same.

Given two linear maps M,N with singular values $s_1 \geq s_2$ and $t_1 \geq t_2$ respectively, the singular values of the composition MN are trapped between s_1t_1 and s_2t_2 (this occurs for the maximum singular values since they give the operator norms of the matrices and these are multiplicative; a similar argument works for the minimum singular values and the inverse maps). Thus the dilation is less than $(s_1t_1)/(s_2t_2)$ i.e., dilatations satisfy

$$D_{M\circ N} \leq D_M \cdot D_N.$$

The dilatation D can be computed in terms of a,b,c,d as follows. The eigenvalues λ_1, λ_2 are roots of the

$$0 = \det(M^T \cdot M - \lambda I),$$

which is the same as

$$0 = (E-\lambda)(G-\lambda) - F^2 = EG - F^2 - (E+G)\lambda + \lambda^2.$$

Thus

$$\begin{aligned}
\lambda_1\lambda_2 &= EG - F^2 \\
&= (a^2+c^2)(b^2+d^2) - (ab+cd)^2 \\
&= a^2b^2 + a^2d^2 + c^2b^2 + d^2c^2 - (a^2b^2 + 2abcd + c^2d^2) \\
&= a^2d^2 + c^2b^2 - 2abcd \\
&= (ad-bc)^2
\end{aligned}$$

Similarly,

$$\lambda_1 + \lambda_2 = E + G = a^2 + b^2 + c^2 + d^2.$$

The values of λ_1, λ_2 can be found using the quadratic formula:

$$\begin{aligned}
\{\lambda_1, \lambda_2\} &= \frac{1}{2}[E + G \pm \sqrt{(E+G)^2 - 4(EG - F^2)}] \\
&= \frac{1}{2}[E + G \pm \sqrt{(E-G)^2 + 4F^2}].
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\lambda_1}{\lambda_2} &= \frac{E+G+\sqrt{(E-G)^2+4F^2}}{E+G-\sqrt{(E-G)^2+4F^2}} \\
&= \frac{(E+G+\sqrt{(E-G)^2+4F^2})^2}{(E+G)^2-(E-G)^2-4F^2} \\
&= \frac{(E+G+\sqrt{(E-G)^2+4F^2})^2}{4(EG+F^2)}.
\end{aligned}$$

and hence

$$D = \frac{s_1}{s_2} = \sqrt{\frac{\lambda_1}{\lambda_2}} \quad = \quad \frac{E+G+\sqrt{(E-G)^2+4F^2}}{2\sqrt{EG+F^2}}.$$

This formula can be made simpler by complexifying. Think of the linear map M on $\mathbb{R}^2$ as a map f on $\mathbb{C}$:

$$x+iy \to ax+by+i(cx+dy) = u(x,y)+iv(x,y) = f(x+iy)$$

Then

$$M = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and we define

$$f_z = \frac{1}{2}(f_x - if_y) = \frac{1}{2}(u_x+v_y) + \frac{i}{2}(v_x - u_y),$$

$$f_{\bar{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}(u_x-v_y) + \frac{i}{2}(v_x + u_y).$$

Some tedious arithmetic now shows that

$$\begin{aligned} 4|f_z|^2 &= (u_x+v_y)^2 + (v_x-u_y)^2 \\ &= u_x^2 + 2u_xv_y + v_y^2 + v_x^2 - 2v_xu_y + u_y^2 \end{aligned}$$

$$\begin{aligned} 4|f_{\bar{z}}|^2 &= (u_x-v_y)^2 + (v_x+u_y)^2 \\ &= u_x^2 - 2u_xv_y + v_y^2 + v_x^2 + 2v_xu_y + u_y^2 \end{aligned}$$

so

$$(|f_z|+|f_{\bar{z}}|)(|f_z|-|f_{\bar{z}}|) = |f_z|^2 - |f_{\bar{z}}|^2 \quad = \quad u_xv_y - v_xu_y = s_1s_2 = \det(M).$$

In particular, if we assume M is orientation preserving and full rank, then $\det(M) > 0$ and we deduce $|f_z| > |f_{\bar{z}}|$. Similarly,

$$\begin{aligned} (|f_z|+|f_{\bar{z}}|)^2 + (|f_z|-|f_{\bar{z}}|)^2 &= 2(|f_z|^2+|f_{\bar{z}}|^2) \\ &= u_x^2+v_x^2+u_y^2+v_x^2 \\ &= E+G \\ &= \lambda_1+\lambda_2 \\ &= s_1^2+s_2^2. \end{aligned}$$

From these equations and the facts $s_1 \geq s_2$, $|f_z| > |f_{\bar{z}}|$ we can deduce

$$s_1 = |f_z|+|f_{\bar{z}}|, \qquad s_2 = |f_z|-|f_{\bar{z}}|,$$

and hence

$$D = \frac{s_1}{s_2} = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}.$$

Note that $D \geq 1$ with equality iff f is a conformal linear map. It is often more convenient to deal with the complex number.

$$\mu = \frac{f_{\bar{z}}}{f_z},$$

which is called the **complex dilatation** (although sometimes we abuse notation and just call thus the dilatation, if the meaning is clear from context). Since $|f_{\bar{z}}| < |f_z|$, we have $|\mu| < 1$ and it is easy to verify that

$$D = \frac{1+|\mu|}{1-|\mu|}, \qquad |\mu| = \frac{D-1}{D+1},$$

so that either D or $|\mu|$ can be used to measure the degree of non-conformality.

We leave it to the reader to check that the map

$$x + iy \to (ax + by) + i(cx + dy)$$

can also be written as

$$(z, \bar{z}) \to \alpha z + \beta \bar{z},$$

where $z = x + iy$, $\bar{z} = x - iy$ and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, satisfy

$$\alpha_1 = \frac{a+d}{2}, \quad \alpha_2 = \frac{a-d}{2}, \quad \beta_1 = \frac{c-b}{2}, \quad \beta_2 = \frac{b+c}{2},$$

In this notation $\mu = \beta/\alpha$ and

$$D = \frac{|\beta| + |\alpha|}{|\alpha| - |\beta|}.$$

As noted above, the linear map f sends the unit circle to an ellipse of eccentricity D. What point on the circle is mapped furthest from the origin? Since

$$s_1 = |f_z| + |f_{\bar{z}}|,$$

the maximum stretching is attained when $f_z z$ and $f_{\overline{w}}\overline{w}$ have the same argument, i.e., when

$$0 < \frac{f_z z}{f_{\bar{z}} \bar{z}} = \frac{z^2}{\mu |z|^2},$$

or

$$\arg(z) = \frac{1}{2} \arg(\mu),$$

Thus $|\mu|$ encodes the eccentricity of the ellipse and $\arg(\mu)$ encodes the direction of its major axis.

If we follow f by a conformal map g, then the same infinitesimal ellipse is mapped to a circle, so we must have $\mu_{g \circ f} = \mu_f$. If f is preceded by a conformal map g, then the ellipse that is mapped to a circle is the original one rotated by $-\arg(g_z)$, so $\mu_{f\circ g} = (|g_z|/g_z)^2\mu_f$. To obtain the correct formula in general we need to do a little linear algebra. Consider the composition $g\circ f$ and let $w = f(z)$ so that the usual chain rule gives

$$(g\circ f)_z = (g_w\circ f)f_z + (g_{\overline{w}}\circ f)\overline{f}_z,$$

$$(g\circ f)_{\overline{z}} = (g_w\circ f)f_{\overline{z}} + (g_{\overline{w}}\circ f)\overline{f}_{\overline{z}}.$$

or in vector notation

$$\begin{pmatrix}(g\circ f)_z\\(g\circ f)_z\end{pmatrix} = \begin{pmatrix}f_z & \overline{f}_z\\ f_{\overline{z}} & \overline{f}_{\overline{z}}\end{pmatrix}\begin{pmatrix}(g_w\circ f)\\(g_{\overline{w}}\circ)\end{pmatrix}$$

The determinate of the matrix is

$$f_z\overline{f}_{\overline{z}} - \overline{f}_z f_{\overline{z}} = f_z\overline{f_z} - \overline{f_{\overline{z}}}f_{\overline{z}} = |f_z|^2 - |f_{\overline{z}}|^2 = J,$$

which is the Jacobian of f, so by Cramer's Rule,

$$(g_w\circ f) = \frac{1}{J}[(g\circ f)_z\overline{f}_{\overline{z}} - (g\circ f)_{\overline{z}}\overline{f}_z],$$

$$(g_{\overline{w}}\circ f) = \frac{1}{J}[(g\circ f)_{\overline{z}}f_z - (g\circ f)_z f_{\overline{z}}],$$

so

$$\begin{aligned}\mu_g\circ f &= \frac{(g\circ f)_{\overline{z}}f_z - (g\circ f)_z f_{\overline{z}}}{(g\circ f)_z\overline{f}_{\overline{z}} - (g\circ f)_{\overline{z}}\overline{f}_z}\\ &= \frac{\mu_{g\circ f}f_z - f_{\overline{z}}}{\overline{f}_{\overline{z}} - \mu_{g\circ f}\overline{f}_z}\\ &= \frac{f_z}{\overline{f_z}}\frac{\mu_{g\circ f} - \mu_f}{1-\mu_{g\circ f}\overline{\mu_f}}.\end{aligned}$$

Now set $h = g\circ f$ or $g = h\circ f^{-1}$ to get

$$\mu_{h\circ f^{-1}}\circ f = \frac{f_z}{\overline{f_z}}\frac{\mu_h - \mu_f}{1-\mu_h\overline{\mu_f}}.$$

Thus if h and f have the same dilatation μ, then $g = h\circ f^{-1}$ is conformal. We will need this in the case when h is more general than an homeomorphism.

2. The geometric definition

The piecewise differentiable definition: h is K-quasiconformal on Ω if there are countable many analytic curves whose union is a closed set Γ of Ω such that h is continuously differentiable on each connected component of $\Omega' = \Omega\setminus\Gamma$ and $D_h \le K$ on Ω'.

A quadrilateral Q is a Jordan domain with two disjoint closed arcs on the boundary. By the Riemann mapping theorem and Caratheodory's theorem, there is a conformal map from Q to a $1\times m$ rectangle that extends continuously to the boundary with the two marked arcs mapping to the two sides of length a. The ratio $M = M(Q) = 1/m$ is called the modulus of the four distinct marked on the boundary and is uniquely determined by Q. The conjugate of Q is the same domain but with the complementary arcs marked. Its modulus is clearly the reciprocal of Q's modulus.

The geometric definition: A homeomorphism h, defined on a planar domain Ω, is K-quasiconformal if the

$$\frac{1}{K}M(Q) \leq M(h(Q)) \leq KM(Q),$$

for every quadrilateral $Q \subset \Omega$.

Our first goal is to check that this definition includes all the "obvious" examples: piecewise C^1 maps with bounded dilatations:

LEMMA 2.1. *Suppose h a homeomorphism of Ω such that there are countable many analytic curves whose union is a closed set Γ of Ω and h is continuously differentiable on each connected component of $\Omega' = \Omega \setminus \Gamma$ and $D_h \leq K$ on Ω'. Then h is K-quasiconformal.*

PROOF. Using conformal maps, it suffices to consider the case when Ω and its image are both rectangles, say $\Omega = [0,a] \times [0,1]$ and $h(\Omega) = [1,b] \times [0,1]$. By integrating over horizontal lines in the first rectangle, we see

$$b \leq \int_0^a (|f_z| + |f_{\bar{z}}|)dx.$$

We have used the piecewise analytic assumtion here to break the integral into finitely many open segments where the fundamental theorem of calculus applies and then use the assumption that h is continous at the endpoints to say the total integral is the sum of these sub-integrals (later in the chapter we will use more difficult arguments to reach a similar conclusion when h fails to be "nice" on a uncountable subset of the segment).

Integrating in the other variable,

$$b \leq \int_0^1 \int_0^a (|f_z| + |f_{\bar{z}}|)dxdy.$$

By Cauchy-Schwarz,

$$\begin{aligned}
b^2 &\leq \left(\int_0^1 \int_0^a (|f_z| + |f_{\bar{z}}|)(|f_z| - |f_{\bar{z}}|)dxdy\right)\left(\int_0^1 \int_0^a \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}dxdy\right) \\
&\leq \left(\int_0^1 \int_0^a (|f_z|^2 - |f_{\bar{z}}|^2)dxdy\right)\left(\int_0^1 \int_0^a \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}dxdy\right) \\
&\leq \left(\int_0^1 \int_0^a J_f dxdy\right)\left(\int_0^1 \int_0^a D_f dxdy\right) \\
&\leq baK,
\end{aligned}$$

and so $b \leq Ka$. The other direction follows by repeating the argument for vertical lines instead of horizontal ones. $\square$

In order for the proof to work we need two things: (1) the area of the range to be bounded above by integrating the Jacobian over the domain and (2) each horizontal line segment S to have an image whose length is bounded above by the integral of $|f_z| + |f_{\bar{z}}|$ over S. This certainly holds if f_z and $f_{\bar{z}}$ are piecewise continuous on a partition of the plane given by countable many analytic curves, as we have assumed, but it holds much more generally. The geometric definition of quasiconformality actually implies that the map h has partials almost everywhere and is absolutely continuous on almost every line. This, in turn, implies the necessary estimates holds. This will be discussed later in Chapter **??**

Next we record the annulus analog of the previous result for rectangles.

COROLLARY 2.2. *If we have a piecewise differentiable K-quasiconformal map f between annuli $A_r = \{1 < |z| < r\}$ and $A_R = \{1 < |z| < R\}$ rectangle with dilatation $\leq K$, then $\frac{1}{K} \log r \leq \log R \leq K \log r$.*

PROOF. Slit A_r with $[1, r]$ for form a quadrilateral $Q \subset A_r$ and let $Q' = f(Q) \subset A_R$. See Figure 2.1 Then $M(A_R) \leq M(Q') \leq KM(Q) = M(A_r)$. The first inequality occurs because of monotonicity of modulus (Lemma 1.2); every separating curve for the annulus connects opposite sides of Q' (but there are connecting curves that don't correspond to closed loops). The other direction follows by considering the inverse map. See Figure 2.2. $\square$

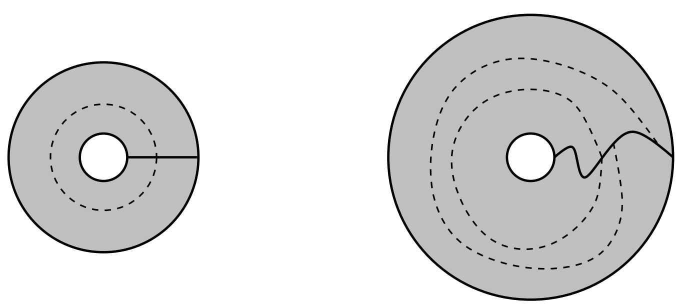


FIGURE 2.1. Notation in the proof of 2.2.

3. Pointwise bounded and equicontinuous

In the next few sections we show that the collection of normalized K-quasiconformal mappings is compact. This has several steps. First must show that this family satisfies the hypotheses of the Arzela-Ascoli theorem, and hence is pre-compact, i.e., every sequence of normalized K-quasiconformal maps contains a subsequence that converges uniformly on compact sets. Second, we have to show that a uniform limit if K-quasiconformal maps is also K-quasiconformal.

LEMMA 3.1. *Suppose* $\Omega \subset \mathbb{C}$ *is a topological annulus of modulus* M *whose boundary consists of two Jordan curves* γ_1, γ_2 *with* γ_2 *separating* γ_1 *from* ∞. *Then* $\operatorname{diam}(\gamma_1) \leq (1-\varepsilon)\operatorname{diam}(\gamma_2)$ *where* $\varepsilon > 0$ *depends only on* M.

PROOF. Rescale so $\operatorname{diam}(\gamma_2) = \operatorname{diam}(\Omega) = 1$ and suppose $\operatorname{diam}(\gamma_1) > 1-\varepsilon$. Then there are points $a \in \gamma_1$ and $b \in \gamma_2$ with $|a-b| \leq \varepsilon$. Let ρ be the metric on Ω defined by $\rho(z) = \frac{1}{|z-a|\log(1/2\varepsilon)}$ for $\varepsilon < |z-a| < 1/2$. Then any curve $\gamma \subset \Omega$ that separates γ_1 and γ_2 satisfies $\int_\gamma \rho ds \geq 1$ and

$$\int \rho^2 dxdy \leq \frac{\pi}{4} \log^{-2} \frac{1}{2\varepsilon}.$$

Thus the modulus of the path family separating the boundary components is bounded above by the right hand side, and the modulus of the reciprocal family connecting the boundary components is bounded below by $\frac{\pi}{4}\log^2 \frac{1}{2\varepsilon}$. Thus $\varepsilon \geq \frac{1}{2}\exp(-\sqrt{\pi M/4})$. □

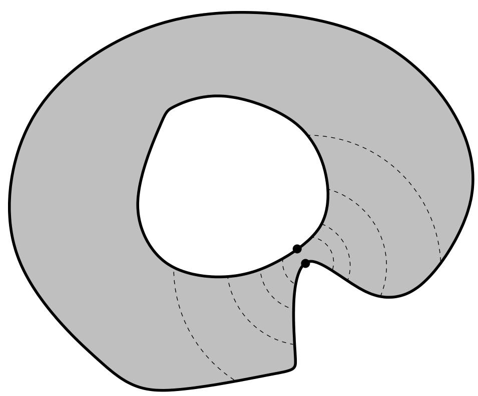


FIGURE 3.1. Proof of Lemma 3.1.

LEMMA 3.2. *Suppose* $f : \mathbb{C} \to \mathbb{C}$ *is a* K*-quasiconformal map that fixes both* 0 *and* 1. *Then* $|f(x)|$ *is bounded with an estimate depending on* $|x|$ *and* K, *but not on* f.

PROOF. By composing with rotations (which do not alter the quasiformal bounds), it is enough to assume $x < 0$. Consider the topological annulus with boundary components consisting of the segment $[x, 0]$ and ray $[1, \infty)$. See Figure **??**. The modulus of the path family separating the two boundary components is bounded below depending only on $|x|$. But if $R = |f(x)|$ then by using the metric $\rho(z) = 1/(|z|\log R)$, we see that the modulus of $f(\mathscr{F})$ is at most $1/\log R$. This is a contradiction if R is too large. □

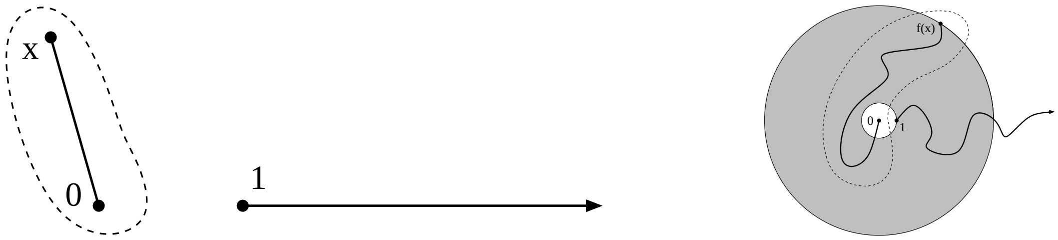


FIGURE 3.2. If $|f(x)| \gg |x|$ then the modulus of the path family separating $[0,x]$ and $[0,\infty)$ must change by more than a factor of K.

THEOREM 3.3. *A K-quasiconformal map of the plane that fixes both* 0 *and* 1 *is locally Hölder continuous.*

PROOF. Suppose f is as in the lemma and $x,y \in D(0,r)$. By Lemma **??**, $D(0,2r)$ is mapped into $D(0,R)$ for some $R = R(r,K)$. Surround $\{x,y\}$ by $N = \lfloor \log_2 \frac{r}{|x-y|} \rfloor$ annuli $\{A_j\}$ of modulus $\log 2$. See Figure **??**. The image annuli $\{f(A_j)\}$ have moduli bounded away from zero, and hence $\operatorname{diam}(f(A_{j+1})) \le (1-\varepsilon)\operatorname{diam}(f(A_j))$ by Lemma 3.1. Therefore

$$|f(x)-f(y)| \le R(1-\varepsilon)^N \le R2^{\log_2(1-\varepsilon)(1+\log_2 R - \log_2|x-y|)} \le C(R)|x-y|^{\log_2(1-\varepsilon)}.$$

□

Later we will compute the actual Hölder exponent as $1/K$.

The proof can generalized to a slightly bigger class than the quasiconformal maps where the dilatation is allowed to grow to ∞ sufficiently slowly. There have been a number of excellent papers written on explicit bounds for this kind of result, but we will only need the "soft" version above. See [**?**], [**?**], [**?**], [**?**], [**?**]. However, here we only give a simple "soft" version of such a result:

THEOREM 3.4. *If f is piecewise differentiable and the dilatation μ satisfies certain estimates of the form*

$$\max_{|x|\le R, r>1/R} \frac{1}{r^2}\int_{D(x,r)} D_f(z)\,dxdy \le \phi(R),$$

then f has modulus of continuity that depends only on ϕ if $\phi \nearrow \infty$ slowly enough as $R \to \infty$.

PROOF. Repeat the proof of Theorem **??**, only now the moduli of the image annuli can tend to zero. However, as long as ϕ grow slowly enough, then

$$\operatorname{diam}(f(A_j)) \le \prod_{j=1}^{N} (1-\varepsilon(\phi(R)))$$

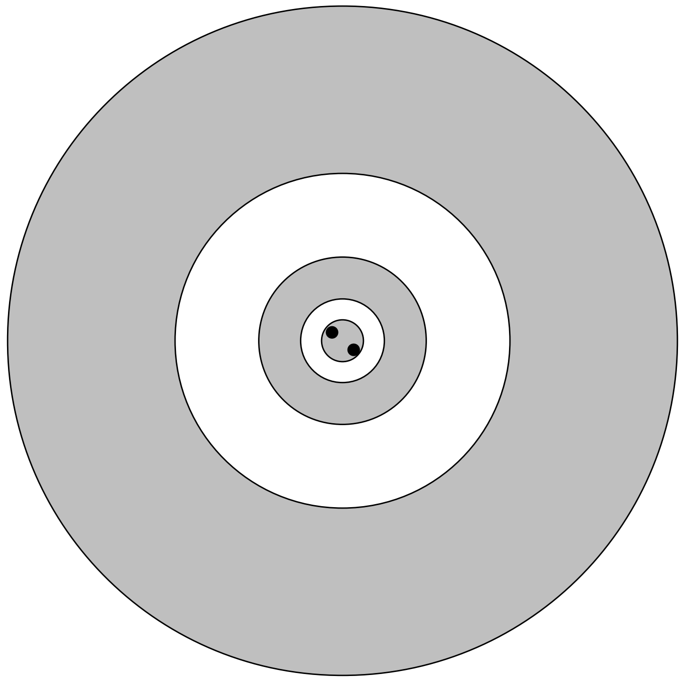


FIGURE 3.3. Annuli of fixed modulus map to annuli with modulus bounded below, and whose diameters shrink geometrically. Thus f is Hölder continuous.

where $\varepsilon(K)$ is as in Lemma 3.1. □

4. Boundary extension

In this section we prove.

THEOREM 4.1. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and onto, then φ is α-Hölder on $\mathbb{D}$, where $\alpha > 0$ only depends on K. Thus φ extends continuously to a homeomorphism of $\mathbb{T} = \partial\mathbb{D}$ to itself.*

The proof is very simuilar to the Hölder estimates for quasiconformal maps in the plane, however, we will also need a trick for converting certain quadrilaterals in the disk into annuli in the plane by reflecting across the circle. The precise statement is:

LEMMA 4.2. *Suppose $Q \subset \mathbb{H}_u$ is a quadrilateral with a pair of opposite sides being intervals $I, J \subset \mathbb{R}$. Let A be the topological annulus formed by taking $Q \cup I \cup J \cup Q^*$ (where Q^* is the reflection of Q across $\mathbb{R}$. Then $M(A) = \frac{1}{2}M(Q)$ (here the modulus of Q refers to the modulus of the path family connecting the two sides of Q that line on the unit circle).*

PROOF. Using conformal invariance, assume Q is in the upper half-plane and A is obtained by reflecting Q across the real line. See Figure **??**. Consider the path family Γ_A in A that connects the two boundary components of A, and the path family Γ_Q in Q that separate the boundary arcs $Q \cap \mathbb{R}$. Then $(\Gamma_A)_+ = \Gamma_Q$ (notation as in Lemma 1.9), so by the Symmetry Rule

$$M(\Gamma_A) = 2M((\Gamma_A)_+) = 2M(\Gamma_Q).$$

The moduli in the lemma are the reciprocals of these moduli, so the result follows. $\square$

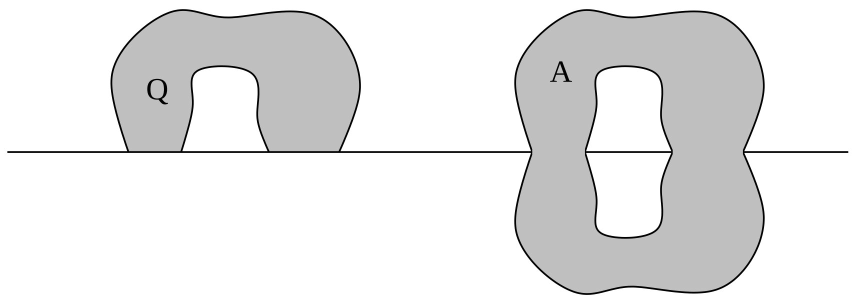


FIGURE 4.1. Reflecting the quadrilateral Q across the line gives an annulus with half the modulus.

PROOF OF THEOREM **??**. We may assume $f(0) = 0$; the general case then follows after composing with a Möbius transformation.

We first suppose φ extends continuously to the boundary. This may seem a bit circular given the final statement of the theorem, but our plan is to prove φ is $\alpha(K)$-Hölder for assuming continuity, and then use a limiting argument to remove the continuity assumption. More precisely, suppose $w, z \in \mathbb{D}$. We will show that

$$|\varphi(z) - \varphi(w)| \leq C|z - w|^{\alpha},$$

for constants $C < \infty$, $\alpha > 0$ that depend only on the quasiconstant K of f. This implies f is uniformly continuous and hence has a continuous extension to the boundary of $\mathbb{D}$.

Let $d = |z - w|$ and $r = \min(1 - |z|, 1 - |w|)$. There are several cases depending on the positions of the points z, w and the relative sizes of d and r. See Figure **??**.

To start, note that if $|z - w| \geq \frac{1}{10}$ we can just take $C = 20$ and $\alpha = 1$. So from here on, we assume $|z - w| < 1/10$.

Suppose $r > 1/4$, so $z, w \in \frac{3}{4}\mathbb{D}$. Surround the segment $[z, w]$ by $N \simeq \log d$ annuli with moduli $\simeq 1$. Then just as in the proof of Theorem **??**, the image annuli have moduli $\simeq 1$ (with a constant depending on K) and hence

$$|f(z) - f(w)| \leq (1 - \varepsilon(K))^N = O(|z - w|^\alpha),$$

for some $\alpha > 0$ depending only on K.

Next suppose $|z| \geq 3/4$ and $d > r$. Then separate $[z, w]$ from 0 by $N \simeq \log d$ disjoint quadrilaterals with a pair of opposite sides being arcs of $\mathbb{T}$, and all with moduli $\simeq 1$. Since $f(0) = 0$ and the image quadrilaterals have moduli $\simeq 1$, there diameters shrink geometrically, so

$$|z - w| = (1 - \varepsilon(K))^N = O(d^\alpha),$$

as desired.

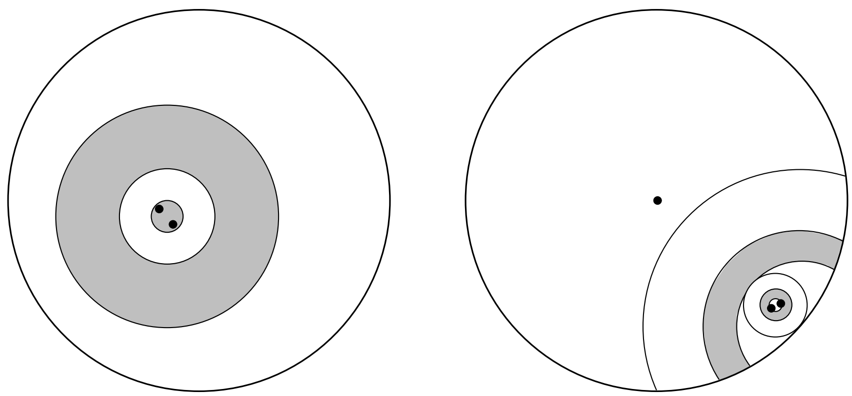


FIGURE 4.2. The proof of Hölder estimates in the disk is similar to the proof in the plane,except that we need to use quadrilaterals, as well as annuli, if the pair of points in near the boundary.

Finally, if $r \leq d$ we combine the two previous ideas: we start by separating $[z, w]$ from 0 by $\simeq \log d$ quadrilaterals with as above. The smallest quadrilateral then bounds a region of diameter approximately r containing $[z, w]$ and we then construct $\simeq \log r/d$ disjoint annuli with moduli $\simeq 1$ that each separate $[z, w]$ from this smallest quadrilateral. See Figure **??**. The same arguments as before now show

$$|z - w| = (1 - \varepsilon(K))^{-\log r}(1 - \varepsilon(K))^{\log r/d} = O(d^\alpha) = O(|z - w|^\alpha).$$

This proves the theorem assuming φ extends continuously to the boundary. Now we have to remove this extra assumption. Assume φ is any K-quasiconformal of $\mathbb{D}$ onto itself, such that $\varphi(0) = 0$. Take r close to 1 and let $\Omega_r = \varphi(\{|z| < r\})$ Then Ω_r is a Jordan domain that satisfies

$$\{|z| < 1 - \delta\} \subset \Omega_r \subset \mathbb{D},$$

with $\delta \to 0$ as $r \nearrow 1$. Let $f_r : \Omega_r \to \mathbb{D}$ be the the conformal map so that $f_r(0) = 0$ and $f_r'(0) > 0$. By Caratheodory's theorem f_r is a homeomorphism from the closure of Ω_r to the closed unit disk, hence the K-quasiconformal map $g_r = f_r \circ \varphi$ is a homeomorphism from the closed unit disk to itself. Thus the previous argument applies to g_r, and we deduce g_r is α-Hölder.

As $r \nearrow 1$, both f_r and f_r^{-1} tend to the identity on compact subsets of $\mathbb{D}$. In particular, for $z, w \in \mathbb{D}$, we have

$$|\varphi(z) - \varphi(w)| = \lim_{r \nearrow 1} |f_r^{1-}(g_r(z)) - f_r^{-1}(g_r(w))| = \lim_{r \nearrow 1} |g_r(z) - g_r(w)| \leq C(K)|z-w|^{\alpha},$$

(by the Schwarz Lemma $g_r(z)$ and $g_r(w)$ remain in a compact subset of $\mathbb{D}$ as $r \nearrow 1$). Thus φ is α-Hölder as well. □

5. Compactness of K-QC maps

We have now verified that normalized K-quasiconformal maps satisify the Arzela-Ascoli theorem, so they form a pre-compact family. To prove compactness, we need to prove:

THEOREM 5.1. *If $\{f_n\}$ is a sequence of K-quasiconformal maps on Ω that converge uniformly on compact subsets to a homeomorphism f, then f is K-quasiconformal.*

This is immediate from:

THEOREM 5.2. *Suppose $\{h_n\}$ are homeomorphisms defined on a domain Ω and $Q \subset \Omega$ is a generalized quadrilateral that is compactly contained in Ω. If $\{h_n\}$ converge uniformly on compact sets to a homeomorphism h on Ω, then $M(h_n(Q)) \to M(h(Q))$*

This, in turn, follows from the more technical looking:

LEMMA 5.3. *Suppose $\{f_n\}$ are conformal maps of $\mathbb{D} \to \Omega_n$ that converge uniformly on compact subsets of $\mathbb{D}$ to a conformal map $f : \mathbb{D} \to \Omega$. Suppose that the boundary of each Ω_n is the homeomorphic image $\partial\Omega_n = \sigma_n(\mathbb{T})$ and that $\{\sigma_n\}$ converges uniformly on $\mathbb{T}$ to a homeomorphism $\sigma : \mathbb{T} \to \partial\Omega$. Then $f_n \to f$ uniformly on the $\overline{\mathbb{D}}$.*

This lemma is an analog of Carathéodory's theorem. That result says that if $\partial\Omega$ is a homeomorphic image of the unit circle under any map, then it is the homeomoprhic image of the unit circle under a conformal map. The lemma aboves says that if there are any homeomorphic parameterizations of $\{\partial\Omega_n\}$ that converge uniformly to a homeomorphic parameterization of $\partial\Omega$, then there are conformal parameterizations with this property. Not surprisingly, the proof of the lemma is also very similar to the proof of Carathéodory's theorem.

PROOF OF LEMMA 4.9. Fix $\varepsilon > 0$ and choose N so large that if we divide $\mathbb{T}$ into N equal sized intervals $\{J_j\}_1^N$, then σ maps each of them to an arc I_j of diameter at most ε. Let $I_j^k = f_k(J_j)$. Because $\sigma_k \to \sigma$ uniformly, the sets $\{\sigma_k(J_j)\}_1^N$ all have diameter at most 2ε, if k is large enough.

Since σ is a homeomorphism, $\partial\Omega = \partial f(\mathbb{D}) = \sigma(\mathbb{T})$ is a Jordan curve, so f extends homeomorphically to the unit circle. Therefore f^{-1} send the endpoints of $\{I_j\}$ to N distinct points on $\mathbb{T}$. Therefore we can choose M so large that $2\pi/M$ is less than the minimal separation between these points, and define M equal length arcs $\{K_j\}$ on $\mathbb{T}$. Clearly, the f image of any one of these is contained in the union of at most two adjacent arcs I_j, and hence has diameter at most 2ε.

We want to show a similar estimate is true for images of K_j under f_k when k is large enough. Choose $\eta > 0$ so small that if $\text{dist}(I_j, I_\ell) < 3\eta$, then $j = \ell$ or I_j and I_ℓ are adjacent; this is possible because the distance between non-adjacent arcs is positive and there are only finitely many possible pairs to consider. Because $\sigma_k \to \sigma$ uniformly, $\sigma_k(J_j)$ is contained in η-neighborhood of I_j for all sufficiently large k. For such k, if

$$\text{dist}(\sigma_k(J_n), \sigma_j(J_m)) \le \eta,$$

then $\text{dist}(I_j, I_k) < 3\eta$. Thus $\sigma_k(J_n)$ and $\sigma_j(J_m))$ are at least distance η apart when J_n and J_m are not the same or adjacent.

Suppose $\eta > 0$. Choose M so that $W = \Omega \setminus f(r_M \cdot \mathbb{D})$, where $r_M = 1 - 1/M$, contains no disks of radius $> \eta$ (why we can do this was explained in the proof of Carathéodory's theorem). We claim that if k is sufficiently large, then $W_k = \Omega_k \setminus f_k(r_M \mathbb{D})$ contains no disks of radius $> 3\eta$. This is because this region is a topological annulus bounded by two closed Jordan curves. One is $\sigma_k(\mathbb{T}) = \partial\Omega_k$ and it contains $\partial\Omega$ inside an η-neighborhood of itself, if k is large enough, because $\sigma_k \to \sigma$ uniformly on $\mathbb{T}$. The other boundary curve is $f_k(r_M \cdot \mathbb{T})$, which contains $f(r_M \cdot \mathbb{T})$ inside an η-neighborhood of itself, since $f_k \to f$ uniformly on the compact set $r_M \cdot \mathbb{T}$. If W_k contained a disk of radius 3η, if follows that W would contain a concentric disk of radius η, a contradiction, so the claim is proved.

Consider an arc K_j and the two arcs K_{j-1}, K_{j+1} adjacent on either side of it. Choose $z \in \mathbb{D}$ with $z/|z| \in K_j$ and $1 - |z| \simeq |K_j$. By Corollary 4.6, we can find points $x_m \in K_m$ for $m = j-1, j, j+1$ so that

$$\begin{aligned} |f_k(w_{j-1}) - f_k(w_{j+1})| &\le |f_k(w_{j-1}) - f_k(z)| + |f_k(z) - f_k(w_{j+1})| \\ &\le C\,\text{dist}(f_k(z), \partial\Omega_k) \\ &\le C\eta. \end{aligned}$$

When this happens, we know w_{j-1} and w_{j+1} lie in the union of two adjacent arcs in $\{J_k\}$. Therefore the intermediate arc K_j must also lie in the same union. Therefore its image under f_k has diameter at most 4ε (the union of two sets of diameter 2ε).

Moreover, fixing K_j, choosing z as above, we also have

$$\begin{aligned} |f_k(w_j) - f(w_j)| &\leq |f_k(w_j) - f_k(z)| + |f_k(z) - f(z)| + |f(z) - f(w_j)| \\ &\leq |f_k(z) - f(z)| + C\,\mathrm{dist}(f_k(z), \partial\Omega_k) \\ &\qquad + C\,\mathrm{dist}(f(z), \partial\Omega) \\ &\leq |f_k(z) - f(z)| + C\eta. \end{aligned}$$

The first term in the last line is less than ε if k is large enough since $f_k \to f$ on compact sets. Therefore $f_k(K_j)$ is contained in a $C\varepsilon + C\eta$ neighborhood of $f(K_j)$, and conversely. Thus $f_k \to f$ uniformly on $\mathbb{T}$, as desired. □

PROOF OF THEOREM **??**. Recall that we want to prove that if quadrilaterals $\{Q_n\}$ converge to Q in the sense that they have parameterizations $\{h_n\}$ that converge uniformly to parameterization h of Q, then the moduli of the $\{Q_n\}$ converge to the modulus of Q.

Let $f : \mathbb{D} \to h(Q)$ and $f_k : \mathbb{D} \to h_k(Q)$ be conformal maps, normalized so $f_k \to f$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 4.9 these maps converge uniformly on the closed disk. The four corners of $h(Q)$ have four distinct f-preimages a, b, c, d on $\mathbb{T}$. Then $f_k(a) \to f(a)$. Because f is homeomorphism, if there was no f_k-preimage of a corner of $h_k(Q)$ near a, this corner could not approach $f(a)$, contradicting $h_k \to k$. Thus a must be a limit of f_k-preimages of corners of $h_k(Q)$. Since modulus is clearly a continuous function of the points $\{a, b, c, d\}$ (e.g., Exercise **??**), we see that $\mathrm{mod}\,(h_k(Q)) \to \ \mathrm{mod}\,(h(Q))$. □

PROOF OF THEOREM **??**. Any quadrilateral $Q \subset \Omega$ has compact closure in Ω so $f(Q) = \lim_n f_n(Q)$ is a quadrilateral in $f(\Omega)$ and $M(f(Q)) = \lim_n M(f_n(Q)) \leq K \lim_n M(Q)$ by Lemma 4.9. The opposite inequality follows by considering the inverse maps, so we see that f is K-quasiconformal. □

LEMMA 5.4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map that fixes both* 0 *and* 1. *Then there is a constant* $0 < C < \infty$*, depending only on K so that if $|z| < 1/C$, then*

$$C^{-1}|z|^K \leq |f(z)| \leq C|z|^{1/K} \leq C|z|^{1/K}.$$

PROOF. Since normalized K-quasiconformal maps form a compact family, there here is a constant $A = A(K)$ so that

$$f(\{|z| = 1\}) \subset \{\frac{1}{A} < |z| < A\}.$$

By rescaling we also get that for any $0 < r < \infty$

$$f(\{|z| = r\}) \subset \{\frac{|f(r)|}{A} < |z| < A|f(r)|\}.$$

Thus if $r < A^{-2}$,

$$\{A|f(r)| < |z| < \frac{1}{A}\}\} \subset f(\{r < |z| < 1\}) \subset \{\frac{|f(r)|}{A} < |z| < A\}\}.$$

Comparing moduli in the first inclusion we get

$$\frac{1}{2\pi}\log\frac{1}{A^2|f(r)|} \leq M(f(\{r < |z| < 1\})) \leq \frac{K}{2\pi}\log\frac{1}{r},$$

which gives

$$|f(r)| \geq r^K/A^2.$$

The second inclusion similarly gives

$$\frac{1}{2\pi}\log\frac{A^2}{|f(r)|} \geq M(f(\{r < |z| < 1\})) \geq \frac{1}{2\pi K}\log\frac{1}{r},$$

which implies $|f(r)| \leq A^2 r^{1/K}$. Taking $C = A^2$ proves the lemma. □

COROLLARY 5.5. *For each $K \geq 1$ there is a $C = C(K) < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and γ is a circle, then there is $w \in \mathbb{C}$ and $r > 0$ so that $f(\gamma) \subset \{z : r \leq |z - w| \leq Cr\}$.*

PROOF. Without loss of generality, we can pre and post-compose so that γ is the unit circle and f fixes $0, 1$. By Lemma **??**, $f(\gamma)$ is then contained in an annulus $\{\frac{1}{C} \leq |z| \leq C\}$, and this gives the result. □

6. Locally QC implies globally QC

The definition of quasiconformality requires us to check the moduli of all quadrilaterals. In this section we prove that it is enough to verify the definition just on all sufficiently small quadrilaterals.

LEMMA 6.1. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ that is K-quasiconformal in a neighborhood of each point of Ω, then f is K-quasiconformal on all of Ω.*

PROOF. Suppose $Q \subset \Omega$ is a quadrilateral that is conformally equivalent via a map φ to a $1 \times m$ rectangle R and $Q' = f(Q)$ is conformally equivalent a $1 \times m'$ rectangle R'. Divide R into M equal vertical strips $\{S_j\}$ of dimension $1 \times m/M$. See Figure **??**. We have to choose M sufficiently large that two things happen.

First choose $\delta > 0$ so that f^{-1} is K-quasiconformal on any disk of radius δ centered at any point of Q' (we can do this since Q' has compact closure in Ω). Next, note that the closure of Q' is a union of Jordan arcs γ

corresponding via $f \circ \varphi^{-1}$ to vertical line segments in R. By the continuity of $f \circ \varphi^{-1}$ there is an $\eta > 0$ so that if $z \in R$ then $f(\varphi^{-1}(D(z,\eta)))$ has diameter $\leq \delta$. By the continuity of the inverse map, there is an $\varepsilon > 0$ so that $x, y \in Q'$ and $|x-y| < \varepsilon$ implies $|\varphi(f^{-1}(x)) - \varphi(f^{-1}(y))| \leq \eta$. Thus for any $\delta > 0$ there is an $\varepsilon > 0$ so that if $x, y \in \gamma \subset Q'$ are at most distance ε apart, then the arc of γ between then has diameter at most δ (and ε is independent of which γ we use).

Choose M so large that each region $Q'_j = f(\varphi^{-1}(R_j))$ contains a disk of radius at most ρ, where ρ will be chosen small depending on ε. Map Ω_j conformally to a $1 \times m'_j$ rectangle S'_j. Note that this rectangle is conformally equivalent to the region $R'_j = \psi(f(\varphi^{-1}(R_j))) \subset R_j$, both with the obvious choice of vertices.

By Lemma **??** there is an absolute constant C so that every for every $y \in [0,1]$, there is a $t \in (0,1)$ with $|t-y| \leq Cm_j$ and so that the horizontal cross-cut of R'_j at height t maps via φ_j^{-1} to a Jordan arc of length $\leq C\rho$. Thus we can divide R'_j by horizontal cross-cuts into rectangles $\{R'_{ij}\}$ of modulus $m'_{ij} \simeq 1$ so that the preimages of these rectangles under ϕ_j are quadrilaterals with two opposite sides of length $\leq C\rho$ and which can be connected inside the quadrilateral by a curve of length $\leq C\rho$.

Taking δ as above, choose ε as above corresponding to $\delta/4$ and choose ρ so that $3C\rho < \min(\varepsilon, \delta/4)$. Then all four sides of the quadrilateral Q'_{ij} have diameter $\leq \delta/4$ and hence Q'_{ij} has diameter less than δ and hence lies in a disk where f^{-1} is K-quasiconformal. Let m_{ij} be the modulus of corresponding preimage quadrilateral $Q_{ij} = f^{-1}(Q'_{ij})$. See Figure **??**.

In S'_j consider the path family Γ'_j that connects the "top" and "bottom" sides of this rectangle and let m'_j denote the modulus of this path family (so $1/m'_j$ is its extremal length).. Let m_{ij} denote the modulus of the path family in the subrectangles S'_{ij} (again we take the path family connecting the top and bottom edges). These are conformally equivalent to path families connecting opposite sides of Q'_{ij} and via f^{-1} to path families in Q_{ij} whose modulus is denoted m_{ij}. Since these quadrilaterals were chosen small enough to fit inside neighborhoods where f is K quasiconformal, we have

$$\frac{m_{ij}}{K} \leq m'_{ij} \leq Km_{ij}.$$

Finally, let Γ_j be the path family that connects the top and bottom of R_j and let Γ'_j be the family that connects the left and right sides of R'.

By the Series Rule

$$\frac{M}{m} = \lambda(\Gamma_j) \geq \sum_i \lambda(\Gamma_{ij}) = \sum_i \frac{1}{m_{ij}}.$$

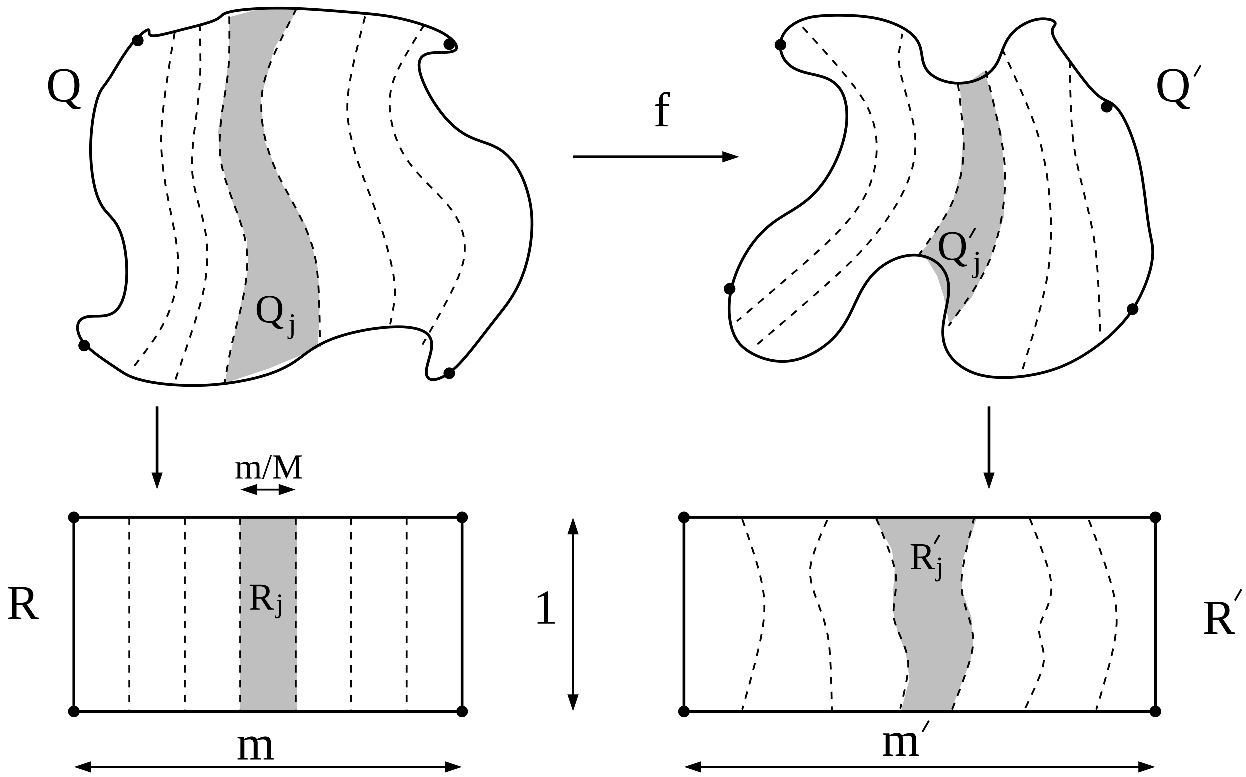


FIGURE 6.1. Notation in the proof of Theorem **??**.

Simlilary,

$$m' = \lambda(\Gamma') \geq \sum_j \lambda(\Gamma'_j) = \sum_j m'_j.$$

We get equality in the Series Rule when a rectangle is cut by vertical lines, so

$$\frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}.$$

Hence

$$\frac{M}{m} \geq \sum_i \frac{1}{m_{ij}} \geq \frac{1}{K} \sum_i \frac{1}{m'_{ij}} = \frac{1}{Km'_j}$$

or

$$\frac{m}{M} \leq Km'_j$$

for every j. Thus

$$m \leq \sum_{j=1}^{M} \frac{m}{M} \leq \sum_j Km'_j \leq Km'.$$

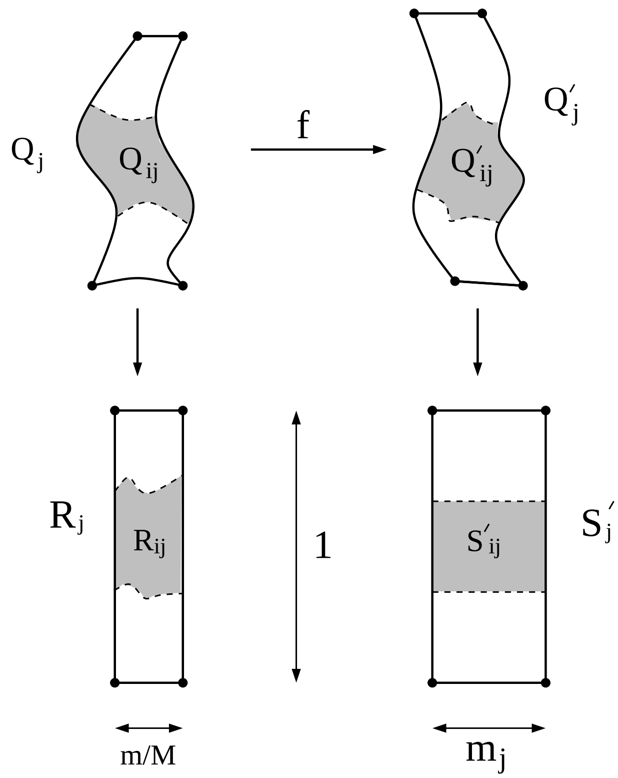


FIGURE 6.2. More notation in the proof of Theorem **??**.

Applying the same result to the inverse map shows f is K-quasiconformal. □

If $K = 1$, then $m = m'$ the last line of the above proof becomes

$$m' = m \leq \sum_j \frac{m}{M} \leq \sum_j m'_j \leq m'.$$

so we deduce

$$\sum_j m'_j = m',$$

whereas in general, we only have $\sum_j m'_j \leq m'$. We want to use this to deduce that 1-quasiconformal map must be conformal. We start with

LEMMA 6.2. *Consider a $1 \times m$ rectangle R that is divided into two quadrilaterals Q_1, Q_2 of modulus m_1 and m_2 by a Jordan arc γ the connects the top and bottom edges of R. Then if $m = m_1 + m_2$, the curve γ is a vertical line segment.*

PROOF. See Figure **??**. Let φ_1, φ_2 be the conformal maps of Q_1, Q_2 onto $1 \times m_1$ and $1 \times m_2$ rectangles R_1, R_2 respectively. Set $\rho = |f_1'|$ on Q_1 and

$\rho = |f_2'|$ in Q_2 and zero elsewhere. Then each horizontal line is cut by γ into pieces one of which connects the left vertical edge of R to γ, and another that connect γ to the right edge of R. The images of these connect the vertical edges of R_1 and R_2 respectively. Thus the images have lengths at least m_1 and m_2 respectively, there length of the image of the entire horizontal segment in Q is $\geq m_1 + m_2$. If we integrate over all horizontal segments in Q, we see

$$\int_R (\rho - 1) dxdy \geq m_1 + m_1 - m = 0.$$

Similarly,

$$\int_R (\rho^2 - 1) dxdy = \text{area}(f_1(Q_1) + \text{area}(f_2(Q_2)) - \text{area}(R) = (m_1 + m_2) - m \leq 0$$

(we would have equality if we knew γ had zero area). Thus

$$\int_Q (\rho - 1)^2 dxdy = \int_Q (\rho^2 - 1) - 2(\rho - 1) dxdy \leq 0.$$

Since $(\rho - 1)^2 \geq 0$, this implies the integral equals zero and hence that that $\rho = 1$ almost everywhere, i.e., f_1 and f_2 are most linear and the curve γ is a vertical line segment. $\square$

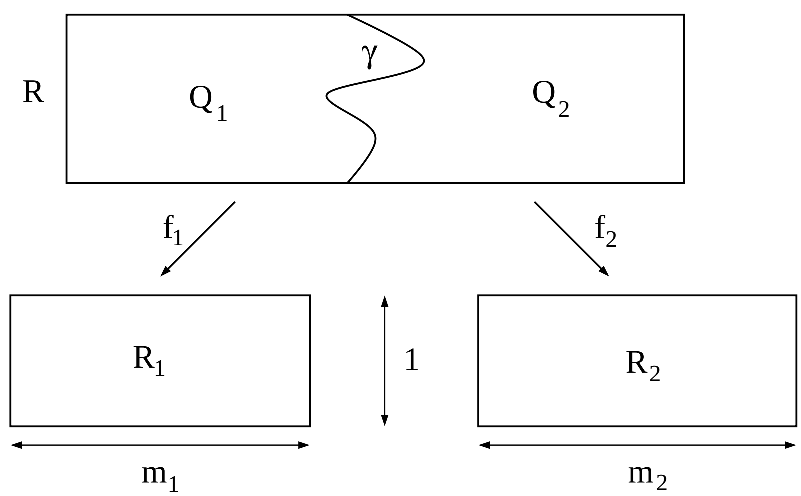


FIGURE 6.3. A partition of a rectangle as in the proof of Lemma **??**.

LEMMA 6.3. *If f is 1-quasiconformal on Ω, then it is conformal on Ω.*

PROOF. If f is 1-quasiconformal in the proof of Theorem **??**, then as noted before Lemma **??**, we must have

$$\frac{M}{m} = \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m_j'} = \sum_i \frac{1}{m_{ij}'}, \qquad m' = \sum_j m_j',$$

Thus the map $\psi = \varphi' \circ f \circ \varphi^{-1}$ between identical rectangles must be the identity map. Thus $f = (\varphi')^{-1} \circ \varphi$ is a composition of conformal maps, hence conformal. $\square$

LEMMA 6.4. *For any $\delta > 0$ and and any $r > 0$ there is an $\varepsilon > 0$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is $(1+\varepsilon)$-quasiconformal and f fixes 0 and 1, then $|z - f(z)| \le \delta$ for all $|z| < r$.*

PROOF. If not, there is a sequence of $(1+\frac{1}{n})$-quasiconformal maps that all fix 0 and 1 and points $z_n \in D(0,r)$ so that $|z_n - f_n(z_n)| > \delta$. However, there is a subsequence that converges uniformly on compact subsets of the plane to a 1-quasiconformal map that fixes 0 and 1 and that moves some point by at least δ. However a 1-quasiconformal map is conformal on $\mathbb{C}$, hence of form $az+b$ and since it fixes both 0 and 1, it is the identity and hence doesn't move any points, a contradiction. $\square$

THEOREM 6.5. *Suppose f is holomorphic on Ω and ϕ is quasiconformal and C^1 on $\mathbb{C}$. Suppose ψ is quasiconformal on Ω and $\mu_\psi = \mu_{\phi \circ f}$ everywhere that $f' \neq 0$. Then there is a holomorphic function g on $\Omega' = \psi(\Omega)$ so that*

$$g \circ \psi = \phi \circ f.$$

PROOF. Let $g = \phi \circ f \circ \psi^{-1}$. Every point where f' is non-zero, there is a disk where the composition is conformal. Thus g is continuous all of Ω' and holomorphic except on a countable set, hence is holomorphic on all of Ω' (since isolated points are removable for bounded holomorphic functions). $\square$

7. Removable sets for quasiconformal maps

When f is continuously differentiable, it is relatively easy to check whether it is quasiconformal; we just compute the complex dilatation $\mu = f_{\bar{z}}/f_z$ and check that $|\mu| < k < 1$ everywhere. For some applications in dynamics, functions arise that that are homeomorphisms f on $\mathbb{C}$, but which are only C^1 on an open set $\Omega = \mathbb{C} \setminus K$ on an open set $\Omega = \mathbb{C} \setminus K$. If we know the dilatation is bounded on just Ω, can we still deduce that f is quasiconformal? If we can, then we say K is removable for quasiconformal mappings.

This depends on the "size" and "shape" of K. If K has interior, then it is easy to construct counterexamples; choose a disk $D \subset K$ and any non-quasiconformal homeomorphism of the disk to itself that is the identity on the boundary and extend it to be the identity off D. If K has positive area, there are also counterexamples corresponding to applications of the measurable Riemann mapping theorem to a dilatation that is a non-zero constant on

K and zero off K. Even if K is quite small, there can be counter examples. For example, given any guage function h such that $h(t) = o(t)$ as $t \searrow 0$, there is a closed Jordan curve γ and a homeomorphism of the sphere that is conformal on both components of $\mathbb{C} \setminus \gamma$ but which is not Möbius (see e.g., [**?**], [**?**], [**?**]). On the other hand, if K has finite or sigma-finite 1-measure then it is removable. These examples show that it is the "shape" rather than the "size" of K that is crucial in most cases of interest.

In this section we give an elegant sufficient condition for K to be removable that is due to Peter Jones and Stas Smirnov [**?**], generalizing an earlier result of Jones [**?**]. We start by observing that quasiconformal images of squares are "roundish" is a precise sense.

LEMMA 7.1. *Suppose Q is a square, $\lambda > 1$ and f is K-quasiconformal on λQ. Then*

$$\operatorname{area}(f(Q)) \geq \varepsilon \operatorname{diam}(f(Q))^2,$$

where $\varepsilon > 0$ depends only on λ and K.

PROOF. By rescaling by conformal linear maps we may assume the square Q is $[-1,1] \times [-1,1]$ and the map f fixes 0 and 1. Choose $x \in \partial Q$; without loss of generality, assume x is in the left halfplane. Consider $A = \lambda Q \setminus ([0,x] \cup [1,\lambda)$. This is a topological annulus and has modulus that is bounded and bounded away from zero independent of x. Thus the same is true of $f(A)$. If $|f(A)| = R \gg 1$, then considering the metric $1/|z| \log R$ on $\{1 < |z| < R\}$ shows that $M(f(A)) \leq 1/\log R \ll 1$, a contradiction. Thus $\operatorname{diam}(f(Q))$ is bounded depending only on K and λ.

The topological annulus $B = \lambda Q \setminus [0,1]$ also has bounded modulus, and hence so does its image under f. Since $0,1$ are fixed every curve surrounding $f([0,1])$ has length at lesat 2, so $\rho = 1/2$ is admissible. Thus $M(f(B)) \leq \operatorname{area}(f(Q))/4$. Therefore $\operatorname{area}(f(Q))$ is bounded below by a constant depending only on λ and K, so the lemma is proven. $\square$

ADD SOME FIGURES TO PROOF

A**Whitney decomposition** of an open set Ω consists of a collection of dyadic squares $\{Q_j\}$ contained in Ω so that

(1) the interiors are disjoint,
(2) the union of the closures is all of Ω,
(3) for each Q_j, $\operatorname{diam}(Q_j) \simeq \operatorname{dist}(Q_j, \partial\Omega)$.

The existence of such a collection is easy to verify be taking the set of dyadic squares Q so that

$$\operatorname{diam}(Q) \leq \frac{1}{4} \operatorname{dist}(Q, \partial\Omega),$$

and that are maximal with respect to this property (i.e., the parent square fails this condition).

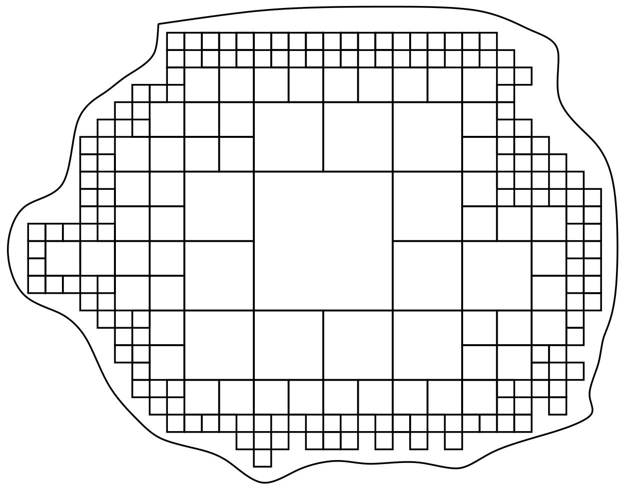


FIGURE 7.1. A Whitney decomposition.

Suppose K is compact, $\delta > 0$ and for each $x \in K$ let γ_x be a Jordan arc in $\Omega = \mathbb{C} \setminus K$ that connects x to $\Omega_\delta = \{z \in \Omega : \text{dist}(z, K) \geq \delta\}$. For a single x, γ_x may consist of several arcs that connect x to Ω_δ. See Figure **??**.

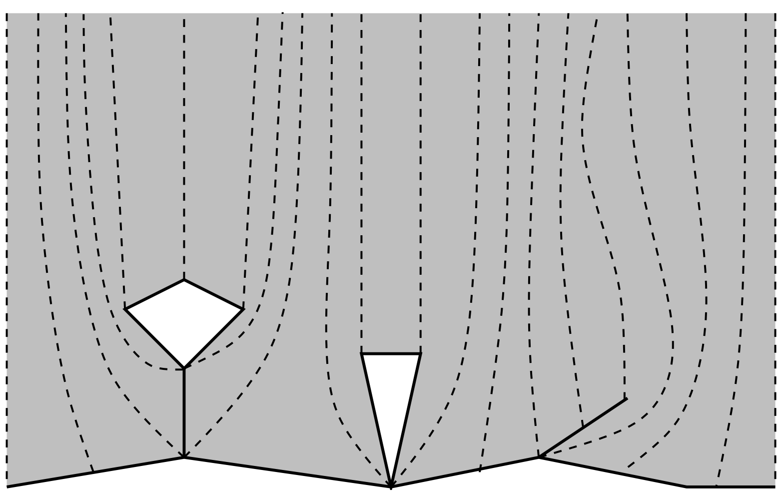


FIGURE 7.2. Each boundary point is connected to a point distance δ from $\partial\Omega$. Some points may be connected by more than one curve.

For each Whitney square $Q \subset \Omega$, let

$$S(Q) = \{x \in K : \gamma_x \cap Q \neq \emptyset\}.$$

This is called the "shadow" of Q on K; the name comes from the special case when K is connected and does not separate the plane and γ_x is a hyperbolic geodesic connecting x to ∞. If we think of ∞ as the "sun" and the geodesics as light rays, then $S(Q)$ is the part of K that blocked from ∞ by Q, i.e., it is Q's shadow. See Figure **??**.

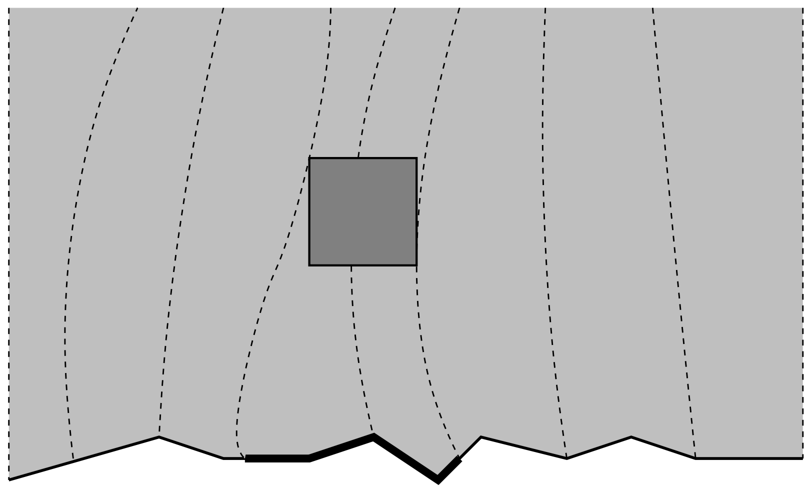


FIGURE 7.3. The shadow of a Whitney square on the boundary.

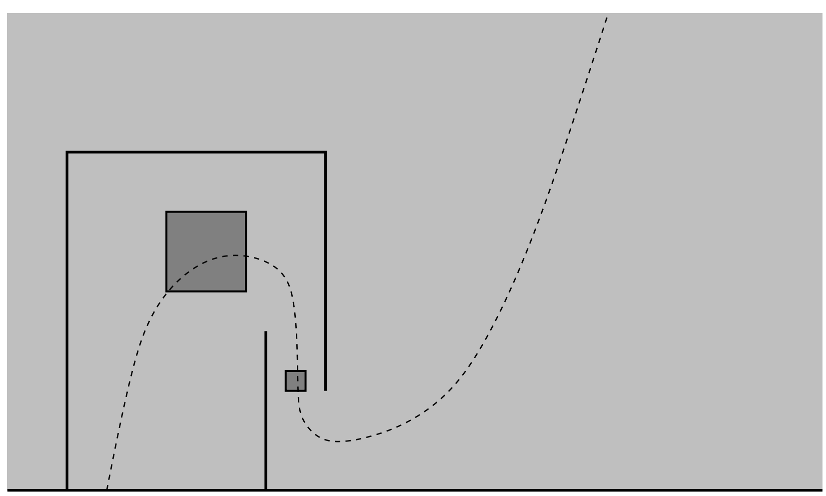


FIGURE 7.4. The paths connecting a Whitney square to its shadow can sometimes hit larger Whitney squares. However this path will hit a largest square, and there after only hit smaller squares.

Let $C(Q)$ be the union of all Whitney squares hit by the arc γ connecting Q to some point of its shadow; this is the "filled shadow" and corresponds to a Carleson square in the unit disk.

We will assume three things about the Whitney squares and their shadows:

(S1) $S(Q)$ is closed.
(S2) $\mathrm{diam}(I(Q)) \to 0$ as $\mathrm{diam}(Q) \to 0$,
(S3) $\lim_{n\to\infty} \sum_{Q:\ell(Q)\leq 2^{-n}} \mathrm{diam}(I(Q))^2 = 0$ where the sum is over all Whitney squares for Ω of side length 2^{-n}.

These will hold in most situations we are interested in. For example, if Ω is simply connected with locally connected boundary, and we take γ_x to be arcs of hyperbolic geodesics connecting some base point $z_0 \in \Omega$ to x, then (S1) and (S2) always holds, (S3) holds if Ω is a John domain (defined below).

THEOREM 7.2. *Suppose Ω has a Whitney decomposition so that the corresponding shadow sets satisfy conditions (S1)-(S3) above. Then $K = \partial\Omega$ is removable for quasiconformal maps.*

f the chain associated to each $x \in \partial\Omega$ consists of adjacent squares (i.e., Q_j touches Q_{j+1}, then the same is true for their images under f, so condition (**??**) is automatically satisfied. Thus we obtain:

COROLLARY 7.3. *Suppose Ω has a Whitney decomposition so that the corresponding shadow sets satisfy conditions (1)-(3) above and all the Whitney chains are connected. The $\partial\Omega$ is removable to quasiconformal homeomorphisms, i.e., any homeomorphism of the plane that is K-QC off $\partial\Omega$ is quasiconformal on the whole plane.*

This is the version given by Jones and Smirnov (restricted to the plane). We have stated the more general version with (**??**) in order to include certain maps arising from groups of circle reflections where we require disconnected Whitney chains, but for which (**??**) is automatically fulfilled.

In both the theorem and the corollary if if the map f is conformal off $\partial Omega$ (i.e., $K = 1$), then we will show that the extension is conformal everywhere. If the map f is K-quasiconformal off $\partial\Omega$ then we only prove that it is C-quasiconformal for some $C < \infty$. However, it follows from this that f is actually K-quasiconformal on the whole plane. Our hypotheses imply that $\partial\Omega$ has zero area and hence $|\mu_f\| \leq (K-1)/(K+1)$ almost everywhere and this implies f is K-quasiconformal if we use the analytic definition of quasiconformality (which we are delaying until a later chapter). The weaker version will be sufficient for our applications.

PROOF OF THEOREM **??**. Suppose that W is any bounded quadrilateral in the plane, say of modulus m and that $W' = F(W)$ has modulus m'. We want to show that $m' \leq Cm$ where $C < \infty$ depends only on K and M as in the statement of the theorem. We will do this by mimicking the proof of Theorem 2.1, that showed that any piecewise differentiable map with bounded dilatation was quasiconformal (in the geometric sense).

Let $\varphi : W \to R = [0,m] \times [0,1]$ and $\psi : W' \to [0,m'] \times [0,1]$ be conformal maps of the quadrilaterals Q, Q' to rectangles R, R' of the same modulus. Let $X = \varphi(\partial\Omega \cap W) \subset R$. The main difficulty with the proof is that we are going to consider three different Whitney decompositions: one for W, one for Ω and one for $U = R \setminus X$. To try to differentiate the different Whitney cubes we we let $\{W_j\}$ denote a Whitney decomposition for W, $\{Q_j\}$ a Whitney decomposition for Ω and $\{U_j\}$ a Whitney decomposition for U.

Fix some $\varepsilon > 0$. Fix a Whitney cube W_j for W. We assume the decomposition is chosen so that $2W_j \subset W$. Suppose $\delta > 0$ is so small (depending on our choice of W_j) that the following conditions all hold:

(1) If Q_k is a Whitney square for Ω with diameter less than δ and the shadow $S(Q_k)$ hits W_j, then $S(Q_k) \subset 2W_j$ and the entire Whitney chain connecting any point $x \in S(Q_k)$ to Q_k is contained in $2W_j$. This is possible by condition (S2) on shadow sets.

(2) Let $\mathscr{S}(W_j)$ denote the collections of all Whitney squares Q_k for Ω so that $\operatorname{diam}(Q_k) \leq \delta$ and $S(Q_k)) \cap W_j \neq \emptyset$. Then

$$\sum_{Q_k \in \mathscr{S}(W_j)} \operatorname{diam}(S(Q_k))^2 \leq \varepsilon \cdot \operatorname{area}(W_j).$$

This holds for small enough δ, because by condition (S3) on shadows, this sum over all Whitney squares for Ω is finite, so removing all the squares bigger than δ gives a sum that tends to 0 as δ tends to zero. Thus we can make is less than $\varepsilon \cdot \operatorname{area}(W_j)$ by taking δ small enough (depending on W_j).

Let $\mathscr{S} = \cup_{W_j} \mathscr{S}(W_j)$ be the collection of all shadow sets of all Whitney squares for Ω that are in some $\mathscr{S}(W_j)$ for some Whitney square W_j of W.

Note that each point $x \in \partial\Omega \cap W_j$ is associated to a Whitney chain that contains a square with diameter comparable to δ. There are only finitely many such squares, so their shadows form a finite collection that covers $\partial\Omega \cap W_j$.

Suppose $L = [a+iy, b+iy]$ is a horizontal segment, compactly contained in the interior of R at height y. We wish to show that

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \leq C\sqrt{mm'}, \tag{7.1}$$

where C depends only on K and M. If we can do this, then by letting $a \to 0$ and $b \to m$ we get

$$m' \leq \lim_{a\to 0, b\to m} |g(b+iy) - g(a+iy)|,$$

and hence

$$m' \leq \lim_{a\to 0, b\to m} \int_0^1 |g(b+iy) - g(a+iy)| dy \leq C\sqrt{mm'},$$

which gives the desired inequality $m' = O(m)$. The reversed inequality, $m = O(m')$, can be deduced from the same argument applied to the other pair of opposite sides of Q, since the corresponding path famlies have the reciprocal moduli. Thus it suffices to prove (**??**).

ADD FIGURE

Since L is compactly contained in the interior of R and X is relatively closed in the interior of R, $L \cap X$ is compact. Thus $\varphi^{-1}(L\cap X)$ is a compact set of W, hence covered by finitely many Whitney squares for W and hence is covered by finitely many shadows sets in $\mathscr{S}$.

Let $\mathscr{X}$ be the image of the elements of $\mathscr{S}$ under φ. Then $L\cap X$ is covered by finitely many elements of $\mathscr{X}$, say $X_1, \dots X_n$. For $k = 1, \dots, n$, let $Y_k = [a_k, b_k]$ be the smallest closed interval in L that contains $X_k \cap L$ (this is the convex hull of $X_k \cap L$, i.e., the interval with the same leftmost and rightmost point as $X_k \cap L$). Then $Y_1, \dots, Y_n$ also cover $L \cap X$ and we can extract a subcover with the property that $Y_j \cap Y_k \neq \emptyset$ implies $|j-k| \leq 1$.

Since the points a_k, b_k are both in the same set X_k, the preimage points $\varphi^{-1}(a_k), \varphi^{-1}(b_k)$ are both in the same element of $\mathscr{S}$. Thus they are both in the shadow set of some Whitney square for Ω and are associated to a two sided chain of distinct Whitney squares $\{Q_m\}_{-\infty}^{\infty}$ of Whitney squares for Ω. If two chains arising in this way, say from Y_k and Y_m with $m > k$, have a Whitney square in common, then we can combine the chains to form a chain connecting a_k to b_m consisting of distinct squares.

After doing this for all intersections, we end up with a finite collection of closed intervals Z_k in L which covers the same set as the union of the Y_k's and such that the two endpoints of each Z_k correspond to a two-sided Whitney chain in Ω and that different intervals use different Whitney squares (no overlapping chains). Moreover, if Z_k has endpoints c_k, d_k and the corresponding chain is $\{Q_n\}$, then

$$|g(c_k) - g(d_k)| \leq \sum_n \mathrm{diam}(\psi(f(Q_n))).$$

The set $V = L \setminus \cup_k Z_k$ consists of finitely many open intervals in $U = R \setminus X$ with their endpoints in X. We break V into countable many subintervals by intersecting it with the Whitney squares for U (without loss of generality, we can assume the endpoints of L occur on the boundary of a Whitney square for U). On each Whitney square U_k for U we define the constant function

$$Dg = \frac{\operatorname{diam}(g(U_k))}{\operatorname{diam}(U_k)}.$$

Then if $L_j = L \cap U_j$,

$$\int_{L_j} Dg dx = \operatorname{diam}(g(U_j))/\sqrt{2}.$$

Thus if Z_L is the union of all the $Z_k \cap l$, we get

$$\int_{L \setminus Z_L} Dg dx \simeq \sum_j \operatorname{diam}(g(U_j)),$$

where the sum is over Whitney squares for U that hit L. Thus

$$|g(b+iy) - g(a+iy)| \lesssim \int_{L \cap U} Dg dx + \sum_n \operatorname{diam}(\psi(f(Q_n))).$$

Now integrate in y to get

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \lesssim \iint_U Dg dx + \sum_n \operatorname{diam}(\psi(f(Q_n)))\mu_n,$$

where μ_n is the Lebesgue measure in $[0,1]$ of the set of lines L_y that use the Whitney square Q_n is at least one of the two-sided chains associated to a interval $Z \subset L_y$. The Lebesgue measure of this set is no more than its diameter, which is no more than the diameter of $X_n = \varphi(S(Q_n))$. Thus

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \lesssim \iint_U Dg dx dy + \sum_n \operatorname{diam}(\psi(f(Q_n))) \operatorname{diam}(X_n),$$

We now estimate each term using the Cauchy-Schwarz inequality. First,

$$\begin{aligned}
&\sum_n \operatorname{diam}(\psi(f(Q_n))) \operatorname{diam}(X_n) \\
&\quad \leq (\sum_n \operatorname{diam}(\psi(f(Q_n)))^2)^{1/2} (\sum_n \operatorname{diam}(X_n)^2)^{1/2} \\
&\quad \leq A \left(\sum_n \operatorname{area}(\psi(f(Q_n))) \right)^{1/2} \left(\sum_{W_k} \sum_{Q_n \in \mathscr{S}(W_k)} \left[\frac{\operatorname{diam}(\varphi(W_k))}{\operatorname{diam}(W_k)} \operatorname{diam}(S(Q_n)) \right]^2 \right)^{1/2}.
\end{aligned}$$

Now use Lemma **??**,

$$\begin{aligned}
&\quad \leq A (\sum_n \operatorname{area}(\psi(f(Q_n))))^{1/2} (\sum_{W_k} \sum_{Q_n \in \mathscr{S}(W_k)} \left[\frac{\operatorname{diam}(\varphi(W_k))}{\operatorname{diam}(W_k)} \right]^2 \varepsilon \operatorname{area}(W_k)))^{1/2} \\
&\quad \leq A \left[\sum_n \operatorname{area}(R')^{1/2} \cdot \varepsilon \cdot \operatorname{area}(R) \right]^{1/2}.
\end{aligned}$$

where A just depends on the distortion estimate for conformal maps (Theorem 1.23) and ε is as small as we wish. Thus this term is small.

The other term is also bounded by Cauchy-Schwarz

$$\begin{aligned}
\iint_U Dg dx &= \sum_k \iint_{U_k} Dg dx dy \\
&\leq \left(\sum_k \iint_{U_k} (Dg)^2 dx dy\right)^{1/2} \left(\sum_k \iint_{U_k} dx dy\right)^{1/2} \\
&\leq \left(\sum_k (\mathrm{diam}(g(U_k))^2\right)^{1/2} \mathrm{area}(R)^{1/2} \\
&\leq C\left(\sum_k (\mathrm{area}(g(U_k))\right)^{1/2} \mathrm{area}(R)^{1/2} \\
&\leq C \,\mathrm{area}(R')^{1/2} \cdot \mathrm{area}(R)^{1/2} \\
&\leq C\sqrt{m'm}.
\end{aligned}$$

Thus

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \lesssim \sqrt{m'm} + O(\varepsilon),$$

Taking $\varepsilon \to$ gives the desired inequality. $\square$

COROLLARY 7.4. *If K satisfies the conditions of Theorem **??**, them K is removable for conformal homeomorphisms, i.e., and homeomorphism of the plane that is conformal off K is conformal everywhere.*

PROOF. Theorem **??** implies that f is quasiconformal on the plane, so the point is to show that we can take the quasiconformal constant to be 1. If we redo the proof assuming f is conformal off $\partial\Omega$, then the piecewise constant function Dg can be replaced by the usual derivative $|g'|$. This leads to the inequality

$$m' \leq \sqrt{m'm},$$

or $m' \leq m$. This, together with the reverse ineqaulity which follows by considering the reciprocal path family in each quadrilateral, implies f is 1-quasiconformal, hence is conformal. $\square$

COROLLARY 7.5. *If f, g are quasiconformal maps of the upper and lower half-planes that agree on the real line, then they define a quasiconformal map on the whole plane.*

PROOF. This is immediate since a line clearly satisfies the Jones-Smirnov criteria: just consider $\mathbb{R}$ as the boundary of the upper half-plane and for $x \in \mathbb{R}$, let γ_x be a vertical line ray. Then the shadow of any square is its vertical projection, and the square of the shadows length is comparable to the area of the square. Thus any compact segment of $\mathbb{R}$ is removable, and since quasiconformality is a local property (Theorem **??**), the whole line is removable. $\square$

COROLLARY 7.6. *If f is a quasiconformal map of the upper half-plane to itself, mapping the real line to itself, then the extension of f to the whole plane by $f(\bar{z}) = \overline{f(z)}$ is quasiconformal in the whole plane.*

PROOF. Immediate from the previous result since composing a quasiconformal map with reflections gives another quasiconformal map. $\square$

COROLLARY 7.7. *Quasicircles are removable.*

PROOF. If $\Gamma = g(\mathbb{R})$ is a quasiconformal image of the reals and f is a homeomorphism that is quasiconformal on each side of Γ, then $h = f \circ g$ is a homeomorphism that is quasiconformal on each side of $\mathbb{R}$, then quasiconformal on the whole plane. Thus $f = h \circ g^{-1}$ is a composition of quasiconformal maps and hence is quasiconformal. $\square$

An open, connected set Ω in $\mathbb{R}^2$ is called a **John domain** if any two points $a, b \in \Omega$ can be connected by a path γ in Ω with the property that $\mathrm{dist}(z, \partial\Omega) \gtrsim \min(|z-a|, |z-b|)$. See Figure **??**

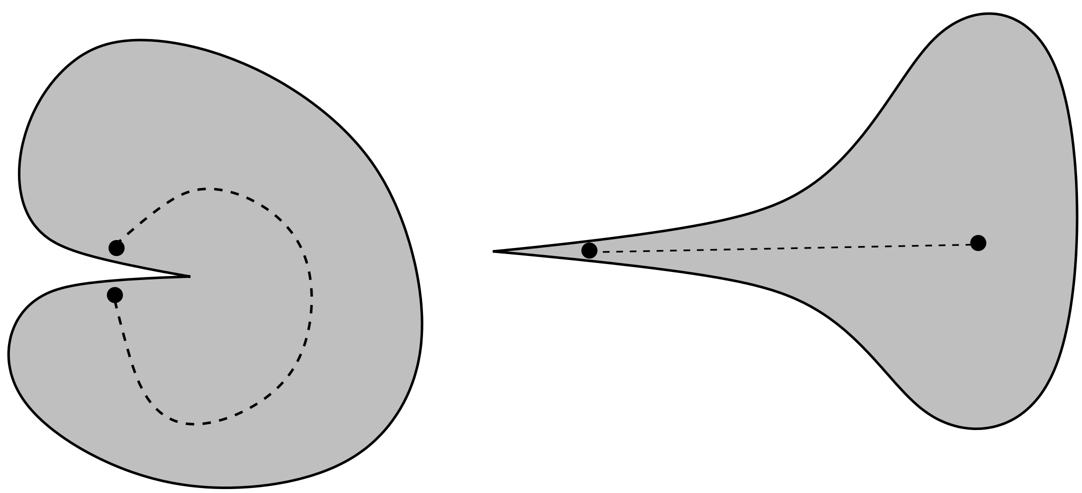


FIGURE 7.5. The domain on the left is a John domain, but the one on the left is not; inward pointing cusps are OK, but outward pointing cusps are not.

LEMMA 7.8. *The Riemann map φ from the unit disk to a bounded John domain satisfies*

$$\mathrm{diam}(\varphi(I(Q))) \le C\,\mathrm{diam}(\varphi(Q)),$$

$$\mathrm{dist}(\varphi(Q), \varphi(I(Q))) \le C\,\mathrm{diam}(\varphi(Q)),$$

for some constant $C < \infty$ and any Whitney square Q and is shadow $I(Q)$.

PROOF. The second inequality follows directly from Lemma 4.4 by considering the path family of radial lines connecting Q to I. To prove the first, consider the Whitney-Carleson boxes Q_1 and Q_2 that are adjacent to Q and of the same size. By Lemma 4.4 each is connected to its shadow by a radial segment whose image under f has length comparable to $\mathrm{diam}(f(Q))$. Thus there is a geodesic crosscut γ of the disk that passes through Q and whose image has length comparable to $\mathrm{diam}(f(Q))$.

Now suppose x is in the shadow of Q. Any curve connecting 0 to x crosses γ, so any curve Γ connecting $f(0)$ and $f(x)$ crosses $f(\gamma)$ and hence contains a point $z \in f(\gamma) \cap \Gamma$ that is at most distance $O(\mathrm{diam}(f(Q))$ from $\partial\Omega$. Thus by the definition of John domain, either

$$\mathrm{dist}(f(0), z) = O(\mathrm{diam}(f(Q))),$$

or

$$\mathrm{dist}(f(x), z) = O(\mathrm{diam}(f(Q))).$$

In a bounded domain, the first can only happen for finitely many Qs; for the remainder, the second must hold and hence $f(I(Q))$ is contained in a $O(\mathrm{diam}(f(Q))$ neighborhood of $f(Q)$. □

COROLLARY 7.9. *Boundaries of John domains are removable.*

PROOF. The conclusions of the Lemma **??** easily imply (1)-(3) in Theorem **??**. □

The Jones-Smirnov result (Theorem **??**) places restrictions on the set E, but none on the mapping (besides being a homeomorphism). An earlier result of Rickman (e.g., [], []) makes an assumption on the mapping, but none on the set K:

LEMMA 7.10 (Rickman's lemma). *Suppose Ω is a planar domain and $K \subset \Omega$ is compact. Suppose f is homeomorphism of Ω that is quasiconformal on $\Omega \setminus K$ and F is quasiconformal on all of Ω. If $f = F$ on K, then f is quasiconformal on all of Ω.*

PROOF. Isolated points of K are clearly removable and there are only countable many such points, so we may assume that every point of K is an accumulation point.

The idea proof is the same as the proof of Theorem **??**: we consider a quadrilateral W and its image $W' = f(W)$ and conformally map each to rectangles of modulus m and m' respectively. Let $G = \psi \circ F \circ \varphi^{-1}$ and $g = \psi \circ f \circ \varphi^{-1}$. Then our assumption implies $g = G$ on X.

As before, we want to prove the estimate (**??**). However, this time we cover X by dyadic squares that are so small that both g and G are quasiconformal on $6Q$ for each square Q used, and both images of Q lie in R'.

The union of these squares plays the role of the set Z in the earlier proof. Given a compact horizontalline segment L in R, we let $\{Y_k\}$ enumerate the convex hulls of sets of the form $L \cap Q$ for Q in our cover of X. Then defining Dg exactly as before on $R \setminus X$, and using $g = G$ on X, we get

$$\begin{aligned} |g(b+iy)-g(a+iy)| &\leq \int_{L\cap U} Dg dx + \sum_k |g(c_k)-g(d_k)| \\ &\leq \int_{L\cap U} Dg dx + \sum_k |G(c_k)-G(d_k)| \\ &\leq \int_{L\cap U} Dg dx + \sum_{Q:Q\cap L\neq\emptyset} \operatorname{diam}(G(Q)). \end{aligned}$$

Integrating over y then gives

$$\int_0^1 |g(b+iy)-g(a+iy)|dy \leq \int_U Dg dx + \sum_Q \operatorname{diam}(G(Q))\ell(Q).$$

The first term is bounded exactly as before and the second is bounded by

$$\begin{aligned} \sum_Q \operatorname{diam}(G(Q))\ell(Q) &\leq [\sum_Q \operatorname{diam}(G(Q))^2]^{1/2}[\sum_Q \ell(Q)^2]^{1/2} \\ &\leq C[\sum_Q \operatorname{area}(G(Q))]^{1/2}[\sum_Q \operatorname{area}(Q)]^{1/2} \\ &\leq C[\operatorname{area}(R')]^{1/2}[\operatorname{area}(R)]^{1/2} \\ &\leq C\sqrt{m'm}. \end{aligned}$$

The rest of the proof is them completed just as before. □

8. Quasisymmetric maps and quasicircles

A homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is called M**-quasisymmetric** if $|h(I)| \leq M|h(J)|$ whenever I and J are adjacent intervals of equal length. Equivalently,

$$\sup_{t\in\mathbb{R},x>0} \frac{h(x+t)-h(t)}{h(t)-h(x-t)} \leq M.$$

A homeomorphism is called **quasisymmetric** if it is M-quasisymmetric for some $M < \infty$. Later we will discuss quasisymmetic map of the unit circle to itself, but for the moment we stick to maps of $\mathbb{R}$ to $\mathbb{R}$.

THEOREM 8.1. *A homeomorphism $h : \mathbb{R} \to$ reals is quasisymmetric if and only if it extends to a quasiconformal mapping of the plane to itself.*

PROOF. First we show that if f is a K-quasiconformal map of the plane that maps $\mathbb{R}$ to itself, then the restriction of f to $\mathbb{R}$ is quasisymmetric. Without loss of generality we may assume $I = [0,1/2]$ and $J = [1/2,1]$ and that f fixes 0 and 1. Consider the modulus of the topological annuus $A = \mathbb{C} \setminus ([0,1] \cup [2,\infty)$ This has a fixed finite, non-zero modulus, so its image $B = f(A) = \mathbb{C} \setminus ([0,x] \cup [1,\infty))$ also has modulus bounded between two positive real numbers that depend only on K. If $x = f(1/2)$ is too close to 0 or 1, then B clear has modulus close to 0 or ∞ respectively, a contradiction. Thus x is bounded away from both 0 and 1 with an estimate depending only on K, and hence h is M-quasiconformal with a constant depending only on K.

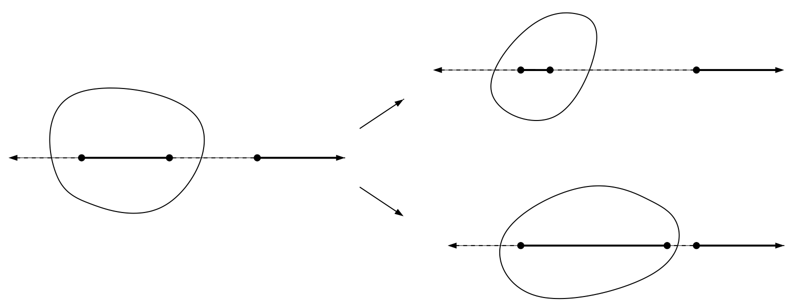


FIGURE 8.1. On the left is the topological annulus A. If $x = f(1/2)$ is too close to 0 or 1, the image annulus has small or large modulus, contradicting $\frac{1}{K}M(A) \leq M(f(A)) \leq KM(A)$. The two possibilities are shown on the right.

Next suppose $h : \mathbb{R} \to \mathbb{R}$ is M-quasisymmetric. We will assume h is increasing; the other case is handled by a similar argument. We will use the fact that the hyperbolic upper half-plane can be tesselated by hyperbolically identical right pentagons. The corresponding picture for the disk is shown in Figure **??**.

Each right pentagon in the tesselation of the upper half-plane determines five hyperbolic geodesics containing its sides, and these deterine ten distinct points on the real line. The h images of these point are also ten distinct points and the same pairs of point determine five new geodesics that define a hyperbolic pentagon (it need not be regular or right). There is a diffeomorphism of the right pentagon to this new one that presevers arc-length along the edges in the sense that on each side of the pentagon length are multiplied by the ratio of the image length over the starting length. This ensures that the diffeomorphisms defined on adjacent pentagons agree on the common sides. These diffeomorphisms come from a compact family of possibilities, thus have uniformly bounded dilatations, and hence define

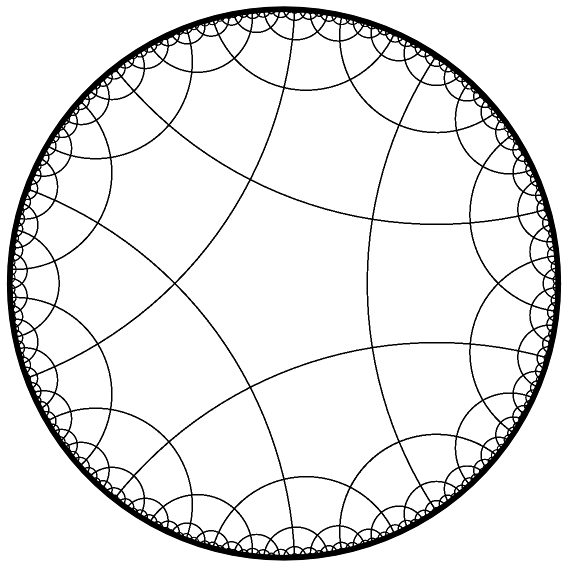


FIGURE 8.2. Hyperbolic space is tesselated by hyperbolically identical right pentagons. There is a corresponding picture on the upper half-plane model.

a quasiconformal map of the half-plane to itself that agrees with h on the boundary.

$\square$

9. Quasicircles

We say that a curve γ satisifies the **3-point condition**, if there is a $M < \infty$ so that given any $x, z \in \gamma$ and y on the smaller diameter arc $\gamma(x,y) \subset \gamma$ between x, y, we have

$$|x - y| \leq M|x - z|,$$

Equaivalently,

$$\operatorname{diam}(\gamma(x,z)) \leq M|x - z|.$$

This condition is also called the **Ahlfors M-condition** or **bounded turning**. It is immediate from Lemma **??** that the image of the real line under any quasiconformal mapping of the plane is bounded turning, and below we shall prove the converse is also true.

The similar looking, but stronger, condition

$$\ell(\gamma(x,z)) \leq M|x - z|$$

where we assume γ is locally rectifiable is called the chord-arc condition. Such curves are called **chord-arc curves** or **Lavrentiev curves**, and form a special, but very important, subclass of the bounded turning curves. It

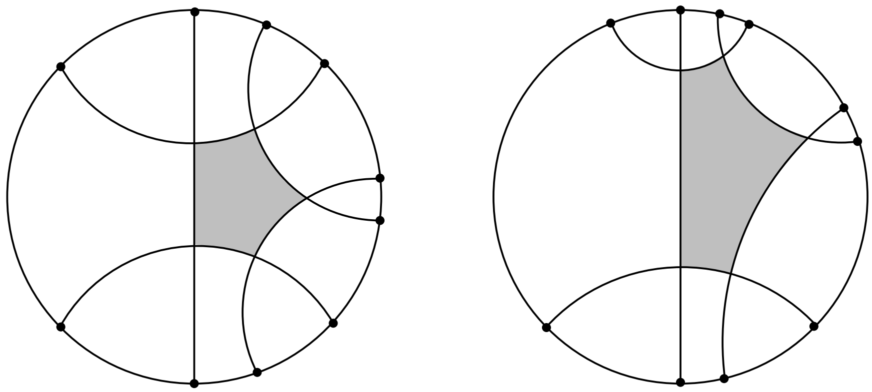


FIGURE 8.3. Two hyperbolic pentagons in the disk, both normalized to have one side on the vertical axis and containing the origin. If the corresponding 10 points on the circle are related by a M-quasisymmetric map, then the sides of pentagons all have comparable hyperbolic length and can be mapped to each other by a diffeomorphism that multiplies lengths on each side and has dilatation bounded in terms of M.

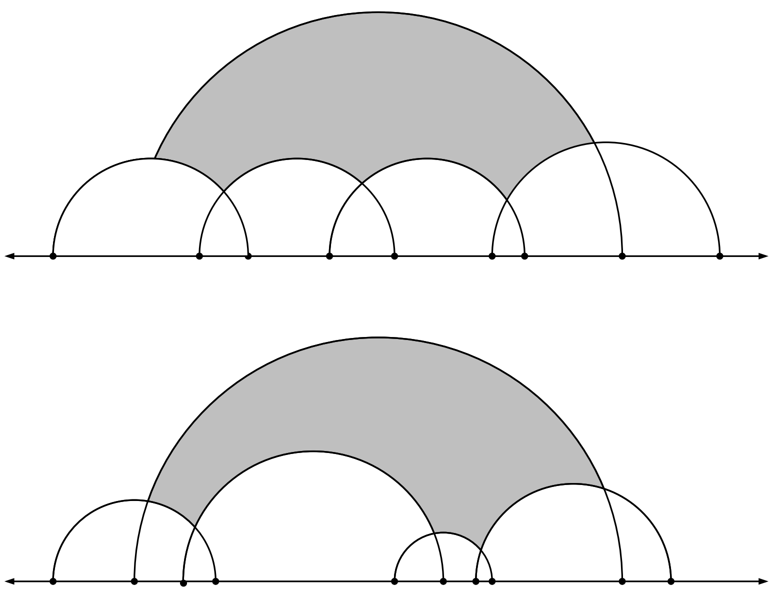


FIGURE 8.4. Sides of a hyperbolic right pentagon determine 5 geodesics and 10 boundary points. The images of these 10 points detremine 5 geodesics, which give a hyperbolic pentatgon. We take any QC map between the pentagons that multiplies hyperbolic arclengh on each edge by a constant (the ratio of the lengths of an edge and it image).

turns out that chord-arc curves are exactly the images of the real line under bi-Lipschitz maps of the plane, but we will not prove this here. See [], [].

To prove that bounded turning curves are quasicircles, we will need the following lemma.

LEMMA 9.1. *Suppose γ is bounded turning with constant M and $0,1 \in \gamma$. Suppose Ω is one of the connected components of $\mathbb{C}\setminus\gamma$ and suppose z is a point on γ between 0 and 1. Let $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ denote the disjoint subarcs of γ from $-\infty$ to 0, from 0 to x, from x to 1 and from 1 to $+\infty$ respectively. Let Γ be the path family joining the arc $\gamma_x \subset \gamma$ from 0 to x to the disjoint half-infinite arc $\gamma_1 \subset \gamma$ joining 1 to ∞. Then $M(\Gamma) \to 0$ as $x \to 0$ with upper and lower bounds that depend only on $|x|$ and M*

PROOF. The 3-point condition implies that
$$\mathrm{dist}(\gamma_2,\gamma_4) \geq \frac{1}{M} - |x|,$$
so for $|x|$ sufficiently small every path in Γ crosses the round annulus
$$\{z : M|x| < |z| < \frac{1}{2M}\} \subset \{z : \mathrm{diam}(\gamma_2) < |z| < \mathrm{dist}(\gamma_2,\gamma_4)\}.$$
For $|x|$ small, this implies $M(\Gamma)$ is small.

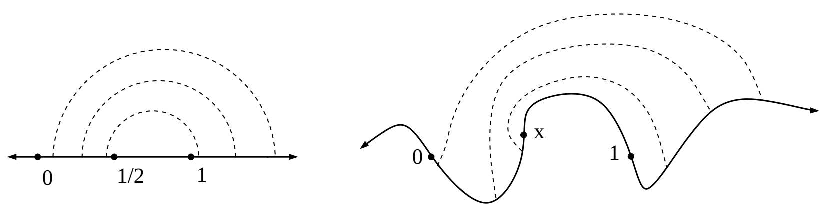


FIGURE 9.1.

To prove a lower bound on $M(\Gamma)$ it suffices to prove an upper bound on the reciprocal modulus of the path family connecting γ_1 to γ_3. By the 3-point condition, these arcs are at least distance $|x|/M$ apart, so the metric $\rho = M/|x|$ on the disk of radius M around the origin is admissible. Hence the modulus of the reciprocal family is at most $\pi M^4/|x|^2$ and so $M(\gamma) \geq |x|^2/M^4\pi$. $\square$

LEMMA 9.2. *If γ has bounded turning, and f,g are the conformal maps from the upper and lower half-planes to the two sides of γ (mapping ∞ to ∞ in both cases), then $h = g^{-1}\circ f$ is a quasisymmetric homeomorphism of the line.*

PROOF. Consider two adjacent intervals of equal length on the real line. After renormalizing by linear maps, we may assume these are $I =$

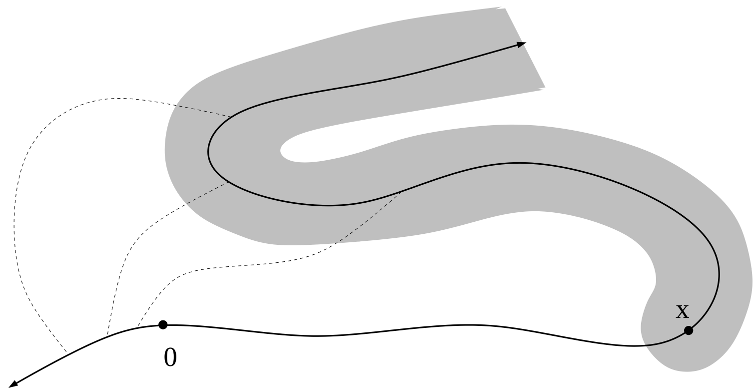


FIGURE 9.2. Since $\operatorname{dist}(\gamma_1, \gamma_3) \geq |x|/2M$, the metric $\rho = 1/2M$ is admissible.

$[0,1/2]$ and $[[1/2,1]$ and that h fixes both 0 and 1. By two applications of Lemma **??**, $f(1/2)$ can't be too close to either 0 or 1, and hence $h(1/2) = g^{-1}(f(1/2))$ can't be too close to 0 or 1 either. Thus h is quasisymetric with a constant that depends only on the 3-point constant.

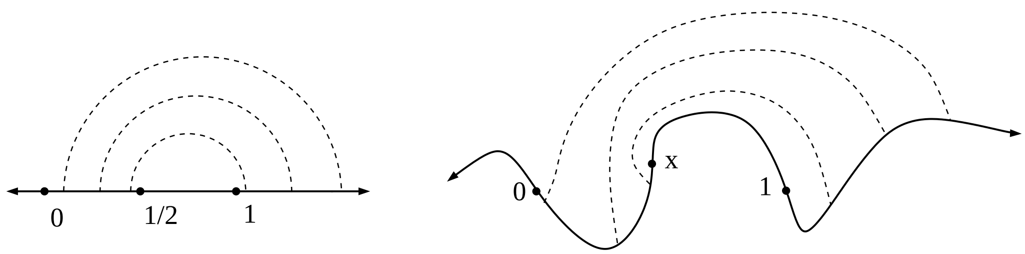


FIGURE 9.3. The path family in the upper half-plane connecting $[0,1/2]$ to $[1,\infty)$ has modulus 1, so its conformal image also has modulus 1. Therefore $x = f(1/2)$ can't be too close to either 0 or 1.

$\square$

LEMMA 9.3. *A curve γ is a quasi-line if and only if it has bounded turning.*

PROOF. If γ is the quasiconformal image of a line, then it satisfies the 3-point condition by Lemma **??**, as mentioned earlier. On the other hand, if γ satisfies the 3-point condition, then $h = g^{-1} \circ f$ (as defined in Lemma 1.23) is quasisymemtric, and hence extends to a quasiconformal map H of the whole plane. Now set $F = f$ on the lower half-plane and $F = g \circ H$ on the upper half-plane. Clearly this is quasiconformal on each half-plane and on the real line $g \circ H = g \circ g^{-1} \circ f = f$ so the two definitions agree. Thus H

is quasymmetric on the whole plane by Corollary 1.23 and $F(\mathbb{R}) = f(\mathbb{R}) = \gamma$. $\square$

CHAPTER 3

Analytic aspects of quasiconformal mappings

1. Piecewise affine maps

We say that a linear map f is **K-quasiconformal** if $D_f \leq K$. The linear map need not be defined on the whole plane. Given two triangles T_1, T_2 with vertices a,b,c and A,B,C, there is a unique affine map $T_1 \to T_2$ taking $a \to A$, $b \to B$ and $c \to C$. The map is orientation preserving if both triangles were labeled in the same orientation.

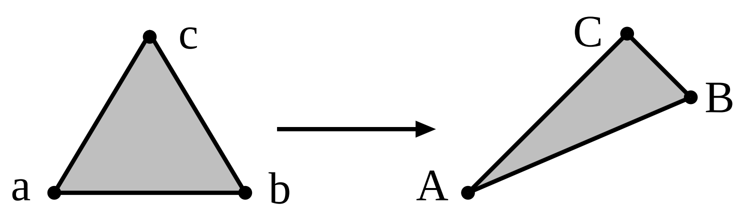


FIGURE 1.1. A pair of similarly oriented, labeled triangles defines a linear map and has an associated dilatation.

There is an obvious affine map between these triangles and we can easily compute its quasiconformal constant of this map as follows. First use a conformal linear map to send each triangle to one of the form $\{0,1,a\}$ and $\{0,1,b\}$. The affine map is then of the form $f(z) \to \alpha z + \beta \bar{z}$ where $\alpha + \beta = 1$ and $\beta = (b-a)/(a-\bar{a})$ and from this we see that

$$K_f = \frac{1+|\mu_f|}{1-|\mu_f|},$$

where

$$\mu_f = \frac{f_{\bar{z}}}{f_z} = \frac{\beta}{\alpha} = \frac{b-a}{b-\bar{a}},$$

If the triangle T' is degenerate, or has the opposite orientation as T, we simply give ∞ as our QC bound K.

2. The mapping theorem

THEOREM 2.1. *If μ is continuous on the plane and $|\mu| \leq k < 1$, then there is a sequence of K-quasiconformal maps $\{f_n\}$ with dilatations μ_n so*

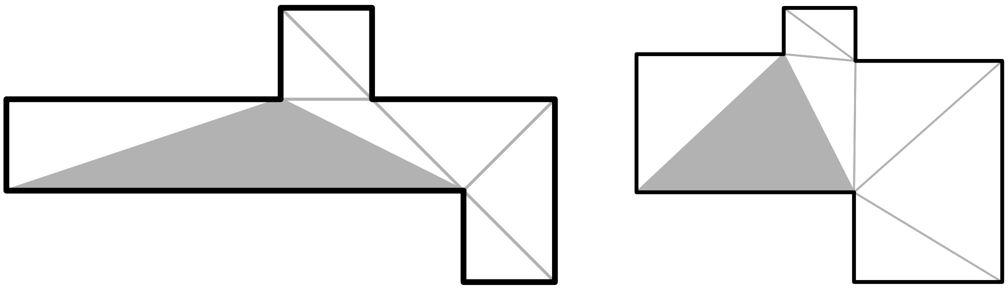


FIGURE 1.2. Two compatible triangulations of different polygons. The most distorted triangle is shaded; this determines an upper bound for the piecewise linear map between the polygons.

that $|\mu_n| \leq k$, $\sup_{\mathbb{C}} |\mu - \mu_n| \to 0$ *and so that* $\{f_n\}$ *converges uniformly on compact sets to a K-quasiconformal map* f.

PROOF. □

COROLLARY 2.2. *Suppose* f *is an entire function and* ψ *is K-quasiconformal map that has a continuous dilatation* μ. *Then there is a K-quasiconformal map* φ *so that* $g = \psi \circ f \circ \varphi$ *is entire.*

PROOF. □

The uniformization theorem states that any simply connected Riemann surface is conformally equivalent to either the 2-sphere $\mathbb{S}$, the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. If the surface is non-compact, the sphere is eliminated and surface must be equivalent to either $\mathbb{C}$ or $\mathbb{D}$. These two choices can be distinguished using extremal length: choose a compact connect set K on the surface and consider the set of rectifiable paths that separate K from ∞. If this family has finite modulus, then the surface is equivalent to the disk and otherwise the modulus is infinite and the surface is equivalent to the disk.

For our applications, we only need to use the uniformization theorem in the case when R is built by attaching Euclidean triangles along their edges in a way that is combinatorially identical to the usual triangulation of the plane by identical equilateral triangles. See Figure **??**.

Subharmonic function play an important role in the Perron process for solving the Dirichlet problem on a planer domain or Riemann surface. Suppose Ω is a Riemann surface and we are given a collection of subregions $\{\Omega_j\}$ on which we can solve the Dirichlet problem (e.g., a collection of disks, where we can use the Poisson formula). If $f \in C(\partial\Omega)$ and $\partial\Omega$ is compact, then f has a lower bound M. Let $\mathscr{F}$ be the collection of subharmonic functions v on Ω that have continuous boundary values less than f on $\partial\Omega$. The collection is non-empty since the constant M is in it.

Let $u = \sup\{v : v \in \mathscr{F}\}$. We claim u is harmonic. It is clearly subharmonic since it is a supremum of subharmonic functions. IN each Ω_n we can solve the Dirichlet problem in Ω_n with boundary data u; if u were not harmonic in Ω_n, replacing u with this solution in Ω_n would give a strictly larger element of $\mathscr{F}$.

The final step is to prove that u has the correct boundary values. This requires some assumption on $\partial\Omega$, since it is not true the Dirichlet problem can be solved for every domain.

EXERCISE: Show that if $\Omega = \mathbb{D} \setminus \{0\}$ and we set $f = 1$ on $\mathbb{T}$ and $f = 0$ at 0, then the function u created by the Perron process is the constant 1, and hence does not solve the Dirichlet problem. Indeed, there is no harmonic function on Ω with the given boundary values.

We say a barrier exists at $x \in \partial\Omega$ if there is a $r > 0$ and a non-negative, harmonic function V on $\Omega' = \Omega \cap D(x,r)$ so that

$$\limsup_{z \to x} V(x) \leq 0,$$

but

$$\liminf_{z \to y} V(x) > 0, y \in \partial\Omega' \setminus \{x\}$$

and $V \geq 1$ on $\{|z - x| = r\} \cap \Omega$.

LEMMA 2.3. *If there is a barrier at x then the Perron solution u extends continuously to x and equals f there.*

PROOF. First consider the special case when f takes values in $[0,1]$ and x is the unique point where f takes the minimal value 0. Suppose $v \in \mathscr{F}$. Since v is subharmonic on $\Omega' = \Omega \cap D(x,r)$ and bounded above by 1, it is bounded above by V □

LEMMA 2.4. *If the dilatation is symmetric with respect to a circle (or line), the corresponding quasiconformal function can be chosen to be symmetric with respect to the same circle (or line).*

COROLLARY 2.5. *If f is piecewise continuous K-quasiconformal on an open set $\Omega \subset \mathbb{C}$ then there is a K-quasiconformal map $g : \mathbb{C} \to \mathbb{C}$ so that $f \circ g$ is conformal on Ω.*

PROOF. The dilatation μ of f is defined on Ω and set it to be zero on the rest of the plane. Apply the construction above to generate a sequence $\{g_n\}$ and limit g. Then $gn \circ f$ □

COROLLARY 2.6. *If $f : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal an onto, and we extend f to a map $\mathbb{C} \to \mathbb{C}$ by reflection*

$$f(1/\bar{z}) = 1/\overline{f(z)},$$

then the extension is K-quasiconformal on the whole plane.

COROLLARY 2.7. *If φ is K-quasiconformal on $\mathbb{C}$ and g is holomorphic on Ω then there is a K-quasiconformal map ψ on Ω such that $f = \varphi \circ g \circ \psi^{-1}$ is holomorphic on $\psi(\Omega)$.*

In this case the functional relation can be rewritten as

$$f \circ \psi = \varphi \circ g.$$

If $\Omega = \mathbb{C}$, such a pair of functions f and g are called **quasiconformally equivalent**. We will examine such pairs in more detail in later sections (see Sections 1.23, 1.23).

COROLLARY 2.8.

We can extend the mapping theorem from continuous dilatations μ to measurable ones, using the following two results that will be proven later (see Theorems **??** and **??**):

PROPOSITION 2.9. *A K-quasiconformal map f defined on a planar domain Ω is differentiable almost everywhere on Ω. The dilatation $\mu_f = f_{\bar{z}}/f_z$ is well defined and less than $k < 1$ almost everwhere.*

PROPOSITION 2.10. *Suppose $\{f_n\}$, f are all K-quasiconformal maps on the plane with dilatations $\{\mu_n\}$, μ_f respectively, that $f_n \to f$ uniformly on compact sets and that $\mu_n \to \mu$ pointwise almost everywhere. Then $\mu_f = \mu$ almost everywhere.*

Assuming this, we can deduce:

THEOREM 2.11. *[Measurable Riemann Mapping Theorem] Given any measurable function μ on the plane with $\|\mu\|_\infty = k < 1$, there is a $K = (k+1)/(k-1)$ quasiconformal map f with dilatation $\mu_f = \mu$ Lebesgue almost everywhere on $\mathbb{C}$.*

PROOF. Given a measurable μ find a sequence of continuous functions $\{\mu_n\}$ with $\mu_n \to \mu$ pointwise and $\sup_{\mathbb{C}} |\mu_n(z)| \leq k = \|\mu\|_\infty < 1$. Let f_n be the quasiconformal map with dilatation μ_n, normalized to fix both 0 and 1. Then since normalized, K-quasiconformal maps form a compact family (Theorem 1.23) there is a subsequence of these maps that converges uniformly on compact sets to a K-quasiconformal map f. This map has a dilatation μ_f. Then $\mu_f = \mu$ by Theorem **??**. □

The main goal of this chapter is toprove the two propositions above, and complet the proof of the mapping theorem. Another goal of this chapter Pompeiu's formula (proven for C^1 functions in Chapter 1.23):

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_{\Omega} \frac{f_{\bar{z}}}{z-w}dxdy. \tag{2.1}$$

However, it is not even clear whether this formula makes sense for a quasiconformal map; since f is continuous, the first integral is well defined, but it is not clear whether the second integral is well defined in general; we need to verify that $f_{\overline{w}}$ is defined.

We expect (but have not yet proved) that

$$\text{area}(f(\Omega)) = \int_\Omega J_f dxdy = \int_\Omega |f_z|^2 - |f_{\overline{z}}|^2 dxdy = \int_\Omega |f_z|^2(1-|\mu_f|^2) dxdy,$$

which would imply f_z and $f_{\overline{z}}$ are in L^2 locally. However, $|z-w|^{-1}$ is not in L^2, so we can't be sure that the area integral in the Pompeiu formula is convergent. However, $|z-w|^{-1}$ it in L^q locally for every $q<2$, so the integral will be bounded if we can show $f_{\overline{z}} \in L^p$ locally for some $p>2$. This is a fundamental result of Bojarski in $\mathbb{C}$ [] and of Gehring [] in dimensions ≥ 2 and we will prove it later in this chapter, using the 2-dimensional version of Gehring's proof.

Another problem with proving the Pompeiu formula for quasiconformal maps is a little more subtle. As noted above, we know the formula is valid for smooth functions and to verify it for general quasiconformal maps, we would like to smooth these functions (say by convolution with a smooth, radial bump function) and pass to a limit. In this case, the smoothed functions converge uniformly to the limit, so the boundary integral term converges as desired, but the integrand of the area integral only converges pointwise and we need some extra condition to insure this integral also converges. In this case, we can use Gehring's result and the L^p boundedness of the Hardy-Littlewood maximal theorem to deduce that the sequence integrands coming from the smooth approximations of f is dominated by fixed L^1 function, so the Lebesgue dominated convergence theorem can be applied to verify Pompeiu's formula. Pompeiu's formula can then we applied to prove the differentiable dependence of f on its dilatation μ.

3. Covering lemmas and maximal theorems

We start with a review of some basic real analysis and them move towards the theorem of Bojarski and Gehring and its consequences.

THEOREM 3.1 (Wiener's Covering Lemma). *Let $\mathscr{B} = \{B_j\}$ be a finite collection of balls in $\mathbb{R}^d$. Then there is a finite, disjoint subcollection $\mathscr{C} \subset \mathscr{B}$ so that*

$$\cup_{B \in \mathscr{B}} B \subset \cup_{B \in \mathscr{C}} 3B.$$

In particular, the Lebesgue measure of the set covered by the subcollection is at least 3^{-d} times the measure covered by the full collection.

The above lemma seems to be first due to Wiener [**?**].

THEOREM 3.2 (Vitali Covering Lemma). *Suppose $E \subset \mathbb{R}^d$ is a measurable set and $\mathscr{B} = \{B_j\} \subset \mathbb{R}^d$ is a collection of balls so that each point of E is contained in elements of $\mathscr{B}$ of arbitrarily small diameter. Then there is a subcollection $\mathscr{C} \subset \mathscr{B}$ so that $E \setminus \cup_{B \in \mathscr{C}} B$ has zero d-measure.*

THEOREM 3.3 (Lebesgue Dominated Convergence theorem). *Suppose $g \in L^2(\mu)$ and $\{f_n\}$ satisfy $|f_n| \leq g$ and $\lim f_n = f$ pointwise. Then $\lim \int f_n d\mu = \int f d\mu$.*

THEOREM 3.4 (Egorov's Theorem). *Suppose μ is a finite positive measure and $\{f_n\}$ is a sequence of measurable functions that converge to f pointwise almost everywhere on a set E with respect to μ. Then for every $\varepsilon > 0$ there is a subset $F \subset E$ so that $\mu(E \setminus F) < \varepsilon$ and $f_n \to f$ uniformly on F.*

LEMMA 3.5 (The Calderon-Zygmund lemma). *) Suppose Q is a square, $u \in L^1(Q, dxdy)$ and suppose*

$$\alpha > \frac{1}{\text{area}(Q)} \int_Q |u| dxdy.$$

Then there is a countable collection of pairwise disjoint open dyadic subsquares of Q so that

$$\alpha \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} |u| dxdy < 4\alpha, \tag{3.1}$$

$$|u| \leq \alpha \text{ almost everywhere on } Q \setminus \cup_j Q_j, \tag{3.2}$$

$$\sum \text{area}(Q_j) \leq \frac{1}{\alpha} \int_Q |u| dxdy \tag{3.3}$$

PROOF. We say a subsquare of Q has property P is the first conclusion above holds and we define a collection of subsquares by iteratively dividing squares that do not have property P into four, equal sized disjoint subsquares, and stopping when property P is achieved. If the average of u over a square is less than α then average over each of the four subsquares is $< 4\alpha$, so every stopped square has property P. Any point not in a stopped square is a limit of squares where the average of u is $< \alpha$, so by the Lebesgue differentiation theorem $u \leq \alpha$ at almost every such point. Finally,

$$\int_Q |u| dxdy \geq \sum_j \alpha \, \text{area}(Q_j),$$

which proves the third property. $\square$

Hardy-Littlewood maximal function.
Marcinkiewicz interpolation
L^p boundedness of maximal function
Maximal function bounds maximal function of convolution with radial L^1 bump function.

4. Absolute continuity on lines

The main type of K-quasiconformal maps used in this text are piecewise C^1 functions that satisfy

$$|f_{\bar{z}}| \leq k|f_z|, \tag{4.1}$$

where $k-(K-1)/(K+1)$. By itself, this equation is not enough to guarantee a map is quasiconformal. For example, suppose $g:[0,1]\to[0,1]$ is the usual Cantor singular function.e., a continuous function that increases from 0 to 1 on $[0,1]$ and is constant on each complementary component $\{I_j\}$ of the Cantor middle-$\frac{1}{3}$ set E. Then the map $f(x,y)=(x+g(x),y)$, is a homeomorphism of $[0,1]\times[0,1]$ to $[0,2]\times[0,1]$ that is a translation (hence conformal) on each rectangle $I_j\times[0,1]$, where I_j is a complementary interval of the Cantor set. Thus $f_{\bar{z}}=0$ almost everywhere, but there are several way to check that f is not quasiconformal.

EXERCISE : Find rectangles whose modulus is increased by arbitrarily large factors by f.

EXERCISE: Find a path family Γ of zero modulus, so that $f(\Gamma)$ has positive modulus.

EXERCISE: Show that f map some set of zero area to positive area (later we will prove quasiconformal maps can't do this).

The problem with this example is that it is not absolutely continuous on horizontal lines, and so f cannot be recovered by integrating its partials.

A function f is called **absolutely continuous** on a line L if for every $\varepsilon>0$ there is a $\delta>0$ so that $m_1(E)<\delta$ implies $m_1(f(E))<\varepsilon$ where m_1 denotes 1-dimensional Hausdorff measure.

THEOREM 4.1. *If f is quasiconformal, then f is absolutely continuous on almost every line in any given direction.*

PROOF. After a Euclidean similarity, we may consider horizontal lines in $Q=[0,1]^2$. Define

$$A(y)=\text{area}(f([0,1]\times[0,y])).$$

Then $A(0)=0$, $A(1)=\text{area}(f(Q))<\infty$ and A is increasing. Thus A is continuous except on a countable set and has a finite derivative almost everywhere. Fix a value of y where both this things happen, and we will show that f is absolutely continuous on the horizontal line $L_y=[0,1]\times\{y\}$. The main idea is that if this failed, then modulus estimates relating length to area will force $A'(y) = \infty$.

Consider the long, narrow rectangle $R = [0,1] \times [y, y+\frac{1}{n}]$ and divide it into $m << n$ disjoint $\frac{1}{m} \times \frac{1}{n}$ sub-rectangles $\{R_j\}$. Let $R'_j = f(r_j)$ and the the "left", "right", and "bottom" edges of R'_j be the images under f of corresponding edges of R_j. Let b_j be length of $f(L_y \cap \partial R_j)$, i.e., the length of the bottom edge of R'_j. This number might be finite or infinite. Fix $\varepsilon > 0$. In the first case, by taking n large enough, we can insure that any curve in $f(R_j)$ than joins the images of the vertical sides of R_j has length $\geq b_j - \varepsilon$. In the second case, we can insure these curves all have length $\geq 1/\varepsilon$. In both case this follows because as $n \to \infty$, any curve in $f(R_j)$ joining the opposite "vertical" sides limits on the bottom edge and hence the liminf of the lengths of such curves is at least the length of the bottom edge of R'_j.

By quasiconformality we know

$$M(R'_j) \geq M(R_j)/K = \frac{m}{Kn},$$

and using the metric $\rho = 1$ on R'_j, shows

$$M(R'_j) \leq \frac{\text{area}(R'_j)}{b_j^2}.$$

Thus by Cauchy-Schwarz,

$$\begin{aligned}
(\sum_{j=1}^m b_j)^2 &\leq (\sum_{j=1}^m b_j^2 m)(\sum_{j=1}^m \frac{1}{m}) \\
&\leq m \sum_{j=1}^m \frac{\text{area}(R'_j)}{M(R'_j)} \\
&\leq m \sum_{j=1}^m \frac{\text{area}(R'_j)}{m/Kn} \\
&\leq \sum_{j=1}^m \text{area}(R'_j) Kn \\
&\leq K \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \\
&\to KA'(y).
\end{aligned}$$

If any of the b_j's is infinite, so is $A'(y)$, so $f(L_y)$ has finite length for our choice of y. Given a compact set $E \subset L_y$, suppose E is hit by N of the rectangles R_j and that m has been chosen so large that $N/m \leq 2m_1(E)$. Then repeating the argument above, but only summing over the j's so that the bottom edges of R_j hit E,

$$\begin{aligned}
(\sum_j b_j)^2 &\leq (\sum_j b_j^2 m)(\sum_j \frac{1}{m}) \\
&\leq N\sum_j \frac{\text{area}(R'_j)}{M(R'_j)} \\
&\leq N\sum_j \frac{\text{area}(R'_j)}{m/Kn} \\
&\leq \frac{N}{m}\sum_{j=1}^{m} \text{area}(R'_j)Kn \\
&\leq Km_1(E)\frac{A(y+\frac{1}{n})-A(y)}{1/n} \\
&\to Km_1(E)A'(y).
\end{aligned}$$

Thus $m_1(E)$ small, implies $\sum b_j$ is small, and hence $f(E)$ has small 1-dimensional measure. Hence f is absolutely continuous on L_y, as desired. $\square$

Basic theorems of real analysis say that if f is absolutely continuous on a line L , then its partial derivative along that lines exists almost everywhere and

$$f((b)-f(a)=\int_a^b f_n ds,$$

where $a,b \in L$ and f_n is the partial in the direction from a to b. Since we have shown that quasiconformal maps are absolutely continuous on almost every horizontal and almost every vertical line, we see that the partial f_x, f_y exist almost everywhere and hence $f_z, f_{\bar{z}}, \mu_f = f_{\bar{z}}/f_z$ are all well defined almost everywhere. Next we want to say that at a point w where these all exist, we have

$$f(z)=f(w)+f_z(w)(z-w)+f_{\bar{z}}(w)(\bar{z}-\overline{w})+o(|z-w|),$$

i.e., f is differentiable at w. However, as explained in most calculus texts, the existence of partial derivatives at at a point does not imply a function is differentiable there (consider $f(x,y)=x^2y/(x^2+y^2)$ at the origin).

However, a remarkable theorem of Gehring and Lehto [**?**], says that is implication is true almost everywhere for homeomorphisms. Our proof follows that in [**?**].

THEOREM 4.2. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ and has partials almost everywhere, then it is differentiable almost everywhere.*

PROOF. By Egorov's theorem the limits

$$f_x(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

$$f_y(z) = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{h},$$

are uniform and converge to a continuous functions on a compact set $E \subset \Omega$ so that area$(\Omega \setminus E)$ is as small as we wish.

Almost every point of E is a point of density for the intersection of E with both the vertical and horizontal lines through z_0, so if suffices to proof differentiability at such points. For simplicity we assume 0 is such a point. The proof follows the usual case in calculus where we assume the partials are continuous, except that here we have to replace continuous on a neighborhood of 0 with continuous on a set E that is measure dense around 0.

Because of the continuity and uniform convergence on E, for any $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|f_x(0) - f_x(z)|, |f_y(0) - f_y(z)| < \varepsilon,$$

if $z \in E \cap D(0,\delta)$-neighborhood of 0 and

$$|f_x(z) - \frac{f(z+h) - f(z)}{h}|, |f_y(z) - \frac{f(z+ih) - f(z)}{h}| < \varepsilon,$$

if $z \in E \cap D(0,\delta)$ and $h \in [-\delta, \delta]$.

Note that

$$\begin{aligned} f(z) - f(0) - xf_x(0) - yf_y(0) &= [f(z) - f(x) - yf_y(0)] + [f(x) - f(0) - xf_x(0)] \\ &\quad + [yf_y(x) - f_y(0)] \\ &= I + II + III. \end{aligned}$$

If $|z| < \delta$ and $x \in E$, then by the inequalities above, $I < \varepsilon|y|$, $II < \varepsilon|x|$ and $III < \varepsilon y$, so the term on the far left is bounded by $3\varepsilon|z|$, which proves differentiability if $x \in E$. A similar proof works if $iy \in E$.

Fix $\varepsilon > 0$ and choose δ so small that if $0 < x < \delta$, then $E \cap (\frac{x}{1+\varepsilon}, x) \neq \emptyset$ (this must be possible since $E \cap \mathbb{R}$ has density 1 at 0) and $E \cap (\frac{iy}{1+\varepsilon}, y) \neq \emptyset$. Thus if $0 < |x|, |y| \leq \delta/(1+\varepsilon)$ can find points $x_1, x_2 \in E \cap (\frac{x}{1+\varepsilon}, (1+\varepsilon)x)$ and $iy_1, iy_2 \in E \cap i(\frac{y}{1+\varepsilon}, (1+\varepsilon)y)$ and so that $x + iy$ is inside the rectangle $R = (x_1, x_2) \times (y_1, y_2)$. Since f is a homeomorphism (all we need is that it is continuous and open), $|f|$ takes its maximum on the boundary, so

$$\begin{aligned}\sup_{z=x+iy\in R} &|f(z)-f(0)-xf_x(0)-yf_y(0)| \\ &\leq \sup_{w=u+iv\in\partial R} |f(zw-f(0)-xf_x(0)-yf_y(0)| \\ &\leq 3\varepsilon|w| + \sup_{w=u+iv\in\partial R} |x-u||f_x(0)|+|y-v||f_y(0)| \\ &\leq 3\varepsilon(1+\varepsilon)|z|+\varepsilon|f_x(0)||z|+\varepsilon|f_y(0)||z|.\end{aligned}$$

□

COROLLARY 4.3. *A K-quasiconformal map f defined on a planar domain Ω is differentiable almost everywhere on Ω.*

PROOF. This is immediate from Theorems **??** and **??**. □

LEMMA 4.4. . *If f is K-quasiconformal then*

$$\int_Q J_f dxdy \leq \int \leq \text{area}(f(Q)) \leq \pi \,\text{diam}(f(Q))^2,$$

for every square Q.

PROOF. We only use the quasiconformal hypothesis to deduce f is differentiable almost everywhere; the result holds for all such maps. At any point x where f is differentiable we can choose a small square Q_x containing x such that

$$\text{area}(f(Q')) \geq (1-\varepsilon)J_f(x)\,\text{area}(Q'),$$

and by the Lebesgue differentiation theorem, for almost every x we have

$$\int_{Q'} J_f dxdy \leq (1+\varepsilon)J_f(x)\,\text{area}(Q'),$$

for all small enough squares centered at x. Combining these two estimates and using the Vitali covering theorem to extract a collection of disjoint squares $\{Q_j\}$ with centers x_j and with these properties that cover almost every point of Q, we get

$$\begin{aligned}\int_Q J_f dxdy &\leq \sum_j \int_{Q_j} J_f dxdy \\ &\leq (1+\varepsilon)J_f(x_j)\,\text{area}(Q_j) \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\text{area}(f(Q_j)) \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\text{area}(f(Q)).\end{aligned}$$

Taking $\varepsilon \searrow 0$, gives $\text{area}(f(E)) \geq \int_E J_f dxdy$. The inequality area $\leq \pi\,\text{diam}^2$ is obvious. □

Since $|f_z|^2 \le J_f/(1-k^2)$, we also get

COROLLARY 4.5. *If f is K-quasiconformal then*

$$\int_Q |f_z|^2 dxdy \le \frac{\pi}{1-k^2} \operatorname{diam}(f(Q))^2,$$

for every square Q.

Next we turn to

LEMMA 4.6. *If f is K-quasiconformal, then*

$$\frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)} \gtrsim \operatorname{diam}(f(Q))^2.$$

with a uniform constant for every square Q.

PROOF. The path family connecting opposite sides of a square Q has modulus 1, so the image of this family in $f(Q)$ has modulus between K and $1/K$. This implies the shortest path in $f(Q)$ connecting the same sides has length $\simeq \operatorname{diam}(f(Q))$, so the integral of $|f_z| + |f_{\bar z}|$ along any horizontal segment crossing Q is at least $C \operatorname{diam}(f(Q))$ for some fixed $C > 0$ (depending only on K). Since $|f_z| \le |f_z| + |f_{\bar z}| \le 1(1+k)|f_z|$, the same is true for the integral of $|f_z|$. Integrating over all horizontal segments crossing Q gives

$$\int_Q |f_z| dxdy \gtrsim \operatorname{diam}(Q) \operatorname{diam}(f(Q)).$$

Hence

$$\frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)} \gtrsim \frac{[\operatorname{diam}(Q) \operatorname{diam}(f(Q))]^2}{\operatorname{area}(Q)} \gtrsim \operatorname{diam}(f(Q))^2.$$

$\square$

Note that for K-quasiconformal maps, $|\mu_f| \le k = (K-1)/(K+1)$ and

$$|f_z|(1-k^2) \le |f_z|^2(1-|\mu|^2) \le |f_z|^2 - |f_{\bar z}|^2 = J_f \le |f_z|^2,$$

so that J_f and $|f_z|^2$ are the same up to a bounded factor. Thus

$$\int_Q |f_z|^2 dxdy \le \int \lesssim \operatorname{diam}(f(Q))^2 \lesssim \frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)}$$

or

$$\int_Q |f_z|^2 dxdy \le C \frac{(\int_Q |f_z| dxdy)^2}{\operatorname{area}(Q)}$$

for some constant C that depends only on the quasiconformal constant of f (and not on the choice of the square Q). This is called a reverse Hölder inequality and we shall see in the next section that it has profound implications for the behavior of f_z.

5. Gehring's inequality and Bojarski's theorem

Hölder's inequality says that

$$\int fgd\mu \leq (\int f^p d\mu)^{1/p} (\int g^q d\mu)^{1/q},$$

where $1 \leq p,q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Applying this to a non-negative function on a square Q we get

$$(\frac{1}{\text{area}(Q)} \int_Q v^p dxdy) \geq (\frac{1}{\text{area}(Q)} \int_Q v dxdy)^p,$$

with equality if and only if v is a.e. constant. Thus the "reverse Hölder inequality "

$$(\frac{1}{\text{area}(Q)} \int_Q v^p dxdy) \leq (K \frac{1}{\text{area}(Q)} \int_Q v dxdy)^p,$$

can only hold if $K \geq 1$. If it holds for single Q, this does not say much, except that $v \in L^p \cap L^1$. However, if it holds (with the same K) for all Q's we can deduce that $v \in L^{p+\varepsilon}$ for some $\varepsilon > 0$. This remarkable "self-improvement" estimate is due to Gehring [], although the proof we give follows the presentation in Garnett's book [**?**].

We start with a technical lemma.

LEMMA 5.1. *Suppose that $p > 1$, $v \geq 0$, $E_\lambda = \{z : v(z) > \lambda\}$, and*

$$\int_{E_\lambda} v^p dxdy \leq A\lambda^{p-1} \int_{E_\lambda} v dxdy,$$

for all $\lambda \geq 1$. Then there is $r > p$ and $C < \infty$ so that

$$(\int_Q v^r dxdy)^{1/r} \leq C(\int_Q v^p dxdy)^{1/p}.$$

PROOF. This is basically just arithmetic with distribution functions. Note that it suffices to assume $\text{area}(Q) = 1$ and $\int_Q v^p dxdy = 1$. Then

$$\begin{aligned}
\int_{E_1} v^r dxdy &= \int_{E_1} v^p v^{r-p} dxdy \\
&= (r-p)\int_{E_1} v^p (1 + \int_1^v \lambda^{r-p-1} d\lambda) dxdy \\
&= (r-p)\int_{E_1} v^p + (r-p)\int_1^\infty \lambda^{r-p-1} \int_{E_\lambda} v^p dxdyd\lambda \\
&\leq (r-p)\int_{E_1} v^p + A(r-p)\int_1^\infty \lambda^{r-2} \int_{E_\lambda} vdxdyd\lambda \\
&\leq (r-p)\int_{E_1} v^p + A(r-p)\int_{E_1} v(\int_0^v \lambda^{r-2} d\lambda) dxdy \\
&\leq (r-p)\int_{E_1} v^p + A\frac{r-p}{r-1}\int_{E_1} v^r dxdy \\
&\leq (r-p)\int_{E_1} v^p + \frac{1}{2}\int_{E_1} v^r dxdy
\end{aligned}$$

where the last inequality holds if r is close enough to p (depending on A and p). Subtracting the last term of the last step from the first step gives

$$\int_{E_1} v^r dxdy \leq 2(r-p)\int_{E_1} v^p dxdy.$$

Off E_1 we have $v \leq 1$ so $v^r \leq v^p$ and hence

$$\int_Q v^r dxdy \leq (1 + 2(r-p))\int_Q v^p dxdy.$$

Because of our normalizations, this proves the lemma. $\square$

Next we show the reverse Hölder inequality implies the distribution function hypothesis of the previous lemma, and hence Gehring's inequality.

THEOREM 5.2. *Let $p > 1$. If $v(x) \geq 0$ and $v \in L^p(Q, dxdy)$, and if the "reverse Hölder inequality"*

$$(\frac{1}{\text{area}(Q)}\int_Q v^p dxdy) \leq (K\frac{1}{\text{area}(Q)}\int_Q vdxdy)^p,$$

holds for all subsquares of a square Q_0, then there is an $r > p$ so that

$$(\frac{1}{\text{area}(Q_0)}\int_{Q_0} v^r dxdy)^{1/r} \leq (C(K,p,r)\frac{1}{\text{area}(Q_0)}\int_{Q_0} vdxdy),$$

PROOF. We need only verify the hypothesis of Lemma **??**. Fix λ and set $\beta = 2K\lambda$. We will split the integral

$$\int_{E_\lambda} v^p dxdy = \int_{E_\lambda \setminus E_\beta} v^p dxdy + \int_{E_\beta} v^p dxdy$$

into two pieces. The second piece is trivial to bound by the correct estimate because

$$\int_{E_\lambda \setminus E_\beta} v^p dxdy \leq \beta^{p-1} \int_{E_\lambda \setminus E_\beta} vdxdy \leq (2K\lambda)^{p-1} \int_{E_\lambda} vdxdy.$$

To bound the other piece of the integral, we use the Calderon-Zygmund lemma (Lemma **??**) to find a sequence of disjoint squares $\{Q_j\}$ so that

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy < 2\beta^p,$$

and $v \leq \beta$ almost everywhere off $\cup Q_j$. Thus $E_\beta \setminus \cup Q_j$ has measure zero and

$$\int_{E_\beta} v^p dxdy \leq \sum_j \int Q_j v^p dxdy \leq 2\beta^p \sum \text{area}(Q_j).$$

We now make use of the reverse Hölder hypothesis to write

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy \leq (\frac{K}{\text{area}(Q_j)} \int_{Q_j} vdx)^p,$$

hence

$$\begin{aligned} \text{area}(Q_j) &\leq \frac{K}{\beta} \int_{Q_j} vdxdy \\ &\leq \frac{K}{\beta} (\int_{Q_j \cap E_\lambda} vdxdy + \lambda \,\text{area}(Q_j) \\ &\leq \frac{K}{\beta} \int_{Q_j \cap E_\lambda} vdxdy + \frac{1}{2} \text{area}(Q_j). \end{aligned}$$

Solving for area(Q_j) gives

$$\begin{aligned} \text{area}(Q_j) &\leq \frac{2K}{\beta} \int_{Q_j} vdxdy \\ &\leq \frac{1}{\lambda} \int_{Q_j} vdxdy. \end{aligned}$$

Thus by the defining property of the Q_j's,

$$\begin{aligned}
\int_{E_\beta} v^p dxdy &\leq \sum_j \int_{Q_j} v^p dxdy \\
&\leq 2\beta^p \sum_j \text{area}(Q_j) \\
&\leq 2\beta^p \lambda^{-1} \sum_j \int_{Q_j \cap E_\lambda} v dx \\
&\leq 2^{p+1} K^p \lambda^{p-1} \int_{E_\lambda} v dx.
\end{aligned}$$

Thus the hypothesis of Lemma **??** holds with $A = (2K)^{p-1} + 2^{p+1}K^p$, and we deduce that $v \in L^r(Q, dxdy)$ for some $r > p$. $\square$

To apply Gehring's inequality to the partial derivatives of quasiconformal maps, we have to show that these partial satisfy a reverse Hölder inequality. What we want is

$$\int_Q |f_z|^2 dxdy \leq \frac{C}{\text{area}(Q)} (\int_Q |f_z| dxdy)^2,$$

with a uniform C for all squares in the plane. This was proven in the previous section.

Thus we have proven the theorem of Bojarski and Gehring mentioned earlier:

THEOREM 5.3. *If $1 \leq K < \infty$, there is a $p > 2$ and $A, B < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $Q \subset \mathbb{C}$ is a square, then*

$$(\frac{1}{\text{area}(Q)} \iint_Q |f_z|^p dxdy)^{1/p} \leq A(\frac{1}{\text{area}(Q)} \int_Q |f_z|^2 dxdy)^{1/2} \leq B \frac{\text{diam}(f(Q))}{\text{diam}(Q)}$$

LEMMA 5.4. *If f fixes $0, 1, \infty$, then*

$$\int_Q |L_f(x) - 1| dxdy \leq \varepsilon \, \text{diam}(Q),$$

where $L_f = |f_z| + |f_{\bar{z}}|$ and $\varepsilon \to 0$ as $\|\mu_f\|_\infty \to 0$.

PROOF. Fix a square Q with sides parallel to the axes, let $\ell(Q)$ denote its side length and let S_1, S_2 denote the two vertical sides of S Use fact that as $\|\mu\|_\infty \to 0$, f tends to the identity and

$$\begin{aligned} 0 &\le \left(\frac{1}{\mathrm{area}(Q)}\int_Q |v-1|dxdy\right)^2 \le \frac{1}{\mathrm{area}(Q)}\int_Q |v-1|^2 dxdy \\ &\le \frac{1}{\mathrm{area}(Q)}\int_Q (v^2-1) - \frac{2}{\mathrm{area}(Q)}\int_Q (v-1)dxdy \\ &\le \frac{1}{\mathrm{area}(Q)}\int_Q (KJ_f-1) - \frac{2}{\mathrm{area}(Q)}\int_Q (v-1)dxdy \\ &= \frac{1}{\mathrm{area}(Q)}\int_Q (K-1)J_f dxdy + \frac{1}{\mathrm{area}(Q)}\int_Q (J_f-1)dxdy \\ &\quad - \frac{2}{\mathrm{area}(Q)}\int_Q (v-1)dxdy \\ &\le O(\|\mu\|_\infty)\frac{\mathrm{area}(f(Q))}{\mathrm{area}(Q)} + \frac{\mathrm{area}(f(Q))-\mathrm{area}(Q)}{\mathrm{area}(Q)} - 2\left(\frac{\mathrm{dist}(S_1,S_2)}{\ell(Q)}-1\right). \end{aligned}$$

Since f converges uniformly to the identity on Q as $\|\mu\|_\infty \to 0$, each term in the last line tends to zero. $\square$

COROLLARY 5.5. *If Ω has a piecewise C^1 boundary and f is quasiconformal on Ω, then*

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_\Omega \frac{f_{\bar z}}{z-w}dxdy. \tag{5.1}$$

PROOF. Smooth and take a limit using the L^p boundedness of the the Hardy-Littlewood maximal theorem and the Lebesgue dominated convergence theorem. $\square$

COROLLARY 5.6. *If f is quasiconformal, then f maps sets of zero area to zero area and*

$$\mathrm{area}(f(E)) = \int_E J_f dxdy.$$

PROOF. Since $\nu(E) = \mathrm{area}(f(E))$ and $\nu(E) = \int_E J_f dxdy$ are both non-negative Borel measures, it suffices to show that they are equal for some convenient basis of sets, say squares with sides parallel to the coordinate axes. Let Q be such a square.

We have already proved the "$\ge$" direction in Lemma **??**. To prove the other direction, we use the fact that $J_f \in L^p(Q, dxdy)$ for some $p > 1$. Define a smoothed version f_n of f by convolving f with a smooth, non-negative bump function φ_n of total mass 1 and support in $D(0, \frac{1}{n})$. Since f is continuous on $\mathbb{C}$, $f_n \to f$ uniformly on Q. Since convolution is linear, the partials of f_n are the partials of f convolved with φ_n and therefore the supremum over n of these partials is bounded by the Hardy-Littlewood maximal function of

f_z, i.e.,

$$\sup_n |(f_n)_z(x)| \leq \mathscr{H}L(f_z)(x),$$

and similarly for $f_{\bar{z}}$. Since the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p < \infty$, and $f_z, f_{\bar{z}} \in L^p$ for some $p > 1$, we see that $\{((f_n)_z)\}$, $\{((f_n)_{\bar{z}})\}$ are dominated by an L^p function and hence by an L^2 function on Q (since $L^p \subset L^2$ on bounded sets). Thus the sequence of Jacobians $\{J_{f_n}\}$ is dominated by an L^1 function on Q, so by the Lebesgue dominated convergence theorem,

$$\int_Q J_{f_n} dxdy \to \int_Q J_f dxdy.$$

Moreover, since f_n is smooth

$$\int_Q J_{f_n} dxdy \geq \text{area}(f_n(Q)),$$

(equality may not hold since we don't known f_n is 1-to-1, and the integral computes area with multiplicity) and since $f_n \to f$ uniformly, $f_n(Q)$ eventually contains any compact subset of $f(Q)$ and hence

$$\limsup_n \text{area}(f_n(Q)) \geq \text{area}(f(Q)).$$

Thus $\text{area}(f(Q)) \leq \int_Q J_f dxdy$, as desired. □

LEMMA 5.7. *Suppose* $\{g_n\} \in L^p(R, dxdy)$ *for some* $p > 2$ *and*

$$\lim_n \iint_R \frac{g_n(z)}{z - w} dxdy = 0$$

for all $w \in R$. *Then* $\lim_n \iint_R g_n dxdy = 0$.

PROOF. Fix rectangles $R'' \subset R' \subset R$, each compactly contained in the interior of the next. Using the Cauchy integral formula for the constant function 1 on the curve $\partial R'$ we see that we can uniformly approximate the constant function 1 on R'' by a finite sum $s(z) = \sum \frac{a_k}{z - w_k}$ with $w_k \in \partial R'$ and $\sum |a_k|$ is uniformly bounded. Then

$$\begin{aligned}\iint_R g_n(z) dxdy &= \iint_R g_n(z)s(z)dxdy + \iint_R g_n(z)(1 - s(z))dxdy \\ &= o(1) + \iint_{R''} g_n(z)(1 - s(z))dxdy + \iint_{R \setminus R''} g_n(z)(1 - s(z))dxdy.\end{aligned}$$

For a fixed n, the first integral can be made as close to zero as we wish by taking s close to 1 on R''. The second integral can be made small by taking $\text{area}(R \setminus R'') \to 0$; this implies the L^p norm of g_n on $R \setminus R''$ tends to zero (hence so does its L^1 norm) whereas the L^q norm of s remains uniformly bounded (it is a convex combination of L^q functions with bounded norm).

Thus we can make $\iint_R g_n dxdy$ as small a we wish if n is large, proving the lemma. □

LEMMA 5.8. *If $\{g_n\}$ are K-quasiconformal maps that converge uniformly on compact sets to a quasiconformal map g, then for any rectangle R.*

$$\iint_R [(g_n)_z - g_z]dxdy \to 0,$$

$$\iint_R [(g_n)_{\bar{z}} - g_{\bar{z}}]dxdy \to 0.$$

and $(g_n)_z \to g_z$ and $(g_n)_{\bar{z}} \to g_{\bar{z}}$ weakly.

PROOF. First consider the $\bar{z}$-derivative. Let $h_n = (g_n)_{\bar{z}} - g_{\bar{z}}$. By the Pompeiu formula and the fact that $g_n \to g$ uniformly on R, we deduce that

$$\lim_{n\to\infty} \iint_R \frac{h_n(z)}{z-w} dxdy = 0$$

for any $w \in R$. That

$$\iint_R h_n dxdy \to 0,$$

follows from Lemma **??**. To prove weak conference, take any continuous f of compact support and uniformly approximate it to within ε by a function $\tilde{f}$ that is constant on finite union of rectangles. Then

$$\iint f h_n dxdy = \iint (f - \tilde{f}) h_n dxdy + \iint \tilde{f} h_n dxdy.$$

The first integral is bounded by $\varepsilon \iint |h_n| dxdy$, which is small since $\|h_n\|_1 \leq C\|h_n\|_p$ is uniformly bounded on a large ball containing the support of both f and $\tilde{f}$. The second integral tends to zero since is a finite linear combination of integrals of h_n over rectangles.

The result for z-derivatives follows from the same proof applied to the complex conjugates of g and $\{g_n\}$, using the fact that $(\bar{f})_{\bar{z}} = \overline{f_z}$. □

6. Convergence of maps implies convergence of dilatations

We are now ready to use the second analytic fact that we assumed earlier in the proof of the measureable Riemann mapping theorem (Theorem **??**).

THEOREM 6.1. *Suppose $\{f_n\}$, f are all K-quasiconformal maps on the plane with dilatations $\{\mu_n\}$, μ_f respectively, that $f_n \to f$ uniformly on compact sets and that $\mu_n \to \mu$ pointwise almost everywhere. Then $\mu_f = \mu$ almost everywhere.*

PROOF. We restrict attention to some domain Ω with compact closure. We know that $f_{\bar z} = \mu_f f_z$ almost everywhere and we know that f_z is non-zero almost everywhere, so it suffices to show that

$$f_{\bar z}(w) - \mu(w) f_z(w) = 0,$$

almost everywhere. To prove this it suffices to show that the integral of $f_{\bar z}(w) - \mu(w) f_z(w)$ over any rectangle R is zero (this is an application of the Lebesgue differentiation theorem: at almost every point an integrable function is the limit of its averages over rectangles shrinking down to that point). We re-write this function as

$$\begin{aligned} f_{\bar z}(w) - \mu(w) f_z(w) &= [f_{\bar z}(w) - (f_n)_{\bar z}(w)] \\ +[(f_n)_{\bar z}(w) - \mu_n (f_n)_z(w)] \\ &\qquad +[\mu_n(w)(f_n)_z(w) - \mu(w)(f_n)_z(w)] \\ &\qquad +[\mu(w)(f_n)_z(w) - \mu(w) f_z] \\ &= I + II + III + IV. \end{aligned}$$

Term II equals zero almost everywhere, so we need only show that the other three terms tend to zero as n tends to ∞.

Case I: This is Lemma **??**.

Case III: We use Cauchy-Schwarz to show the integral of the third term is bounded by

$$\left(\iint_R (\mu - \mu_n)^2 dxdy\right)^{1/2} \left(\iint_R |(f_n)_x|^2 dxdy\right)^{1/2},$$

The first integrand tends to zero pointwise and is bounded above by 2 almost everywhere, so the integrals tend to zero by the Lebesgue dominated convergence theorem. On the other hand

$$\left(\iint_R |(f_n)_x|^2 dxdy\right)^{1/2} \simeq \operatorname{diam}(f_n(R)),$$

by Lemma **??**, and since $\{f_n\}$ converges uniformly on compact sets, this remains bounded. Thus the integral of III is bounded above by a term tending to zero times a term that is uniformly bounded, and hence it tends to zero.

Case IV: The same lemma as in case I, but applied to $f_z = (\bar f)_{\bar z}$, and using the fact that $(\bar f)_{\overline{w}} = \overline{(f_z)}$, show that

$$\iint_R (f_z - (f_n)_z) dxdy \to 0$$

for every rectangle R. Now approximate μ in the $L^q(R, dxdy)$ norm by a function ν that is constant on a finite collection of disjoint squares (such functions are dense in L^q) and we deduce

$$\int_n \iint_R \mu((f_z - (f_n)_z) dxdy = \lim_n \iint_R (\mu - \nu)((f_z - (f_n)_z) dxdy \le \lim_n \|\mu - \nu\|_q \|(f_z - (f_n)_z)\|_p.$$

The first term is as small as we wish and the second is uniformly bounded, so the product is as small as we wish. Thus the limit must be zero, as desired. □

This completes the proof of the measurable Riemann mapping theorem in the general case.

CHAPTER 4

Estimates for quasiconformal mappings

1. The Ahlfors formula

The dependence of f on its dilatation μ is non-linear (there is an explicit power series relationship between the two in terms of certain singular integral operators, see, e.g., [**?**]), but it is possible to give a linear approximation that is valid when $\|\mu\|_\infty$ is small, namely

$$f(w) = w - \frac{1}{\pi}\int_{\mathbb{R}^2} \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty^2),$$

for all $|w| \leq 1$, where

$$R(z,w) = \frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z} = \frac{w(w-1)}{z(z-1)(z-w)}.$$

The goal of this section is to prove this formula. The proof is basically a manipulation of the Pompeiu formula

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_\Omega \frac{f_{\bar z}}{z-w}dxdy$$

where we use our L^p estimates on $f_z, f_{\bar z}$ to put certain terms into the error term. We start by showing f is close to the identity in a precise L^p sense when $\|\mu\|$ is small.

LEMMA 1.1. *If $k<1$ is small enough then there is a constant $C_3 = C_3(k)$ so that the following holds. Suppose $\|\mu\|_\infty \leq k < 1$. Then*

$$\|f_z^\mu - 1\|_{p,1} \equiv (\int_{B_1} |f_z^\mu - 1|^p dxdy)^{1/p} \leq C_3\|\mu\|_\infty.$$

for all $2 \leq p \leq p(k)$.

PROOF. First assume μ is supported in $D(0,R)$ and let $\varepsilon = \|\mu\|_\infty$. It is proven on page 100 of [**?**] that $\|F_z^\mu - 1\|_p \leq C\|\mu\|_p \leq C\varepsilon R^{2/p}$ if $p < p(k)$.

Since $f^\mu = F^\mu/F^\mu(1)$, Lemma **??** implies

$$\begin{aligned} \|f_z^\mu - 1\|_{p,1} &= \|\frac{F_z^\mu}{F^\mu(1)} - 1\|_{p,1} \\ &= |1-\frac{1}{F^\mu(1)}| + \frac{1}{F^\mu(1)}\|F_z^\mu - 1\|_{p,1} \\ &\leq 2C_2\varepsilon + \frac{C(R)}{1-C\varepsilon}\varepsilon \\ &\leq C\varepsilon. \end{aligned}$$

Now write $\check{f}(z) = 1/f(1/z)$. We want to show

$$\|\check{f}_z^\mu - 1\|_{p,R} \leq C(R)\varepsilon, \tag{1.1}$$

when μ has support in B_R. Just as above, it suffices to show $\|\check{F}_z^\mu - 1\|_{p,R} \leq C\varepsilon$. Note that $\check{F}^\mu$ is analytic on $\{z : |z| < 3r\}$ where $r = 1/(3R)$. For an analytic function f on a ball $B(x,r)$ it is easy to see by the mean value property and Hölder's inequality that

$$|f(x)| \leq \frac{1}{\pi r^2}\int_{B(x,r)} |f| \leq \frac{1}{(\pi r^2)^{1/p}}\|f\|_{L^p(B(x,r))}.$$

Thus by the maximum principle,

$$\begin{aligned} \int_{|z|<r} |\check{F}_z^\mu(z) - 1|^p dxdy &\leq C(r)\sup_{|z|=2r} |\check{F}_z^\mu(z) - 1|^p \\ &\leq C(r)\int_{r<|z|<3r} |\check{F}_z^\mu(z) - 1|^p dxdy. \end{aligned}$$

On the other hand, changing variables from z to $1/z$ gives

$$\begin{aligned} \int_{r<|z|<R} |\check{F}_z^\mu(z) - 1|^p dxdy &= \int_{1/R<|z|<1/r} |\frac{z^2 F_z^\mu(z)}{F^\mu(z)^2} - 1|^p \frac{dxdy}{|z|^4} \\ &= \int_{1/R<|z|<1/r} |\frac{z^2(F_z^\mu(z)-1)}{F^\mu(z)^2} + \frac{z^2 - F^\mu(z)^2}{F^\mu(z)^2}|^p \frac{dxdy}{|z|^4} \\ &\leq C\int_{1/R<|z|<1/r} |\frac{z^2(F_z^\mu(z)-1)}{F^\mu(z)^2}|^p + |\frac{z^2 - F^\mu(z)^2}{F^\mu(z)^2}|^p \frac{dxdy}{|z|^4} \\ &\leq C(R)\int_{1/R<|z|<1/r} |F_z^\mu(z) - 1|^p dxdy \\ &\quad + C(R)\int_{1/R<|z|<1/r} |z - F^\mu(z)|^p dxdy \\ &\leq C(R)\varepsilon^p. \end{aligned}$$

Since the integral over $\{|z| < 3r\}$ was dominated by a constant (depending only on R) times this estimate, we have proven (**??**).

The general case now follows just as in [**?**]. Write $f = \check{g} \circ h$ where $\mu_h = \mu_f$ inside the unit disk and $\mu_h = 0$ outside the unit disk. Then

$$\|f_z - 1\|_{p,1} \le \|[(\check{g}_z - 1) \circ h]h_z\|_{p,1} + \|h_z - 1\|_{p,1}.$$

The second term is bounded by $C\varepsilon$ by the first paragraph and the first term is bounded using

$$\begin{aligned}
\|[(\check{g}_z - 1) \circ h]h_z\|_{p,1}^p &= \int_{B_1} |(\check{g}_z - 1) \circ h|^p |h_z|^p dxdy \\
&\le \frac{1}{1-k^2} \int_{h(B_1)} |\check{g}_z - 1|^p |h_z \circ h^{-1}|^{p-2} dxdy \\
&\le \frac{1}{1-k^2} (\int_{h(B_1)} |\check{g}_z - 1|^{2p} dxdy \int_{B_1} |h_z|^{2p-4} dxdy)^{1/2}
\end{aligned}$$

Clearly $h(B_1) \subset \{|z| < R\}$ for some R depending only on k. Thus using (**??**), the first integral is bounded by

$$\int_{B_R} |\check{g}_z - 1|^{2p} dxdy \le C\varepsilon^{2p},$$

(assuming $2p < p(k)$; but since $p(k) \to \infty$ as $k \to 0$ this holds for some $p > 2$ if k is small enough). On the other hand

$$\int_{B_1} |h_z|^{2p-4} dxdy \le C(\int_{B_1} |h_z|^{2p} dxdy)^{1-2/p} \le \|\mu_h\|_{p,1} + \|1\|_{p,1} \le C,$$

since $\|h_z - 1\|_p \le C\|\mu_h\|_p$. $\square$

LEMMA 1.2. *With notation as above,*

$$f(w) = w - \frac{1}{\pi}\int_{B_1} f_{\bar{z}}(z) R(z,w) dxdy - \frac{1}{\pi}\int_{B_1} \frac{\check{f}_{\bar{z}}(z)}{\check{f}(z)^2} z S(z,w) dxdy,$$

where $R(z,w) = (\frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z})$ *and* $S(z,w) = \frac{w^2}{1-wz} - \frac{w}{1-z}$.

PROOF. This is where we use the Pompeiu formula. Assume $|w| < 1$ and apply the formula to the unit disk to get

$$f(w) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z-w} dz - \frac{1}{\pi}\iint_{\mathbb{D}} \frac{f_{\bar{z}}}{z-w} dxdy. \tag{1.2}$$

Replace z by $1/z$ in the boundary integral. We claim that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z}) \frac{dz}{z(1-zw)} &= A + Bw + \frac{w^2}{2\pi i}\int_{\mathbb{T}} \frac{dz}{\check{f}(z)(1-zw)} \\
&= A + Bw - \frac{w^2}{2\pi}\int_{\mathbb{D}} \frac{\check{f}_{\bar{z}}(z) dxdy}{\check{f}(z)^2(1-zw)}.
\end{aligned}$$

To see this, first suppose that on $\{|z|=1\}$ $f(z)=z^n$. Then $f(\frac{1}{z})=z^{-n}$, and

$$\frac{1}{z(1-zw)}=\frac{1}{z}(1+zw+(zw)^2+\dots),$$

so

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z})\frac{dz}{z(1-zw)}\\ &\quad=\frac{1}{2\pi i}\int_{\mathbb{T}} z^{-n-1})(1+zw+z^2w^2+\dots)dz\\ &\quad=\frac{1}{2\pi i}\int_{\mathbb{T}} z^{-n-1})(z^{-n-1}+z^{-n}w+z^{n+1}w^2++z^{-n-1+j}w^j+\dots)dz.\end{aligned}$$

This integral is only non-zero for the term containing z^{-1}; this corresponds to $j=n$, so for $n\geq 0$,

$$\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z})\frac{dz}{z(1-zw)}=w^n$$

and the integral is zero for $n<0$. By a similar argument, if $n\geq 2$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\check{f}(\frac{1}{z})^{-1}zdz}{(1-zw)} &= \frac{1}{2\pi i}\int_{\mathbb{T}} z^{-n+1})(1+zw+z^2w^2+\dots)dz\\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} z^{-n-1})(z^{-n+1}+z^{-n}w+z^{n+2}w^2++z^{-n+1+j}w^j+\dots)dz.\\ &= w^{n-2}\end{aligned}$$

and for $n<2$ the integral is zero. So the two integrals differ by a factor of w^2 in general and for $n=0,1$ the integral on the right can gives $1,w$ but the integral on the left gives 0. Thus

$$\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z})\frac{dz}{z(1-zw)}=A+Bw+\frac{w^2}{2\pi i}\int_{\mathbb{T}}\frac{dz}{\check{f}(z)(1-zw)}. \tag{1.3}$$

We now rewrite the line integral as an area integral using Pompeiu's formula again with the change of variable $\alpha = wz$

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=1}\frac{\check{f}(z)^{-1}zdz}{1-zw} &= -\frac{1}{2\pi i w^2}\int_{|z|=1}\frac{\check{f}(z)^{-1}wzwdz}{zw-1}\\
&= -\frac{1}{2\pi i w^2}\int_{\alpha=|w|}\frac{\check{f}(\frac{\alpha}{w})^{-1}\alpha d\alpha}{\alpha-1}\\
&= -\frac{1}{w^2}[\check{f}(1)+\frac{1}{\pi}\int_{|\alpha|<|w|}\frac{[\check{f}(\frac{\alpha}{w})^{-1}\alpha]_{\bar{z}}dadb}{\alpha-1}\\
&= -\frac{1}{w^2}[\check{f}(1)+\frac{1}{\pi}\int_{|z|<1}\frac{[\check{f}(z)^{-1}wz]_{\bar{z}}w^2dxdy}{zw-1}\\
&= -\frac{1}{w}+\frac{1}{\pi}\int_{|z|<1}\frac{\check{f}_{\bar{z}}(z)zdxdy}{\check{f}^2(z)(zw-1)}
\end{aligned}$$

Therefore,

$$\frac{1}{2\pi i}\int_{\mathbb{T}}f(\frac{1}{z})\frac{dz}{z(1-zw)} = A+Bw+w-\frac{w^2}{2\pi}\int_{\mathbb{D}}\frac{\check{f}_{\bar{z}}(z)dxdy}{\check{f}(z)^2(1-zw)} \tag{1.4}$$

as claimed.

Since we know $f(0)=0$ and $f(1)=1$ we can solve for the values of A and B. When we do this we get

$$\begin{aligned}
f(w)=w \quad &- \quad \frac{1}{\pi}\int_{|z|<1}f_{\bar{z}}(z)(\frac{1}{z-w}-\frac{w}{z-1}+\frac{w-1}{z})dxdy\\
&- \quad \frac{1}{\pi}\int_{|z|<1}\frac{\check{f}_{\bar{z}}(z)}{\check{f}(z)^2}(\frac{w^2z}{1-wz}-\frac{wz}{1-z})dxdy.
\end{aligned}$$

As a check, the reader can set $w=0,1$ and verify that both integrands vanish in these cases. $\square$

LEMMA 1.3. *There is a $0<k<1$ and a $C_4<\infty$ so that the following holds. Suppose that f is a quasiconformal mapping of the plane to itself which preserves $\mathbb{H}_u$, fixing $0,1$ and ∞ and the Beltrami coefficient of f is μ with $\|\mu\|_\infty \le k$. Then*

$$|f(w)-[w-\frac{1}{\pi}\int_{\mathbb{R}^2}\mu(z)R(z,w)dxdy]| \le C_4\|\mu\|_\infty^2,$$

for all $|w|\le 1$, where

$$R(z,w)=\frac{1}{z-w}-\frac{w}{z-1}+\frac{w-1}{z}=\frac{w(w-1)}{z(z-1)(z-w)}.$$

PROOF. Consider (**??**). If the first integral, we use

$$f_{\bar z} = \mu f_z = \mu(f_z - 1) + \mu.$$

The first term has L^p norm $O(\|\mu\|_\infty)$ by Lemma 1.23, so using Hölder's inequality shows that the first integral equals

$$\begin{aligned}
&\frac{1}{\pi}\int_{|z|<1} \mu(z)\Big(\frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z}\Big)dxdy + \|\mu\|_\infty^2 \\
&\qquad = \frac{1}{\pi}\int_{|z|<1} \mu(z)\frac{w(w-1)}{z(z-1)(z-w)}dxdy + \|\mu\|_\infty^2 \\
&\qquad = \frac{1}{\pi}\int_{|z|<1} \mu(z) R(z,w)dxdy + \|\mu\|_\infty^2
\end{aligned}$$

where

$$R(z,w) = \frac{w(w-1)}{z(z-1)(z-w)}.$$

Using the same estimates, second integral is equal to

$$\frac{1}{\pi}\int_{|z|<1} \mu\Big(\frac{1}{z}\Big)\bar z^{-2}\Big(\frac{w^2 z}{1-wz} - \frac{wz}{1-z}\Big)dxdy + \|\mu\|_\infty^2$$

where $\check\mu(z) = \mu(\frac{1}{z})(z/\bar z)^2$. If we replace z by $1/z$ in the second integral, the integral over the disk transforms into the integral over its complement

$$\begin{aligned}
&\frac{1}{\pi}\int_{|1/z|<1} \check\mu\Big(\frac{1}{z}\Big)\Big(\frac{w^2/z}{1-w/z} - \frac{w/z}{1-1/z}\frac{-dxdy}{|z|^4} \\
&\qquad = \frac{1}{\pi}\int_{|z|>1} \mu(z)(\bar z^2/z^2)\Big(\frac{w^2}{z-w} - \frac{w}{z-1}\frac{1}{z^2\bar z^2}dxdy \\
&\qquad = \frac{1}{\pi}\int_{|z|>1} \mu(z)\Big(\frac{w^2(z-1)-(z-w)w}{(z-w)(z-1)z^2}dxdy \\
&\qquad = \frac{1}{\pi}\int_{|z|>1} \mu(z)\Big(\frac{w(w-1)}{(z-w)(z-1)z}dxdy.
\end{aligned}$$

This integrand has the same form as before which proves the lemma.

NEEDS CHECKING AND FIXING

By Lemma **??** we can write

$$f(w) \quad = \quad w - \frac{1}{\pi}\int_{B_1} f_{\bar z}(z)R(z,w)dxdy - \frac{1}{\pi}\int_{B_1}\frac{\check f_{\bar z}(z)}{\check f(z)^2} zS(z,w)dxdy,$$

where $S(z,w) = \frac{w^2}{1-wz} - \frac{w}{1-z}$ and as before $\check{f}(z) = 1/f(1/z)$. Using $f_{\bar{z}} = \mu f_z = \mu + \mu(f_z - 1)$, the first integral equals

$$\begin{aligned}\int_{B_1} \mu(z)R(z,w)dxdy \quad &+ \quad \int_{B_1} \mu(z)(f_z(z)-1)R(z,w)dxdy \\ &= \quad \int \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty \|f_z - 1\|_{p,1}\|R\|_{q,1}) \\ &= \quad \int \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty^2),\end{aligned}$$

by Lemma **??** and the fact that $R \in L^q$, for every $q < 2$ (with a bound depending on q, but not on w for $|w| \leq 1$).

The second integral is estimated by writing $\check{f}_{\bar{z}} = \check{\mu} + \check{\mu}(\check{f}_z - 1)$ where $\check{\mu}(z) = (z/\bar{z})^2\mu(1/z)$. Repeating the argument above shows the second integral is equal to

$$\begin{aligned}\int_{B_1} \frac{\check{\mu}(z)}{\check{f}(z)^2} &+ \frac{\check{\mu}(z)(\check{f}_z(z)-1)}{\check{f}(z)^2} zS(z,w)dxdy \\ &= \int_{B_1} \mu(\frac{1}{z})[\frac{1}{\bar{z}^2} + \frac{z^2 - \check{f}(z)^2}{\bar{z}^2\check{f}(z)^2}]zS(z,w)dxdy \\ &\qquad + \int_{B_1} \frac{\check{\mu}(\check{f}_z - 1)}{\check{f}(z)^2} zS(z,w)dxdy \\ &= \int_{B_1} \mu(\frac{1}{z})\frac{1}{\bar{z}^2} zS(z,w)dxdy + I + II\end{aligned}$$

Using Lemma **??**, we see

$$\frac{1}{C}|z|^{1/\alpha} \leq |\check{f}(z)| \leq C|z|^\alpha,$$

$$|z - \check{f}(z)| \leq C\|\mu\|_\infty |z|^\alpha,$$

so we can estimate I by

$$\begin{aligned}I \quad &\leq \quad |\int_{B_1} \mu(1/z)\frac{z^2 - \check{f}(z)^2}{\bar{z}^2\check{f}(z)^2} zS(z,w)dxdy| \\ &\leq \quad C\|\mu\|_\infty^2 \int_{B_1} |z|^{2\alpha - 1 - \frac{2}{\alpha}} S(z,w)dxdy \\ &\leq \quad C\|\mu\|_\infty^2 C(\alpha),\end{aligned}$$

if $2\alpha - 1 - \frac{2}{\alpha} > -2$ (recall that we may take α as close to 1 as we wish, if k is small enough). To estimate II, note that for $\frac{1}{p} + \frac{1}{q} = 1$, Lemma **??** implies

$$\begin{aligned} II &= \int_{B_1} \frac{\check{\mu}(z)(\check{f}_z(z)-1)}{\check{f}(z)^2} zS(z,w)dxdy \\ &\leq C\|\mu\|_\infty \|\check{f}_z - 1\|_p \|\frac{zS(z,w)}{\check{f}(z)^2}\|_q \\ &\leq \|\mu\|_\infty^2 \|z^{1-\frac{2}{\alpha}} S(z,w)\|_q. \end{aligned}$$

Fix some $q < 2$, and take k so small that $\alpha > 2q/(2+q)$, which implies the L^q norm is finite (with bound depending only on α, hence only of k). Thus,

$$f(w) = w - \frac{1}{\pi}\int_{B_1} \mu(z)R(z,w)dxdy - \frac{1}{\pi}\int_{B_1} \mu(\frac{1}{z})\frac{1}{\bar{z}^2} zS(z,w)dxdy + O(\|\mu\|_\infty^2).$$

Changing variables from z to $1/z$ in the second integral converts the integrand to the same form as the first (but now over $\{|z| > 1\}$). Hence,

$$f(w) = w - \frac{1}{\pi}\int_{\mathbb{R}^2} \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty^2),$$

as desired. □

COROLLARY 1.4. *If $\mu(t)$ is continuous in the L^∞ norm, then $f_{\mu(t)}(z)$ is a C^1 curve in $\mathbb{C}$.*

PROOF. Think of the path $\gamma(t) = f_{t\mu}(z)$. The key point is the formula

$$|\gamma(t) - (\gamma(0) + \gamma'(0)t)| \leq Ct^2,$$

holds on an interval $[-\delta, \delta]$ and with a constant C that do not depend on t or z or μ (except for $\|\mu\|_\infty$). Thus using the same estimate at $\gamma(t)$ and time $-t$ gives

$$|\gamma(0) - (\gamma(t) - \gamma'(t)t)| \leq Ct^2.$$

Thus adding the estimates and dividing by t gives

$$|\gamma'(0) - \gamma'(t)| \leq Ct$$

Thus $f_{t\mu}(z)$ has a continuous derivative in t when t is real. When t is multivariable, the same argument shows we have continuous partial derivatives, and this implies differentiability by the usual calculus argument (e.g., see Rudin's book [**?**]). □

2. Teichmuller-Wittich theorem

3. Bilipschitz bounds

LEMMA 3.1. *Suppose A,B are disjoint, planar sets and*

$$\int_A \frac{dxdy}{|z-w|^2} \leq C < \infty,$$

for all $w \in B$. If φ is a K-quasiconformal mapthat is conformal off A, then φ is M-bi-Lipschitz on B with M depending only on C and K, i.e., for all $w,z \in B$,

$$0 < \frac{1}{M(C,K)} \leq \frac{|\varphi(z)-\varphi(w)|}{|z-w|} \leq M(C,K) < \infty.$$

PROOF. This is more-or-less immediate from results of Bojarski, Lehto, Teichmüller and Wittich [**?**], [**?**], [**?**], [**?**] although we shall give specific references to the more recent paper [**?**] which also gives the higher dimensional versions of the two dimensional results we will use.

First we prove that φ is asymptotically conformal at ∞. Let $w \in B$. Denote by $\mu(z)$ the dilatation of φ. This function is supported on A. Thus, we get

$$\int_{|z|>|w|} \frac{|\mu(z)|dxdy}{|z|^2} \leq \int_{A\cap\{|z|>|w|\}} \frac{dxdy}{|z|^2} \leq \int_A \frac{4dxdy}{|z-w|^2} \leq 4C.$$

Hence, (see for example [**?**, Chapter V, Theorem 6.1]) there is a $c \neq 0$ so that

$$\lim_{z\to\infty} \frac{\varphi(z)}{z} = c.$$

Now suppose $z,w \in B$ and let $r = |z-w|$. Note that

$$\{\xi : |\xi - z| = R\} \subset \{\xi : R - |z| \leq |\xi| \leq R + |z|\}.$$

If R is large enough, φ maps the round annulus $A(z,r,R) = \{\xi : r < |\xi - z| < R\}$ to a topological annulus A' whose outer boundary is contained in the annulus $A(\varphi(z), |c|R/2, 2|c|R)$ and whose inner boundary is a closed Jordan curve γ. By taking R large enough, we can assume γ hits the disk $D(\varphi(z), |c|R/4)$. Therefore,

$$\mathrm{mod}(A') = \mathrm{mod}(A(\varphi(z), \mathrm{diam}(\gamma), R)) + O(1) = \log R - \log \mathrm{diam}(\gamma) + O(1).$$

On the other hand, Corollary 2.10 of [**?**] says that

$$\begin{aligned} \mathrm{mod}(A') \quad &= \mathrm{mod}(A(z,r,R)) + O\left(\int_{r<|\xi-z|<R} \tfrac{|\mu(\xi)|}{|\xi|^2} dxdy\right) \\ &= \mathrm{mod}(A(z,r,R)) + O\left(\int_{A\cap\{r<|\xi-z|<R\}} \tfrac{1}{|\xi|^2} dxdy\right) \\ &= \log R - \log r + O(1). \end{aligned}$$

Thus, $\log \operatorname{diam}(\gamma) = \log r + O(1)$, or $\operatorname{diam}(\gamma) \simeq r$.Since φ is quasiconformal, the segment S connecting z and w maps to a quasi-arc and hence satisfies the Ahlfors three-point condition, so $|\varphi(z) - \varphi(w)| \simeq \operatorname{diam}(\varphi(S))$, and since quasiconformal maps are quasisymmetric $\operatorname{diam}(\varphi(S)) \simeq \operatorname{diam}(\gamma)$. Thus, $|\varphi(z) - \varphi(w)| \simeq |z - w|$, as desired. □

4. Thin support

If the dilatation μ of a quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is small, then we expect f to be close to conformal, hence close to linear. There are at least two reasonable senses in which we can ask μ to be small: that $\|\mu\|_\infty$ is small or that $\{z : \mu(z) \neq 0\}$ is small. In this section we consider the latter possibility.

To be more precise, we say a measurable set $E \subset \mathbb{R}^2$ is (ε, φ)-thin if $\varepsilon > 0$ and

$$\operatorname{area}(E \cap D(z, 1)) \leq \varepsilon \varphi(|z|)$$

where $\varphi : [0, \infty) \to [0, \pi]$ is a bounded, decreasing function, such that

$$\int_0^\infty \varphi(r) r^n dr < \infty,$$

for every $n > 1$. If $a > 0$, the function $\varphi(r) = \exp(-ar)$ satisfies this condition, and this example suffices for many applications.

Recall that a quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ is often normalized by post-composing by a conformal linear map in one of two ways. First, we can assume $f(0) = 0$ and $f(1) = 1$. We call this the 2-point normalization. Second, if the dilatation of f is supported on a bounded set, then f is conformal in a neighborhood of ∞ and then we can choose R large and post-compose with a linear conformal map so that

$$|f(z) - z| = O(\frac{1}{|z|}),$$

for $|z| > R/2$. We say that such an f is normalized at ∞. This is also called the **hydrodynamical normalization** of f. We will first prove an estimate for the hydrodynamical normalization and then deduce one for the 2-point normalization.

THEOREM 4.1. *Suppose $F : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $E = \{z : \mu(z) \neq 0\}$ is bounded (so F is conformal near ∞) and F is normalized so*

$$|F(z) - z| \leq M/|z|,$$

near ∞. Assume E is (ε, φ)-thin. Then for all $z \in \mathbb{C}$,

$$|F(z) - z| \leq \frac{\varepsilon^\beta}{|z| + 1},$$

where β depends only on K and φ. In particular, as $\varepsilon \to 0$, F converges uniformly to the identity on the whole plane.

COROLLARY 4.2. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, $F(0) = 0$, $F(1) = 1$, and $E = \{z : \mu(z) \neq 0\}$ is (ε, φ)-thin. Then*

$$(1 - C\varepsilon^{\beta})|z-w| - C\varepsilon^{\beta} \le |f(z) - f(w)| \le (1 + C\varepsilon^{\beta})|z-w| + C\varepsilon^{\beta}, \tag{4.1}$$

where C and β only depend on $k = \|\mu\|_\infty$ and φ.

Similar estimates are known, e.g., compare to the well known result of Teichmüller and Wittich (e.g., Theorem 7.3.1 of [**?**], [**?**], [**?**]) or estimates of Dyn'kin [**?**]. The version stated above is intended for specific applications to holomorphic dynamics, as in [**?**] and [**?**] (a particular consequence used in the latter paper is given as Lemma **??**). Because the quasiconformal maps used in these references satisfy the strong ε-thin condition, it seemed desirable to have a self-contained proof of the estimate above.

We will use the following facts about quasiconformal maps proved earlier:

LEMMA 4.3 (Characterization of quasicircles). *For each $K \ge 1$ there is a $C = C(K) < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and $r > 0$ so that $f(\gamma) \subset \{z : r \le |z-w| \le Cr\}$.*

THEOREM 4.4 (Borjarki's theorem). *If $1 \le K < \infty$, there is a $p > 2$ and $A, B < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $Q \subset \mathbb{C}$ is a square, then*

$$\Big(\frac{1}{\text{area}(Q)} \iint_Q |f_z|^p dxdy\Big)^{1/p} \le A\Big(\frac{1}{\text{area}(Q)} \int_Q |f_z|^2 dxdy\Big)^{1/2} \le B\frac{\text{diam}(f(Q))}{\text{diam}(Q)}$$

LEMMA 4.5 (Pompeiu's formula). *If Ω has a piecewise C^1 boundary and f is quasiconformal on Ω, then*

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \iint_{\Omega} \frac{f_{\bar z}}{z-w} dxdy. \tag{4.2}$$

If Ω is a topological annulus in the plane with boundary components γ_1, γ_2 that are closed Jordan curves, then mod (Ω) refers to the modulus of the path family in Ω that separates the boundary components. This is the same as the extremal length of the path family that connects the boundary components (also called the extremal distance between the boundary components). If $A(a,b) \equiv \{z : a < |z| < b\}$ then it is standard fact that mod $(A) = \frac{1}{2\pi} \log \frac{b}{a}$. Let

$$D_f = \frac{|f_z| - |f_{\bar z}|}{|f_z| + |f_{\bar z}|},$$
$$J_f = |f_z|^2 - |f_{\bar z}|^2 = (|f_z| - |f_{\bar z}|)(|f_z| + |f_{\bar z}|),$$

denote the distortion and Jacobian of f respectively. Note that $D_f \geq 1$ and f is conformal if and only if $D_f \equiv 1$.

LEMMA 4.6. *Suppose f is a K-quasiconformal map from $A_m = A(1, e^m)$ to $A_M = A(1, e^M)$. Then*

$$M \geq m - \frac{1}{2\pi} \int_{A(1,e^m)} (D_f(z) - 1) \frac{dxdy}{r^2}.$$

PROOF. Let Γ_M be the path family connecting the boundary components of A_M. If $\tilde{\rho}$ is admissible for this family then

$$\rho(z) = \tilde{\rho}(f(z))(|f_z| + |f_{\bar{z}}|)$$

is admissible for Γ_m, the path family connecting the boundary components of A_m. Therefore the modulus of Γ_m satisfies

$$\text{mod}\,(\Gamma_m) \leq \int_{A_m} \tilde{\rho}(f(z))^2 (|f_z| + |f_{\bar{z}}|)^2 dxdy.$$

Applying this formula to the inverse of f shows that for any admissible ρ for Γ_m,

$$\begin{aligned}
\text{mod}\,(f(\Gamma_m)) &\leq \int_{A_m} \rho(z)^2 \frac{1}{(|f_z| - |f_{\bar{z}}|)^2} J_f dxdy \\
&\leq \int_{A_m} \rho(z)^2 \frac{1}{(|f_z| - |f_{\bar{z}}|)^2} (|f_z|^2 - |f_{\bar{z}}|^2) dxdy \\
&\leq \int_{A_m} \rho(z)^2 \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} dxdy \\
&\leq \int_{A_m} \rho(z)^2 D_f(z) dxdy.
\end{aligned}$$

Applying this with the admissible metric $\rho(z) = \frac{1}{m|z|}$, we get

$$\begin{aligned}
\frac{2\pi}{M} = \quad \text{mod}\,(f(\Gamma_m)) &\leq \frac{1}{m^2} \int_{A_m} \frac{D_f(z)}{|z|^2} dxdy \\
&= \frac{1}{m^2} [\int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy + \int_{A_m} \frac{1}{|z|^2} dxdy] \\
&= \frac{1}{m^2} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy + \frac{2\pi}{m}.
\end{aligned}$$

Rearranging gives

$$m - M \leq \frac{M}{2\pi m} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy,$$

or

$$M \geq m - \frac{M}{2\pi m} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy.$$

If $M > m$, the lemma is trivially true. If $M \le m$, then because the integral in non-negative, the inequality above becomes

$$M \geq m - \frac{1}{2\pi}\int_{A_m} \frac{D_f(z)-1}{|z|^2} dxdy.$$

Thus in either case the lemma holds. □

LEMMA 4.7. *Suppose f is a K-quasiconformal map from $A_m = A(1,e^m)$ to $A_M = A(1,e^M)$. Then*

$$M \le m + \frac{1}{2\pi}\int_{A_m)} (D_f - 1)\frac{dxdy}{r^2}.$$

PROOF. If we cut A_m with a radial slit and let $g = \log(f)$, then g maps A_m to a quadrilateral with its vertical sides on $\{x = 0\}$ and $\{x = M\}$. This quadrilateral has area $2\pi M$. If we integrate over the radial segments in A_m, we get

$$M \quad \le \quad \int_1^{\exp(m)} (|g_z| + |g_{\bar z}|) dr$$

so integrating over all angles and using $rdrd\theta = dxdy$ gives

$$2\pi M \quad \le \quad \int_0^{2\pi}\int_1^{\exp(m)} (|g_z| + |g_{\bar z}|) drd\theta \le \int_{A_m} (|g_z| + |g_{\bar z}|)\frac{dxdy}{r}.$$

Thus by Cauchy-Schwarz,

$$\begin{aligned}
(2\pi M)^2 \quad &\le \quad (\int_{A_m} (|g_z| + |g_{\bar z}|)(|g_z| - |g_{\bar z}|)dxdy)(\int_{A_m} \frac{|g_z| + |g_{\bar z}|}{|g_z| - |g_{\bar z}|}\frac{dxdy}{r^2}) \\
&\le \quad (\int_{A_m} J_g dxdy)(\int_{A_m} D_g \frac{dxdy}{r^2}) \\
&\le \quad 2\pi M(\int_{A_m} D_f \frac{dxdy}{r^2}),
\end{aligned}$$

where in the last line we have used the facts that $g(A_m)$ has area $2\pi M$ and $D_g = D_f$ (since $\log z$ is conformal on the slit annulus). Thus

$$\begin{aligned}
M \quad &\le \quad \frac{1}{2\pi}\int_{A_m} 1 + (D_f(z) - 1)\frac{dxdy}{|z-w|^2} \\
&= \quad m + \frac{1}{2\pi}\int_{A_m} (D_f(z) - 1)\frac{dxdy}{|z-w|^2}.
\end{aligned}$$

□

The following simply combines the last two results.

COROLLARY 4.8. *Suppose f is a K-quasiconformal map from $A_m = A(1,e^m)$ to $A_M = A(1,e^M)$. Then*

$$M = m + O(\frac{1}{2\pi}\int_{A_m} \frac{D_f(z)-1}{r^2} dxdy.)$$

A special case of this is:

COROLLARY 4.9. *Suppose f is a K-quasiconformal map from $A_m = A(1,e^m)$ to $A_M = A(1,e^M)$. Suppose $D_f(z) \le D$ on A_m. Suppose μ is the dilatation of f, that $E = \{z : \mu(z) \neq 0\}$ and that $E_k = E \cap \{2^{k-1} < |z| < 2^k\}$. If we choose an integer n so that $m \le 2^n$, then*

$$M = m + O((D-1)\sum_{k=0}^{n} 2^{-2k} \operatorname{area}(E_k)).$$

Next we apply these estimates to quasiconformal maps with dilatations that have small suppport in a precise sense.

LEMMA 4.10. *Suppose F is a K-quasiconformal map with dilatation μ, that μ has bounded support, and that F has the hydrodynamical normalization at ∞. Let $E = \{z : \mu(z) \neq 0\}$ and suppose for some $t > 0$, E satisfies*

$$\int_{E\setminus D(w,t)} \frac{dxdy}{|z-w|^2} \le a,$$

for every $w \in \mathbb{C}$. Then there is a $C - C(K,a) < \infty$, depending only on K and a, so that for every $w \in \mathbb{R}^2$ and $r \ge t$,

$$\frac{1}{C} \le \frac{1}{r}\operatorname{diam}(F(D(w,r)) \le C.$$

PROOF. We need only prove this for $r = t$ since for $r > t$, we can simply apply the lemma after setting $t = r$ (the integral just gets smaller).

The mapping $G(z) = F(tx)/t$, satisfies the same estimates as F, but with t replaced by 1. If we prove the lemma for G, it follows for F, so it suffices to assume $t = 1$.

By assumption we can choose $R > 100$ so that $|f(z) - z| \le 1$, for $|z| > R/8$. The result is clear if $|w| > R/2$, so we may assume $|w| \le R/2$. Fix such a w. Let $m = \log R$, so $R = e^m$, and consider the annulus $A = \{z : 1 < |z-w| < e^m\}$. $F(A)$ is a topological annulus and can be conformally mapped to $A_M = \{1 < |z| < e^M\}$ for some $M > 1$. By Corollary **??**,

$$M = m + O(\int_{A_m} \frac{D_f - 1}{|z-w|^2} dxdy).$$

By our assumptions, this becomes

$$M = m + O(\frac{K-1}{2\pi}\int_{A_m} \mathbf{1}_E(z)\frac{dxdy}{|z-w|^2}) = m + O(Ka),$$

where $\mathbf{1}_E$ denotes the indicator function of E (the function that is one on E and zero off E) and we have used the fact that E has finite planar area and $|z-w|^{-1} \leq 1$ on A_m (recall w is the center of the annulus and the inner radius is at least 1.).

he Hardy-Littlewood inequality on rearrangement of integrals states that

$$\int^{\mathbb{R}^2} f(z)g(z)dxdy \leq \int^{\mathbb{R}^2} f^*(z)g^*(z)dxdy$$

where f^*, g^* are the symmetric, decreasing rearrangements of f and g. This inequality implies that we will maximize the integral by setting $w=0$, so

$$\begin{aligned} \lambda(f(A))\log^2\frac{R}{r} &\leq \log\frac{R}{r} + K\varepsilon \int_{|z|>1} \varphi(|z|)dxdy|z|^2 \\ &= \log\frac{R}{r} + O(\varepsilon). \end{aligned}$$

Thus

$$\lambda(f(A)) \leq (\log\frac{R}{r})^{-1} + \frac{K\varepsilon}{\log^2 R/r}$$

which implies

$$M(f(A)) \geq \log\frac{R}{r} - K\varepsilon$$

By Corollary **??**, the boundary components of $f(A_m)$ are each closed curves that are contained in round annuli (with concentric circles) of bounded modulus (depending on K). Thus $f(A_m)$ is contained in a topological annulus A' with circular boundaries γ_1, γ_2 (not necessarily concentric) whose diameters are comparable to the diameters of the boundary components of $f(A_m)$. By monotonicity of modulus, the modulus of the annulus A' (denoted $M'/2\pi$) is larger than the modulus $M/2\pi$ of $f(A)$, hence $M' \geq M$. Moreover, we claim

$$M' \leq \log\frac{\mathrm{diam}(\gamma_2)}{\mathrm{diam}(\gamma_1)}.$$

This is well known to hold with equality if the circles γ_1, γ_2 are concentric. If they are not, then we can apply a Möbius transformation that maps the outer circle, γ_2, to itself and moves the inner circle, γ_1 to circle concentric with γ_2. This make the Euclidean diameter of γ_1 larger and preserves the modulus between the circles, and this proves the claimed inequality. Thus

$$M \leq M' \leq \log\frac{\mathrm{diam}(\gamma_2)}{\mathrm{diam}(\gamma_1)},$$

or

$$\mathrm{diam}(\gamma_1) \leq \mathrm{diam}(\gamma_2)\cdot e^{-M} = \mathrm{diam}(\gamma_2)\cdot e^{-m+O(KA)}.$$

Since $|f(z)-z| \le 1$ on $\{|z|=R\}$ we know $\mathrm{diam}(\gamma_2) \simeq R = e^m$. Using this and the fact $M = m + O(Ka)$ prove above gives

$$\mathrm{diam}(f(\{|z-w|=1\})) \simeq \mathrm{diam}(\gamma_1) = O(e^{Ka}).$$

To get the other direction, we choose γ_1, γ_2 to be circles that bound an annulus inside $f(A_m)$, again with diameters comparable to the diameters of the corresponding components of $\partial f(A_m)$. We then use monotonicity again, and argue as before, but now we note that since f is close the identity for $|z| > R/2$, the curve γ_1 is not too close to γ_2, i.e., the distance between them is comparable to R. Thus in the argument above, where we moved γ_1 be be concentric with γ_2, its Euclidean diameter was only changed by a bounded factor. Thus

$$\mathrm{diam}(\gamma_1) \gtrsim \mathrm{diam}(\gamma_2) \cdot e^{-M} = \mathrm{diam}(\gamma_2) \cdot e^{-m-O(KA)} \gtrsim e^{-O(KA)}..$$

This proves the lemma. □

If F is as above, Bojarski's theorem (Theorem **??**) says there is a $p = p(K) > 2$ so that the L^p norm of F_z is uniformly bounded on every unit radius disk. If a region can be covered by n such disks then the L^p norm is $O(n^{1/p})$ with a uniform constant, i.e.,

COROLLARY 4.11. *If F satisfies the conditions of Lemma **??**, and $p = p(K) > 2$ is as above, then $\|F_z \cdot \mathbf{1}_{D(z,r)}\|^p = O(r^{2/p})$ uniformly for all $z \in \mathbb{C}$.*

PROOF OF THEOREM **??**. Suppose the support of μ is contained in $D(0,R)$. The main idea is to use the Pompeiu formula

$$F(w) = \frac{1}{2\pi i}\int_{|z|=r}\frac{F(z)}{z-w}dz - \frac{1}{\pi}\iint_{|z|<r}\frac{F_{\overline{z}}}{z-w}dxdy. \tag{4.3}$$

Because of our assumptions on F, the first integral is

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{z+O(1/|z|)}{z-w}dz = w + O(1/r).$$

The left-hand side of (**??**) and the second integral are both constant for $r > R$, so the first integral must equal w for all $r > R$. Thus

$$F(w) = w - \frac{1}{\pi}\iint_{|z|<r}\frac{F_{\overline{z}}}{z-w}dxdy = w - \frac{1}{\pi}\iint_{|z|<r}\frac{\mu F_z}{z-w}dxdy.$$

Since $|F_{\overline{z}}| = |\mu F_Z| \le k|F_z|$, we get

$$|F(w)-w| \le \frac{k}{\pi}\int_E |\frac{F_z}{z-w}|dxdy.$$

where $k = (K-1)/(K+1)$ is our upper bound for $|\mu|$.

The estimate in the theorem already holds if $|w| \geq R$, so assume $|w| < R$. Let $r = \max(1, |w|/2)$. We will estimate the integral

$$\int_E |\frac{F_z}{z-w}| dxdy,$$

by cutting $D(0,R)$ into three pieces:

$$\begin{aligned} D_1 &= \{z : |z-w| \leq 1\} \\ A &= \{z : 1 \leq |z-w| \leq r\} \\ X &= D(0,R) \setminus (D_1 \cup A), \end{aligned}$$

and showing the integral over each piece is $O(\varepsilon^\beta/|w|)$ for some $\beta > 0$ depending only on K.

First consider D_1. With p as in Corollary **??**, the L^p norm of F_z over D_1 is uniformly bounded, so using Hölder's inequality with the conjugate exponents, we get

$$\int_{D_1} |\frac{F_z}{z-w}| dxdy = O(\| \frac{\mathbf{1}_{E \cap D(w,1)}}{|z-w|} \|_q). \tag{4.4}$$

Since $E \cap D(w,1)$ has area at most $\varphi(|w|) \leq \varphi(r)$, the L^q norm on the right side of (**??**) is bounded above by what happens when $E \cap D(w,1)$ is a disk of radius $s \simeq (\varepsilon\varphi(r))^{1/2}$ centered at w. In this case we get the bound (using polar coordinates and recalling $1 < q < 2$)

$$O([\int_0^s r^{-q} r dr]^{1/q}) = O(s^{(2-q)/q}) = O((\varepsilon\varphi(r))^{\frac{1}{q}-\frac{1}{2}}).$$

Since φ tends to zero faster than any polynomial, this is $= O(\varepsilon^{\frac{1}{q}-\frac{1}{2}} \frac{1}{|w|})$. This is the desired estimate with $\beta = \frac{1}{q} - \frac{1}{2} > 0$.

Next consider the integral over A:

$$\begin{aligned} \int_A |\frac{F_z}{z-w}| dxdy &= \int_A \mathbf{1}_E(z) |F_z| dxdy \\ &= (\int_A \mathbf{1}_E(z)^q dxdy)^{1/q} (\int_A |F_z|^p dxdy)^{1/p} \\ &= O(\text{area}(E \cap A))^{1/q} \|F_z \mathbf{1}_A\|_p \\ &= O((\varepsilon r^2 \varphi(r))^{1/q}) r^{2/p} \\ &= O(\varepsilon^{1/q} \frac{1}{|w|}), \end{aligned}$$

since φ decays faster than any power.

To estimate the integral over X, write

$$X = \cup_{k=1}^R X_k = \cup_{k=1}^R X \cap A_k = \cup_{k=1}^R X \cap \{z : k-1 \leq |z| < k\},$$

Then

$$\begin{aligned}\int_{X_k} \mathbf{1}_E(z)|F_z|dxdy &= (\int_{A_k} \mathbf{1}_E(z)^q dxdy)^{1/q}(\int_{A_k} |F_z|^p dxdy)^{1/p} \\ &= (\text{area}(E \cap A_k))^{1/q}(\int_{A_k} |F_z|^p dxdy)^{1/p} \\ &= (\varepsilon k \varphi(k))^{1/q}(O(k))^{1/p} \\ &= O(\varepsilon^{1/q}\varphi(k))^{1/q}k^{1+1/p}) \\ &= O(\varepsilon^{1/q}k^{-2}),\end{aligned}$$

again since φ decays faster than any power. Summing over k gives the desired estimate. This proves the theorem with $\beta = \frac{1}{q} - \frac{1}{2} > 0$. □

he argument in Section 1.23 only shows that $q < 2$ with some bound depending on K, but

$$\beta = \varepsilon^{\frac{1}{q}-\frac{1}{2}},$$

where $q < 2$ is the conjugate exponent of p. We have only proved $p > 2$, but a sharp estimate was given by Astala [**?**]: we can take any $p \in (2, 2K/(k-1))$.

The proof given above shows that the conclusion of Theorem **??** still holds if $\int_0^\infty \varphi(r)r^n dr < \infty$ for some (large) finite n that depends on K (in particular, it depends on the value $p > 2$ so that $F_z \in L^p$ in Bojarski's theorem). Similarly, we can assume less if we simply want a uniform bound on $|F(w) - w|$, rather than the $O(1/|z|)$ estimate above. We leave these generalizations to the reader.

PROOF OF COROLLARY **??**. First we note that it suffices to prove this with the additional assumption that μ has bounded support, for a general quasiconformal f is the pointwise limit of such maps (truncate μ_f, apply the measurable Riemann mapping theorem and show the truncated maps converge uniformly on compact subsets to f).

So assume $\mu = \mu$ has bounded support, say inside the disk $D(0,R)$. Then f is conformal outside $D(0,R)$, so we can post-compose by a conformal linear map L to get a quasiconformal map

$$F(z) = z + O(\frac{1}{z}),$$

or

$$|F(z) - z| \leq C/|z|,$$

outside $D(0, 2R)$ with a constant that does not depend on F (this follows from the distortion theorem for conformal maps). We apply Theorem **??** to get

$$|F(z) - z| \leq C\varepsilon^\beta,$$

for all z with constants C, β that depend only on k. Note that

$$f(z) = \frac{F(z)-F(0)}{F(1)-F(0)},$$

and that

$$|F(1)-F(0)-1| \leq C\varepsilon^{\beta},$$

so,

$$|f(z)-f(w)| \quad = \quad |\frac{F(z)-F(w)}{F(1)-F(0)}| = \frac{|z-w|+O(\varepsilon^{\beta})}{1+O(\varepsilon^{\beta})},$$

and this implies (**??**). □

The following consequence of Theorem **??** is used in [**?**].

LEMMA 4.12. *Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is K-quasiconformal, it fixes* 0 *and* 1*, maps $\mathbb{R}$ to $\mathbb{R}$, and is conformal in the strip $\{x+iy : |y| < 1\}$. Let $E = \{z : \mu(z) \neq 0\}$ and suppose E is (ε, φ)-thin. If ε is sufficiently small (depending on k and φ), then $0 < \frac{1}{C} \leq |f'(x)| \leq C < \infty$ for all $x \in \mathbb{R}$, where C depends on K, φ and ε is otherwise independent of f. If we fix K and φ and let $\varepsilon \to 0$ then $C \to 1$.*

PROOF. For each $x \in \mathbb{R}$, f is conformal on the disk $D(x,1) \subset S$, so Koebe's $\frac{1}{4}$-theorem says that

$$|f'(x)| \simeq \operatorname{dist}(f(x), \partial f(D(x,1))).$$

However taking $z = x$ and $w \in \partial D(x,1)$ in (**??**) shows that

$$\operatorname{dist}(f(x), \partial f(D(x,1))) \simeq 1.$$

This gives the first claim. When ε is small, then (**??**) implies that

$$(1-\delta)S \subset f(S) \subset (1+\delta),$$

where $\delta > 0$ tends to zero with ε (for fixed k and a). Thus as $\varepsilon \to 0$, f converges uniformly to the identity on S. In particular, f' converges uniformly to 1 on $\mathbb{R}$. □

Exercises

CHAPTER 5

Constructing Eremenko-Lyubich funtions

The singular set of a entire function f is the closure of its critical values and finite asymptotic values and is denoted $S(f)$. The Eremenko-Lyubich class $\mathscr{B}$ consists of transcendental entire functions such that $S(f)$ is a bounded set. The Speiser class $\mathscr{S} \subset \mathscr{B}$ are those functions for which $S(f)$ is a finite set. If $f \in \mathscr{B}$ then $\Omega = \{z : |f(z)| > R\}$ and $f|_\Omega$ must satisfy certain simple topological conditions when R is sufficiently large. A model (Ω, F) is an open set Ω and a holomorphic function F on Ω that satisfy these same conditions. In this chapter we show any model can be approximated by an Eremenko-Lyubich function in a precise sense. In many cases, this allows the construction of functions in $\mathscr{B}$ with a desired property to be reduced to the construction of a model with that property, and this is often much easier to do. We shall see examples if this in the next chapter.

1. Models

Suppose $\Omega = \cup_j \Omega_j$ is a disjoint union of unbounded simply connected domains such that

(1) sequences of components of Ω accumulate only at infinity,
(2) $\partial\Omega_j$ is connected for each j (as a subset of $\mathbb{C}$).

Such an Ω will be called a model domain. If $\overline{\Omega} \cap \{|z| \leq 1\} = \emptyset$, we say the model domain is disjoint type. The connected components $\{\Omega_j\}$ of Ω are called tracts. Given a model domain, suppose $\tau : \Omega \to \mathbb{H}_r = \{x + iy : x > 0\}$ is holomorphic so that

(1) The restriction of τ to each Ω_j is a conformal map $\tau_j : \Omega_j \to \mathbb{H}_r$,
(2) If $\{z_n\} \subset \Omega$ and $\tau(z_n) \to \infty$ then $z_n \to \infty$.

Given such a $\tau : \Omega \to \mathbb{H}_r$, we call $F(z) = \exp(\tau(z))$ a model function.

The second condition on τ is a careful way of saying that the conformal map on each component sends ∞ to ∞. Even after making this condition, we still have a (real) 2-dimensional family of conformal maps from each component of Ω to $\mathbb{H}_r$ determined by choosing where one base point in each component will map in $\mathbb{H}_r$. A choice of both a model domain Ω and a model function F on Ω will be called a model.

Given a model (Ω, F) we let

$$\Omega(\rho) = \{z \in \Omega : |F(z)| > e^\rho\} = \tau^{-1}(\{x+iy : x > \rho\}),$$

and

$$\Omega(\delta, \rho) = \{z \in \Omega : e^\delta < |F(z)| < e^\rho\} = \tau^{-1}(\{x+iy : \delta < x < \rho\}).$$

If Ω has connected components $\{\Omega_j\}$ we let $\Omega_j(\rho) = \Omega(\rho) \cap \Omega_j$ and similarly for $\Omega_j(\delta, \rho)$.

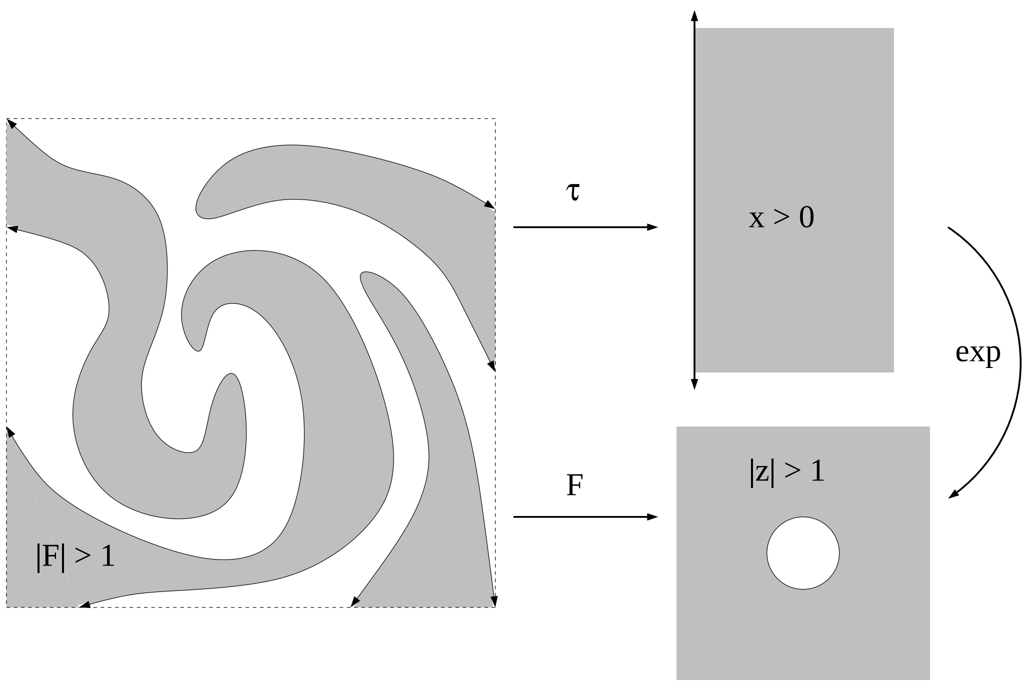


FIGURE 1.1. A model consists of an open set Ω which may have a number of connected components called tracts. Each tract is mapped conformally by τ to the right half-plane and then by the exponential function to the exterior of the unit disk. The composition of these two maps is the model function F. In this paper, we are interested in knowing if a holomorphic model function on Ω can be approximated by holomorphic function on the entire plane.

Each function f in the Eremenko-Lyubich class that satisfies $S(f) \subset \mathbb{D}$ gives rise to a model by taking $\Omega = \{z : |f(z)| > 1\}$ and $\tau(z) = \log f(z)$. The log is well defined since each component of Ω is simply connected and f is non-vanishing on Ω. Eremenko and Lyubich proved in [**?**] that τ defined in this way is a conformal map from each component of Ω to $\mathbb{H}_r$.

We call a model arising in this way an Eremenko-Lyubich model. If f is in the Speiser class, we call it a Speiser model.

THEOREM 1.1 (All models occur). *Suppose* (Ω, F) *is a model and* $0 < \rho \leq 1$. *Then there is* $f \in \mathscr{B}$ *and a quasiconformal* $\varphi : \mathbb{C} \to \mathbb{C}$ *so that* $F = f \circ \varphi$ *on* $\Omega(2\rho)$. *In addition,*

(1) $|f \circ \varphi| \leq e^{2\rho}$ *off* $\Omega(2\rho)$ *and* $|f \circ \varphi| \leq e^{\rho}$ *off* $\Omega(\rho)$. *Thus the components of* $\{z : |f(z)| > e^{\rho}\}$ *are in* 1*-to-*1 *correspondence to the components of* Ω *via* φ.
(2) $S(f) \subset D(0, e^{\rho})$.
(3) *the quasiconstant of* φ *is* $O(\rho^{-2})$ *with a constant independent of* F *and* Ω,
(4) φ^{-1} *is conformal except on the set* $\Omega(\frac{\rho}{2}, 2\rho)$.

We sketch the proof of Theorem **??** quickly here to give the basic idea, and give the details in later sections. Let $W = \mathbb{C} \setminus \overline{\Omega(\rho)}$. It is simply connected, non-empty and not the whole plane, so there is a conformal map $\Psi : W \to \mathbb{D}$. Since Ψ maps ∂W to the unit circle, if we knew that $F = f|_{\Omega}$ for some entire function f, then $B = e^{-\rho} \cdot F \circ \Psi^{-1}$ would be an inner function on $\mathbb{D}$ (i.e., a holomorphic function on $\mathbb{D}$ so that $|B| = 1$ almost everywhere on the boundary).

The proof of Theorem **??** reverses this observation. Given the model and the corresponding domain W and conformal map Ψ we construct a Blaschke product B (a special type of inner function) on the disk so that $G = B \circ \Psi$ approximates $F = e^{\tau}$ on $\partial\Omega(\rho)$ (the precise nature of the approximation will be described later). This step is fairly straightforward using standard estimates of the Poisson kernel on the disk. We then "glue" G to F across ∂W to get a quasi-regular function g that agrees with F on $\Omega(2\rho)$ and agrees with G on W. This takes several (individually easy) steps to accomplish. We then use Stoilow factorization (Theorem **??**) to define a quasiconformal mapping $\phi : \mathbb{C} \to \mathbb{C}$ so that $f = g \circ \phi$ is holomorphic on the whole plane. The only critical points of g correspond to critical points of B, and critical points introduced into $\Omega(\rho, 2\rho)$ by the gluing process. We will show that both types of critical values have absolute value $\leq e^{\rho}$. A different argument shows that any finite asymptotic value of f must correspond to a limit of B along a curve in $\mathbb{D}$, so all finite asymptotic values of f are also bounded by e^{ρ}. Thus $f \in \mathscr{B}$. Since g is only non-holomorphic in $\Omega(\rho, 2\rho)$, we will also get that ϕ^{-1} is conformal everywhere except in $\Omega(\rho, 2\rho)$.

2. Reduction of Theorem ?? to the case $\rho = 1$

We start the proof of Theorem **??** with the observation that it suffices to prove the result for $\rho = 1$.

To do this we define two quasiconformal maps, ψ_ρ and φ_ρ. Define

$$L(x) = \begin{cases} x, & 0 < x < \rho/2, \\ (\frac{2-\rho}{\rho})(x-\rho/2)+\rho/2 & \rho/2 \le x \le \rho, \\ x/\rho & \rho \le x \le 2\rho. \end{cases}$$

This is a piecewise linear map that sends $[\rho/2,\rho]$ to $[\rho/2,1]$ and sends $[\rho,2\rho]$ to $[1,2]$. The slope on both intervals is less than $2/\rho$. For $z = x+iy \in \mathbb{H}_r$, define

$$\sigma_\rho(z) = \begin{cases} L(x)+iy & 0 < x \le 2\rho, \\ z+2-2\rho & x > 2\rho. \end{cases}$$

This is quasiconformal $\mathbb{H}_r \to \mathbb{H}_r$ with quasiconstant $K \le 2/\rho$. Then set

$$\psi_\rho(z) = \begin{cases} z, & z \notin \Omega \\ \tau_j^{-1} \circ \sigma_\rho \circ \tau_j(z), & z \in \Omega_j. \end{cases}$$

Note that ψ_ρ is the identity near $\partial\Omega$, so ψ_ρ is quasiconformal on the whole plane by the Royden gluing lemma, e.g., Lemma 2 of [**?**], Lemma I.2 of [**?**] on page 303, or [**?**].

The use of Royden's lemma can be avoided in the argument above by using a slightly more complicated L that is the identity on $[0,\rho/2]$ and maps $[\rho/2,\rho]$ linearly to $[\rho/2,1]$ and maps $[\rho,2\rho]$ linearly to $[1,2]$. Then we only need the gluing lemma across the smooth curves in $\partial\Omega(\rho/2)$, and this is easier than the general case (and the dependence on ρ is only slightly worse).

Next, define

$$\varphi_\rho(z) = \begin{cases} z, & |z| < e^{\rho/2} \\ \exp(\sigma_\rho(\log(z))), & |z| \ge e^{\rho/2} \end{cases}.$$

Note that even though $\log(z)$ is multi-valued, the function σ_ρ does not change the imaginary part of its argument, so the exponential of $\sigma_\rho(\log(z))$ is well defined. This is clearly a quasiconformal map of the plane with quasiconstant $2/\rho$. Note also that these functions were chosen so that if $F = \exp\circ\tau$ is the model function associated to Ω and τ, then on Ω_j

$$\begin{aligned} F\circ\psi_\rho &= \exp\circ\tau_j\circ\tau_j^{-1}\circ\sigma_\rho\circ\tau_j \\ &= \exp\circ\sigma_\rho\circ\log\circ\exp\circ\tau_j \\ &= \varphi_\rho\circ F. \end{aligned} \tag{2.1}$$

Now apply Theorem **??** to the model (Ω,F) with $\rho = 1$ to get a $f \in \mathscr{B}$ and a quasiconformal map $\Phi : \mathbb{C} \to \mathbb{C}$ so that $f\circ\Phi = F$ on $\Omega(2)$ and $S(f) \subset D(0,e^1)$. Let $g_\rho = \varphi_\rho^{-1}\circ f\circ\Phi\circ\psi_\rho$. This is an entire function pre and post-composed with quasiconformal maps of the plane, so it is quasiregular. By the measurable Riemann mapping theorem, there is a quasiconformal $\Phi_\rho : \mathbb{C} \to \mathbb{C}$ so that $f_\rho = g_\rho \circ \Phi_\rho^{-1}$ is entire and clearly

$$S(f_\rho) = S(g_\rho) \subset \varphi_\rho^{-1}(S(f)) \subset \varphi_\rho^{-1}(D(0,e)) = D(0,e^\rho).$$

For $z \in \Omega(2\rho)$, $\psi_\rho(z) \in \Omega(2)$, so using this and (**??**)

$$\begin{aligned} f_\rho \circ \Phi_\rho(z) &= g_\rho(z) \\ &= \varphi_\rho^{-1}(f(\Phi((\psi_\rho(z))))) \\ &= \varphi_\rho^{-1}(F(\psi_\rho(z))) \\ &= F(z). \end{aligned}$$

Similarly, $|f_\rho \circ \Phi_\rho| = |g_\rho|$ is bounded by $e^{2\rho}$ off $\Omega(2\rho)$. The quasiconstant of Φ_ρ is, at worst, the product of the constants for Φ, ψ_ρ and φ_ρ, which is $K_1 \cdot 4\rho^{-2}$, where K_1 is the upper bound for the quasiconstant in Theorem **??** in the case $\rho = 1$.

Finally, our construction in the next section will show that Φ is conformal except on $\Omega(1,2)$ and that F has a quasiregular extension to the plane that is holomorphic except on $\Omega(1,2)$ and is bounded by e off $\Omega(1)$ and by e^2 off $\Omega(2)$. This implies that g_ρ is holomorphic except on $\Omega(\rho/2, 2\rho)$ (since ψ_ρ is holomorphic off $\Omega(\rho/2, 2\rho)$ and φ_ρ^{-1} is holomorphic off $\{e^{\rho/2} < |z| < e^2\}$.) This, in turn, implies that Φ_ρ is conformal except on $\Omega(\rho/2, 2\rho)$, as desired. Thus f_ρ satisfies Theorem **??** for the model (Ω, F) and the given $\rho > 0$.

3. The proof of Theorem ??

In this section we give the proof of Theorem **??** for $\rho = 1$, stating certain facts as lemmas to be proven in later sections.

Let $W = \mathbb{C} \setminus \overline{\Omega(1)}$. This is an open, connected, simply connected domain that is bounded by analytic arcs $\{\gamma_j\}$ that are each unbounded in both directions. See Figure **??**. The same comments hold for the larger domain $W_2 = \mathbb{C} \setminus \overline{\Omega(2)}$.

Let $L_1 = \{x + iy : x = 1\}$ and $L_2 = \{x + iy : x = 2\}$. The vertical strip between these two lines will be denoted S. Note that L_1 is partitioned into intervals of length 2π by the points $1 + 2\pi i\mathbb{N}$. This partition of L_1 will be denoted $\mathscr{J}$. Note that $\tau_j(\gamma_j) = L_1$, so each curve γ_j is partitioned by the image of $\mathscr{J}$ under τ_j^{-1}. We denote this partition of γ_j by $\mathscr{J}_j$. Because elements of $\mathscr{J}_j$ are all images of a fixed interval $J \in L_1 \subset \mathbb{H}_r$ under some conformal map of $\mathbb{H}_r$, the distortion theorem (e.g., Theorem I.4.5 of [**?**]) implies they all lie in a compact family of smooth arcs and that adjacent

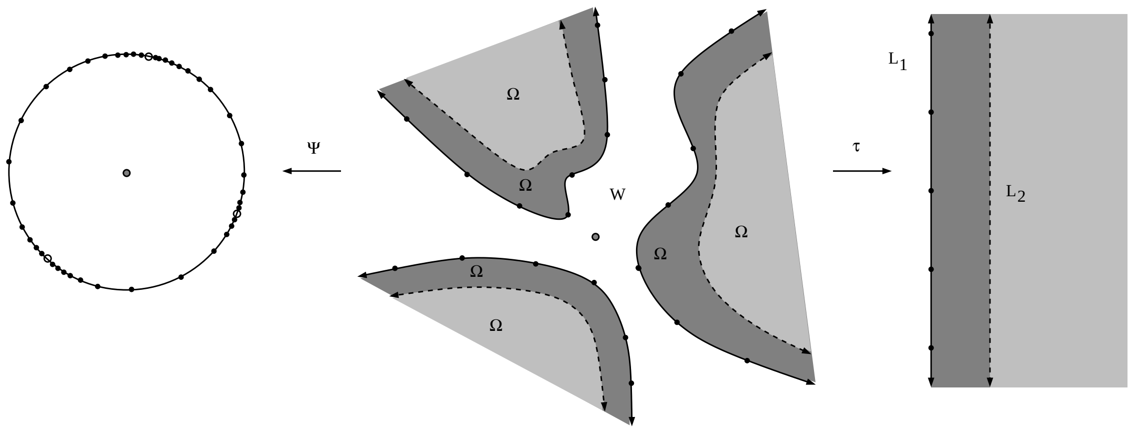


FIGURE 3.1. W is the complement of $\Omega(1)$; it is simply connected and bounded by smooth curves. We are given the holomorphic function $F = e^\tau$ on $\Omega(2)$ and we will define a holomorphic function on W using the Riemann map Ψ of W to the unit disk, and a specially chosen infinite Blaschke product B on the disk. We will then interpolate these functions in $\Omega(2) \setminus \Omega(1)$ by a quasiregular function. Each component of this set is mapped to a vertical strip by τ, and it is in these strips that we construct the interpolating functions. Note that the integer partition on the boundary of the half-plane pulls back under τ to a partition of each component of $\partial\Omega(1)$, and that Ψ maps these to a partition of the unit circle (minus the singular set of Ψ). The Blaschke product B will be constructed so that $B^{-1}(1)$ approximates this partition of the circle.

elements of $\mathscr{I}_j$ have comparable lengths with a uniform constant, independent of j, Ω and F.

Let $\Psi : W \to \mathbb{D}$ be a conformal map given by the Riemann mapping theorem. We claim that Ψ can be analytically continued from W to W_2 across γ_j. Let R_1 denote reflection across L_1 and for $z \in \Omega_j \cap W = \tau_j^{-1}(\{x+iy : 0 < x < 1\})$ let $T = \tau_j^{-1} \circ R_1 \circ \tau_j$; this defines an anti-holomorphic 1-to-1 map from $\Omega_j(0,1)$ to $\Omega_j(1,2)$ that fixes each point of γ_j. We can then extend Ψ by the formula

$$\Psi(T(z)) = 1/\overline{\Psi(z)},$$

(where the right hand side denotes reflection of $\Psi(z)$ across the unit circle). The Schwarz reflection principle says this is an analytic continuation of Ψ to W_2.

Thus Ψ is a smooth map of each γ_j onto an arc I_j of the unit circle $\mathbb{T} = \partial\mathbb{D} = \{|z| = 1\}$. The complement of these arcs is a closed set $E \subset \mathbb{T}$. The set E has zero logarithmic capacity by Corollary 4.5. In particular, E has zero length and can't contain an interval.

The partition $\mathscr{J}_j$ of γ_j transfers, via Ψ to a partition of $I_j \subset \mathbb{T}$ into infinitely many intervals $\{J_k^j\}, k \in \mathbb{N}$. We will let $\mathscr{K} = \cup_{j,k} J_k^j$ denote the collection of all intervals that occur this way. Thus $\mathbb{T} = E \bigcup \cup_{K \in \mathscr{K}} K$.

Because Ψ conformally extends from W to W_2, $|\Psi'|$ has comparable minimum and maximum on each partition element of γ_j (with uniform constants). Thus the corresponding intervals $\{J_k^j\}$ have the property that adjacent intervals have comparable lengths (again with a uniform bound).

Recall from Chapter 1.23 that the hyperbolic distance between two points $z_1, z_2 \in \mathbb{D}$ is defined as

$$\rho(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1 - |z|^2}.$$

Also recall that hyperbolic geodesics are circular arcs in $\mathbb{D}$ that are perpendicular to $\mathbb{T}$, and that points hyperbolic distance r from 0 are Euclidean distance

$$\frac{2}{\exp(2r) + 1} = O(\exp(-r)),$$

from the unit circle.

For any proper sub-interval $I \subset \mathbb{T}$, let γ_I be the hyperbolic geodesic with the same endpoints as I and let a_I be the point on γ_i that is closest to the origin (closest in either the Euclidean or hyperbolic metrics; it is the same point).

Since $\mathscr{K}$ are disjoint intervals on the circle,

$$\sum_{K \in \mathscr{K}} (1 - |a_K|) < \infty,$$

and so

$$B(z) = \prod_{\mathscr{K}} \frac{|a_K|}{a_K} \frac{a_K - z}{1 - \overline{a_K} z},$$

defines a convergent Blaschke product (see Theorem II.2.2 of [**?**]). Thus B is a bounded, non-constant, holomorphic function on $\mathbb{D}$ that vanishes exactly on the set $\{a_n\}$. Also, $|B|$ has radial limits 1 almost everywhere. Moreover, B extends meromorphically to $\mathbb{C} \setminus E$, where E is the accumulation set of its zeros on $\mathbb{T}$; this is the same set E as defined above using the map Ψ (the zeros accumulate at both endpoints of every component of $\mathbb{T} \setminus E$, and since these points are dense in E, the accumulation set of the zeros is the whole singular set E). The poles of the extension are precisely the points in the exterior of the unit disk that are the reflections across $\mathbb{T}$ of the zeros.

Any subset $\mathscr{M}$ of $\mathscr{K}$ also defines a convergent Blaschke product. Fix such a subset. The corresponding Blaschke product $B_{\mathscr{M}}$ induces a partition of each I_j with endpoints given by the set $\{e^{i\theta} : B_{\mathscr{M}}(e^{i\theta}) = 1\}$ and this induces a partition $\mathscr{H}_j$ of each γ_j via the map Ψ. This in turn, induces a partition $\mathscr{L}_j$ of L_1 via τ_j.

We would like to say that the partitions $\mathscr{L}_j$ and $\mathscr{J}$ are "almost the same". The first step to making this precise is a lemma that we will prove in Section **??**:

LEMMA 3.1. *There is a subset $\mathscr{M} \subset \mathscr{K}$ so that if B is the Blaschke product corresponding to $\mathscr{M}$ and $\mathscr{L}_j$ is the partition of L_1 corresponding to B via $\tau_j \circ \Psi^{-1}$, then each element of $\mathscr{J}$ hits at least 2 elements of $\mathscr{L}_j$ and at most M elements of $\mathscr{L}_j$, where M is uniform. In particular, no element of $\mathscr{J}$ can hit both endpoints of any element of $\mathscr{L}_j$ (elements of each partition are considered as closed intervals).*

In Section **??** we will prove

LEMMA 3.2. *Suppose $K = [1+ia, 1+ib] \in \mathscr{L}_j$ and define*

$$\alpha(1+iy) = \frac{1}{2\pi} \arg(B \circ \Psi \circ \tau_j^{-1}(1+iy)),$$

where we choose a branch of α so $\alpha(1+ia) = 0$ (recall that $B(\Psi(\tau_j^{-1}(1+ia))) = 1 \in \mathbb{R}$). Set

$$\psi_1(z) = 1 + i(a(1-\alpha(z)) + b\alpha(z)) = 1 + i(a + (b-a)\alpha(z)).$$

Then ψ_1 is a homeomorphism from K to itself so that $\alpha \circ \psi_1^{-1} : K \to [0,1]$ is linear and ψ_1 can be extended to a quasiconformal homeomorphism of $R = K \times [1,2]$ to itself that is the identity on the $\partial R \setminus K$ (i.e., it fixes points on the top, bottom and right side of R).

The main point of the proof is to show that $\arg(B \circ \Psi \circ \tau_j^{-1}) : K \to [0, 2\pi]$ is biLipschitz with uniform bounds.

Roughly, Lemma **??** says there are more elements of $\mathscr{J}$ than there are of $\mathscr{L}_j$. This is made a little more precise by the following:

LEMMA 3.3. *There is a 1-to-1, order preserving map of $\mathscr{L}_j$ into (but not necessarily onto) $\mathscr{J}$ so that each interval $K \in \mathscr{L}_j$ is sent to an interval J with $\operatorname{dist}(K, J) \leq 2\pi$. Moreover, adjacent elements of $\mathscr{L}_j$ map to elements of $\mathscr{J}$ that are either adjacent or are separated by an even number of elements of $\mathscr{J}$.*

This will be proven in Section **??**. Again, the proof is quite elementary.

Partition $\mathscr{J} = \mathscr{J}_1^j \cup \mathscr{J}_2^j$ according to whether the interval is associated to some element of $\mathscr{L}_j$ by Lemma **??** (i.e., $\mathscr{J}_1^j$ is the image of $\mathscr{L}_j$ under the map in the lemma). The maximal chains of adjacent elements of $\mathscr{J}_2^j$ will be called blocks. By the lemma, each block has an even number of elements. We will say that the block associated to an element $J \in \mathscr{J}_1^j$ is the block immediately above J.

Thus each interval K in $\mathscr{L}_j$ is associated to an interval J' that consists of the corresponding J given by Lemma **??** and its associated block. K and J' have comparable lengths and are close to each other, so the orientation preserving linear map from J' to K defines a piecewise linear map $\tilde{\psi}_2 : \mathbb{R} \to \mathbb{R}$ that is biLipschitz with a uniform constant. Using linear interpolation we can extend this to a biLipschitz map ψ_2 of the strip $S = \{x + iy : 1 < x < 2\}$ to itself that equals $\tilde{\psi}_2$ on L_1 (the left boundary) and is the identity on L_2 (the right side).

Each element $J \in \mathscr{J}_2^j$ is paired with a distinct element $J^* \in \mathscr{J}_2^j$ that belongs to the same block. The two outer-most elements of the block are paired, as are the pair adjacent to these, and so on. Similarly, each point z is paired with the other point z^* in the block that has the same distance to the boundary (the center of the block is an endpoint of $\mathscr{J}$ and is paired with itself).

For each $K \in \mathscr{L}_j$, let J_K be the corresponding element of $\mathscr{J}_1^j$ and let I_K be the union of J_K and its corresponding block. Let $R_K = [1,2] \times I_K$. Let $U_K = R_K \setminus X_K$, where X_K is the closed segment connecting the upper left corner of R_K to the center of R_K. See Figure **??**.

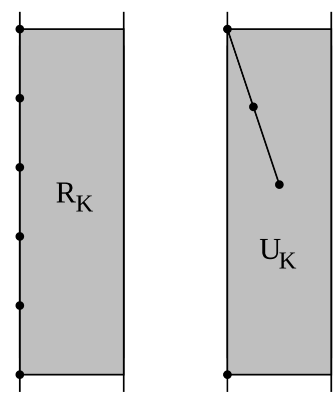


FIGURE 3.2. Definition of U_K

LEMMA 3.4 (Simple folding). *There is a quasiconformal map $\psi_3 : U_K \to R_K$ so that (ψ_3 depends on j and on K, but we drop these parameters from the notation)*

(1) *ψ_3 is the identity on $\partial R_K \setminus L_1$ (i.e., it is the identity on the the top, bottom and right side of R_K),*

(2) ψ_3^{-1} *extends continuously to the boundary and is linear on each element of* $\mathscr{J}$ *lying in* I_K*,*
(3) ψ_3 *maps* I_K *(linearly) to* J_K*,*
(4) *for each* $z \in I_K$*,* $\psi_3^{-1}(z) = \psi_3^{-1}(z^*) \in X_k$ *(i.e.,* ψ_3 *maps opposite sides of* X_k *to paired points in* I_k*),*
(5) *the quasiconstant of* ψ_3 *depends only on* $|I_K|/|J_K|$*, i.e., on the number of elements in the block associated to K. It is independent of the original model and of the choice of j and K.*

We call this "simple folding" because it is a simple analog of a more complicated folding procedure given in [**?**]. In the lemma above, the image domain is a rectangle with a slit removed and the quasiconstant of ψ_3 is allowed to grow with n, the number of block elements. This growth is not important in this paper because here we only apply the folding construction in cases where this number n is uniformly bounded (this will occur in our application because of Lemma **??**). In [**?**], the corresponding values may be arbitrarily large but the folding construction there must give a map with uniformly bounded quasiconstant regardless. The construction in [**?**] removes a collection of finite trees from R_k and does so in a way that keeps the quasiconstant of ψ_3 bounded independent of n (there are also complications involving how the construction on adjacent rectangles are merged).

We want to treat the boundary intervals in $\mathscr{J}_1$ and $\mathscr{J}_2$ slightly differently. The precise mechanism for doing this is:

LEMMA 3.5 (exp-cosh interpolation). *There is a quasiregular map* $\sigma_j : S \to D(0,e^2)$ *so that*

$$\sigma_j(z) = \begin{cases} \exp(z), & z \in J \in \mathscr{J}_1^j, \\ e \cdot \cosh(z-1), & z \in J \in \mathscr{J}_2^j, \\ \exp(z), & z \in \mathbb{H}_r + 2. \end{cases}$$

The quasiconstant of ϕ_j *is uniformly bounded, independent of all our choices.*

This lemma will be proven in Section **??** and is completely elementary.

We now have all the individual pieces needed to construct the interpolation g_j between e^z on L_2 and $B \circ \Psi \circ \tau_j^{-1}$ on L_1. Let U_j be S minus all the segments X_K where $K \in \mathscr{L}_j$ as in Lemma **??**. Define a quasiconformal map $\psi : U_j \to S$ by

$$\psi = \psi_1 \circ \psi_2 \circ \psi_3,$$

and let $g_j = \sigma_j \circ \psi$ map U_j into $D(0,e^2)$. By definition, each ψ_i, $i = 1,2,3$ is the identity on L_2, so $g_j(z) = e^z$ on L_2. For any $K \in \mathscr{L}_j$, the map ψ sends the boundary segments of ∂U_K that lie on some X_K linearly onto elements of $\mathscr{J}_2^j$, so boundary points on opposite sides of X_K get mapped to points that are equidistant from $2\pi i\mathbb{N}$ and cosh agrees at any two such points. Thus g_j extends continuously across each slit X_K. Finally, the map ψ was designed so that g_j is continuous on S and agrees with $B\circ\Psi\circ\tau_j^{-1}$ on L_1. Thus $g_j\circ\tau_j$ continuously interpolates between $B\circ\Psi$ on W and F on $\Omega(2)$ and so defines a quasiregular g on the whole plane with a uniformly bounded constant. Thus by the measurable Riemann mapping theorem there is a quasiconformal $\varphi:\mathbb{C}\to\mathbb{C}$ so that $f=g\circ\varphi$ is entire.

The singular values of f are the same as for g. On $\Omega(2)$, $g=F=e^{\tau}$, so g has no critical points in this region. In U_j, $g=g_j$ is locally 1-to-1, so has no critical points there either. Thus the only critical points of g in $\Omega(1)$ are on the slits X_K, then these are mapped by g onto the circle of radius e around the origin. Thus every critical value of g (and hence f) must lie in $D(0,e)$.

If g has a finite asymptotic value outside $\overline{D(0,e)}$, then it must be the limit of g along some curve Γ contained in a single component of Ω. Then e^z has a finite limit along $\tau(\Gamma)\subset\mathbb{H}_r$; this is impossible, so f has no finite asymptotic values outside $\overline{D(0,e)}$. Thus $S(f)\subset\overline{D(0,e)}$, and so $f\in\mathscr{B}$.

This proves Theorem **??** except for the proof of the lemmas.

4. Blaschke partitions

In this section we prove Lemma **??**. We start by recalling some basic properties of the Poisson kernel and harmonic measure in the unit disk $\mathbb{D}$.

The Poisson kernel on the unit circle with respect to the point $a\in\mathbb{D}$ is given by the formula

$$P_a(\theta)=\frac{1-|a|^2}{|e^{i\theta}-a^2|}=\frac{1-|a|^2}{1-2|a|\cos(\theta-\phi)+|a|^2},$$

where $a=|a|e^{i\phi}$. This is the same as $|\sigma'|$ where σ is any Möbius transformation of the disk to itself that sends a to zero. If $E\subset\mathbb{T}$, we write

$$\omega(E,a,\mathbb{D})=\frac{1}{2\pi}\int_E P_a(e^{i\theta})d\theta,$$

and call this the harmonic measure of E with respect to a. This is the same as the (normalized) Lebesgue measure of $\sigma(E)\subset\mathbb{T}$ where $\sigma:\mathbb{D}\to\mathbb{D}$ is any Möbius transformation sending a to 0. It is also the same as the first hitting distribution on $\mathbb{T}$ of a Brownian motion started at a (although we will not use this characterization).

Suppose $I\subset\mathbb{T}$ is any proper arc, and, as before, let γ_I be the hyperbolic geodesic in $\mathbb{D}$ with the same endpoints as I; then γ_I is a circular arc in $\mathbb{D}$ that is perpendicular to $\mathbb{T}$ at its endpoints. Let a_I denote the point of γ_I that is closest to the origin.

LEMMA 4.1. $\omega(I, a_I, \mathbb{D}) = \frac{1}{2}$.

PROOF. Apply a Möbius transformation of $\mathbb{D}$ that sends a_I to the origin. Then γ_I must map to a diameter of the disk and I maps to a semi-circle. $\square$

Given two disjoint arcs I, J in $\mathbb{T}$, let γ_I, γ_J be the two corresponding hyperbolic geodesics and let a_I^J be the point on γ_I that is closest to J and let a_J^I be the point on γ_J that is closest to I.

LEMMA 4.2. $\omega(I, a_J^I, \mathbb{D}) = \omega(J, a_I^J, \mathbb{D})$

PROOF. Everything is invariant under Möbius maps of the unit disk to itself, so use such a map to send I, J to antipodal arcs. Then the conclusion is obvious. $\square$

LEMMA 4.3. *If $z, w \in \mathbb{D}$ and $I \subset \mathbb{T}$, then*

$$\frac{\omega(I, z, \mathbb{D})}{\omega(I, w, \mathbb{D})} \leq C$$

where the constant C depends only on the hyperbolic distance between z and w.

PROOF. Suppose $\sigma(z) = (z - w)/(1 - \overline{w}z)$ maps w to 0. Then $u(z) = \omega(I, \sigma(z), \mathbb{D})$ is a positive harmonic function on $\mathbb{D}$, so the lemma is just Harnack's inequality applied to u. $\square$

Suppose $I, J, \subset \mathbb{T}$ are disjoint closed arcs and $\text{dist}(I, J) \geq \varepsilon \max(|I|, |J|)$. Then we call I and J ε-separated. This implies the hyperbolic geodesics γ_I, γ_J are separated in the hyperbolic metric (with a lower bounded depending only on ε), but the converse is not true.

LEMMA 4.4. *If $I, J \subset \mathbb{T}$ are ε-separated, then the hyperbolic distance between a_I and a_I^J is bounded, depending only on ε.*

PROOF. Assume I is the longer arc and consider hyperbolic geodesic S that connects a_I^J and a_J^I. Then S is perpendicular to γ_I at a_I^J, so if $1 - |a_I^J| \ll 1 - |a_I|$, S will hit the unit circle without hitting γ_j. See Figure **??**. $\square$

LEMMA 4.5. *Suppose that I, J are ε-separated. Then*

$$\omega(I, a_J, \mathbb{D}) \simeq \omega(J, a_I, \mathbb{D}),$$

where the constant depends only on ε.

PROOF. This follows immediately from our earlier results. $\square$

LEMMA 4.6. *Suppose that I and J are ε-separated and that a_J, a_I are at least distance R apart in the hyperbolic metric. Then*

$$\omega(J, a_I, \mathbb{D}) \leq C(\varepsilon) e^{-R}.$$

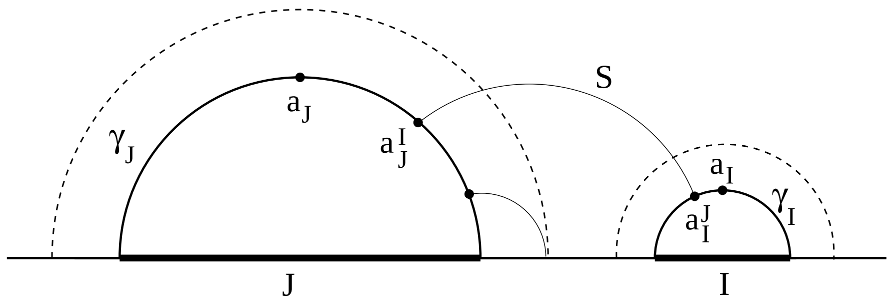


FIGURE 4.1. If the intervals I and J are ε-separated, then a shortest path between γ_I and γ_J must hit each geodesic near the "top" points. A perpendicular geodesic that starts too "low" on γ_J will hit the unit circle without hitting γ_I.

PROOF. Since the intervals are ε-separated, the hyperbolic distance between a_I and a_J is the same as the distance between a_I^J and a_J^I, up to a bounded additive factor. Thus if we apply a Möbius transformation of $\mathbb{D}$ so that $a_J = 0$, a_I is mapped to a point w with $1-|w| = O(e^{-R})$, which implies $\omega(I, a_J, \mathbb{D}) = O(e^{-R})$. Since the intervals are ε-separated, the reverse inequality also holds by Lemma **??**. $\square$

Fix $M < \infty$ and suppose $\mathscr{K}$ is a collection of disjoint (except possibly for endpoints) closed intervals on $\mathbb{T}$ so that any two adjacent intervals have length ratio at most M. We say that two intervals I, J are S steps apart if there is a chain of $S+1$ adjacent intervals $J_0, \dots, J_S$ so that $I = J_0$ and $J = J_S$.

Note that if $I, J \in \mathscr{K}$ are adjacent, then a_I, a_J are at bounded hyperbolic distance T apart (and T depends only on M). Also, if $I, J \in \mathscr{K}$ are not adjacent, then they are ε-separated for some $\varepsilon > 0$ that depends only on M.

LEMMA 4.7. *For any $R > 0$ there is a collection $\mathscr{N} \subset \mathscr{K}$ so that*

(1) *for any $I \in \mathscr{K}$, there is a $J \in \mathscr{N}$ with $\rho(a_J, a_I) \le R$*
(2) *for any $I, J \in \mathscr{N}$, $\rho(a_J, a_I) \ge R$.*

PROOF. Just let $\mathscr{N}$ correspond to a maximal collection of the points $\{a_K\}$ with the property that any two of them are hyperbolic distance $\ge R$ apart. $\square$

Fix a positive integer S. For each $J \in \mathscr{N}$ choose the shortest element of $\mathscr{K}$ that is at most S steps away from J. Let $\mathscr{M} \subset \mathscr{K}$ be the corresponding collection of intervals.

LEMMA 4.8. *Suppose R, S, T are as above and $R \ge 4ST$. If $\mathscr{K}$ and $\mathscr{M}$ are as above, then for all $K \in \mathscr{K}$,*

$$\varepsilon \le \sum_{J \in \mathscr{M}} \omega(K, a_J, \mathbb{D}) \le \mu,$$

where $\varepsilon > 0$ depends only on R and $\mu \to 1/2$ as $S \to \infty$.

PROOF. The left-hand inequality is easier and we do it first. Fix $K \in \mathscr{K}$. There is a $I \in \mathscr{N}$ with $\rho(a_I, a_K) \le R$, and since adjacent elements of $\mathscr{K}$ have points that are only T apart in the hyperbolic metric, there is an element $J \in \mathscr{M}$ with $\rho(a_K, a_J) \le R + ST \le \frac{5}{4}R$. This implies $|J| \simeq |K| \simeq \mathrm{dist}(J,K)$ and these imply $\omega(K, a_J, \mathbb{D}) \ge \varepsilon$ with ε depending only on ρ. Thus every element of $\mathscr{K}$ has harmonic measure bounded below with respect to some point corresponding to a single element of $\mathscr{M}$ and hence the sum of harmonic measures over all elements of $\mathscr{M}$ is also bounded away from zero uniformly.

Now we prove the right-hand inequality. By our choice of R, points a_J corresponding to distinct intervals in $\mathscr{M}$ are at least distance $R/2$ apart. Fix $K \in \mathscr{K}$. There is at most one point within hyperbolic distance $R/4$ of a_K and the harmonic measure it assigns K is at most $1/2$ since the point lies on or outside the geodesic γ_K.

All other points associated to elements of $\mathscr{M}$ are Euclidean distance $\ge \exp(R/8)|K|$ away from K or are within this distance of K, and are within Euclidean distance $\exp(-R/8)|K|$ of the unit circle (this is because of the Euclidean geometry of hyperbolic balls in the half-space). We call these two disjoint sets $\mathscr{M}_1$ and $\mathscr{M}_2$ respectively.

Using Lemma **??** we see that the

$$\sum_{J \in \mathscr{M}_1} \omega(K, a_J, \mathbb{D}) = O\Big(\sum_{J \in \mathscr{M}_1} \omega(J, a_K, \mathbb{D})\Big) = O(\exp(-R/8)).$$

To bound the sum over $\mathscr{M}_2$, we note that each interval in $\mathscr{M}_2$, is the endpoint of a chain of S adjacent intervals that are each at least as long as J. Since

$$|J| \le \exp(-R/8)|K|,$$

and

$$\mathrm{dist}(J,K) \gtrsim |K|,$$

we can deduce

$$\omega(J, a_K, \mathbb{D}) \le O\Big(\frac{1}{S}\Big)\omega(a_K, J, \mathbb{D}),$$

so since the J's are all disjoint intervals,

$$\sum_{J \in \mathscr{M}_2} \omega(K, a_J, \mathbb{D}) = O\Big(\frac{1}{S}\sum_{J \in \mathscr{M}_2} \omega(J, a_K, \mathbb{D})\Big) = O\Big(\frac{1}{S}\Big).$$

Choosing first S large, and then R large (depending on S and separation constant of $\mathscr{K}$), both sums are as small as we wish, which proves the lemma. $\square$

COROLLARY 4.9. *Suppose B is as above and $K \in \mathscr{K}$. Then*

$$\varepsilon \leq \frac{1}{|K|}\frac{\partial B}{\partial \theta} \leq C.$$

PROOF. If I,J are ε-separated, then it is easy to verify that

$$\sup_{z\in J} P_{a_I}(z), \qquad \inf_{z\in J} P_{a_I}(z),$$

are comparable up to a bounded multiplicative factor that depends only on ε. The lemma then follows from our earlier estimates. □

We have now essentially proven Lemma **??**; it just remains to reinterpret the terminology a little. For the reader's convenience we restate the lemma.

LEMMA 4.10 (The Blaschke partition). *There is a subset $\mathscr{M} \subset \mathscr{K}$ so that if B is the Blaschke product corresponding to $\mathscr{M}$ and $\mathscr{L}_j$ is the partition of L_1 corresponding to B via $\tau_j \circ \Psi^{-1}$, then each element of $\mathscr{J}$ hits at least 2 elements of $\mathscr{L}_j$ and at most M elements of $\mathscr{L}_j$, where M is uniform. In particular, no element of $\mathscr{J}$ can hit both endpoints of any element of $\mathscr{L}_j$ (elements of each partition are considered as closed intervals).*

PROOF. A computation shows that for the Blaschke product

$$B(z) = \prod_n \frac{|a_n|}{a_n}\frac{z-a_n}{1-\bar{a}_n z},$$

the derivative satisfies

$$|\frac{\partial B}{\partial \theta}(e^{i\theta})| = \sum_n P_{a_n}(e^{i\theta}),$$

and the convergence is absolute and uniform on any compact set K disjoint from the singular set E of B (since B is a product of Möbius transformations, and the derivative of a Möbius transformation is a Poisson kernel, this formula is simply the limit of the n-term product formula for derivatives).

Lemma **??** now says we can choose $\mathscr{M}$ so that

$$2\pi\varepsilon \leq \int_J |\frac{\partial}{\partial \theta} B| d\theta \leq \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2}.$$

Since the integral over an element of $\mathscr{L}$ has integral exactly 2π, the lower bound means that an element of $\mathscr{L}$ can contain at most $1/\varepsilon$ elements of $\mathscr{J}$ and hence can intersect at most $2+\frac{1}{\varepsilon}$ elements of $\mathscr{J}$. The upper bound says that each element K of $\mathscr{L}$ must hit at least 2 elements of $\mathscr{J}$. Hence it is not contained in any single element of $\mathscr{J}$, and so no single element of $\mathscr{J}$ can hit both endpoints of K. □

5. Straightening a biLipschitz map

LEMMA 5.1. *Suppose $K = [1+ia, 1+ib] \in \mathscr{L}_j$ and define*

$$\alpha(1+iy) = \frac{1}{2\pi}\arg(B \circ \Psi \circ \tau_j^{-1}(1+iy)),$$

where we choose a branch of α so $\alpha(1+ia) = 0$ (recall that $B(\Psi(\tau_j^{-1}(1+ia))) = 1 \in \mathbb{R}$). Set

$$\psi_1(z) = 1 + i(a(1-\alpha(z)) + b\alpha(z)) = 1 + i(a + (b-a)\alpha(z)).$$

Then ψ_1 is a homeomorphism from K to itself so that $\alpha \circ \psi_1^{-1} : K \to [0,1]$ is linear and ψ_1 can be extended to a quasiconformal homeomorphism of $R = K \times [1,2]$ to itself that is the identity on the $\partial R \setminus K$ (i.e., it fixes points on the top, bottom and right side of R).

PROOF. The linearizing property of ψ_1 is clear from its definition, so we need only verify the quasiconformal extensions property.

Corollary **??** implies α' is bounded above and below by absolute constants. Let $R = K \times [1,2]$ and define an extension of ψ_1 by

$$\psi_1(x+iy) = u(x,y) + iv(x,y) = x + i[(2-x)\psi_1(1+iy) + (x-1)y)].$$

i.e., take the linear interpolation between ψ_1 on L_1 and the identity on L_2. We can easily compute

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y - \psi(y) & (2-x)(b-a)\alpha'(y) + (x-1) \end{pmatrix}.$$

Note that $|y - h(y)| \le |K|$ is absolutely bounded. Also, since $|b-a||\alpha'|$ is bounded above and away from 0, so is v_y. Thus the derivative matrix lies in a compact subset of the invertible 2×2 matrices and hence ψ_1 is quasiconformal (with only a little more work we could compute an explicit bound for the quasiconstant, and even prove that the extension is actually biLipschitz). $\square$

6. Aligning partitions

Now we prove Lemma **??**, which we restate for convenience.

LEMMA 6.1. *There is a* 1*-to-*1*, order preserving map of $\mathscr{L}_j$ into (but not necessarily onto) $\mathscr{J}$ so that each interval $K \in \mathscr{L}_j$ is sent to an interval J with* $\mathrm{dist}(K,J) \le 2\pi$*. Moreover, adjacent elements of $\mathscr{L}_j$ map to elements of $\mathscr{J}$ that are either adjacent or are separated by an even number of elements of $\mathscr{J}$.*

PROOF. For each $K \in \mathscr{K}$ choose $J \in \mathscr{J}$ so that J contains the lower endpoint of K (if two such intervals contain the endpoint, choose the upper one). No interval J is chosen twice, since Lemma **??** says that no J can hit both endpoints of any element of $\mathscr{L}$.

Fix an order preserving labeling of the chosen $\mathscr{J}$ by $\mathbb{N}$ and denote it $\{J_n\}$. By the gap between J_n and J_{n+1} we mean the number of unselected elements of $\mathscr{J}$ that separate these two intervals. The position of J_0 is fixed. If the gap between J_0 and J_1 is even (including no gap), we leave J_1 where it is. If the gap is odd, there is a least one separating interval and we replace J_1 by the adjacent interval in $\mathscr{J}$ that is closer to J_0. If the gap between (the new) J_1 and J_2 is even, we leave J_2 alone; otherwise, we move it one interval closer to J_0. Continuing in this way, we can guarantee that for all $n \geq 0$, gaps are even and each J_n is either in its original position or adjacent to its original position. Thus its distance to the associated element of $\mathscr{K}$ is at most 2π. The argument for negative indices is identical. $\square$

7. Simple Foldings

Now we prove Lemma **??**. This is the step that makes the gluing procedure a little different from a standard quasiconformal surgery.

LEMMA 7.1 (Simple folding). *There is a quasiconformal map $\psi_3 : U_K \to R_K$ so that (ψ_3 depends on j and on K, but we drop these parameters from the notation)*

1. *ψ_3 is the identity on $\partial R_K \setminus L_1$ (i.e., it is the identity on the the top, bottom and right side of R_K),*
2. *ψ_3^{-1} extends continuously to the boundary and is linear on each element of $\mathscr{J}$ lying in I_K,*
3. *ψ_3 maps I_K (linearly) to J_K,*
4. *for each $z \in I_K$, $\psi_3^{-1}(z) = \psi_3^{-1}(z^*) \in X_k$ (i.e., ψ_3 maps opposite sides of X_k to paired points in I_k),*
5. *the quasiconstant of ψ_3 depends only on $|I_K|/|J_K|$, i.e., on the number of elements in the block associated to K. It is independent of the original model and of the choice of j and K.*

PROOF. The proof is a picture, namely Figure **??**. The map is defined by giving compatible finite triangulations of R_k and U_k (compatible means that there is 1-to-1 map between vertices of the triangulations that preserves adjacencies along edges). Such a map defines linear maps between corresponding triangles that are continuous across edges. Since each such map is non-degenerate, it is quasiconformal and hence the piecewise linear map defined between U_k and R_K is quasiconformal (with quasiconstant given by the worst quasiconstant of the finitely many triangles). The other properties are evident. □

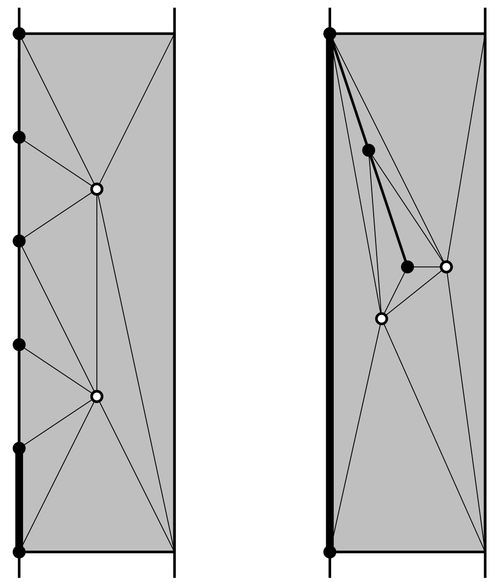


FIGURE 7.1. The pictorial proof of Lemma **??** for $n = 5$.

8. Interpolating between exp and cosh

LEMMA 8.1 (exp-cosh interpolation). *There is a quasiregular map $\sigma_j : S \to D(0, e^2)$ so that*

$$\sigma_j(z) = \begin{cases} \exp(z), & z \in J \in \mathscr{J}_1^j, \\ e \cdot \cosh(z-1), & z \in J \in \mathscr{J}_2^j, \\ \exp(z), & z \in \mathbb{H}_r + 2. \end{cases}$$

The quasiconstant of σ_j is uniformly bounded, independent of all our choices.

PROOF. As with the previous lemma, the proof is basically a picture; see Figure **??**. Suppose $J \in \mathscr{J}$ and let $R = J \times [1,2]$. The exponential map sends R to the annulus $A = \{e < |z| < e^2\}$, with the left side of R mapping to the inner circle and the top and bottom edges of R mapping to the real segment $[e, e^2]$.

Now define a quasiconformal map $\phi : A \to D(0, e^2)$ that is the identity on $\{|z| = e^2\}$ and on $[e, e^2]$, but that maps $\{|z| = e\}$ onto $[-e, e]$ by $z \to \frac{1}{2}(z + \frac{e^2}{z})$ (this is just a rescaled version of the Joukowsky map $\frac{1}{2}(z + \frac{1}{z})$ that maps the unit circle to $[-1, 1]$, identifying complex conjugate points).

In $\mathbb{H}_r + 2$ and in rectangles of the form $J \times [1,2]$ for $J \in \mathscr{J}_1$ we set $\sigma_j(z) = \exp(z)$. In the rectangles corresponding to elements of $\mathscr{J}_2$ we let $\sigma_j(z) = \phi(\exp(z))$. This clearly has the desired properties. $\square$

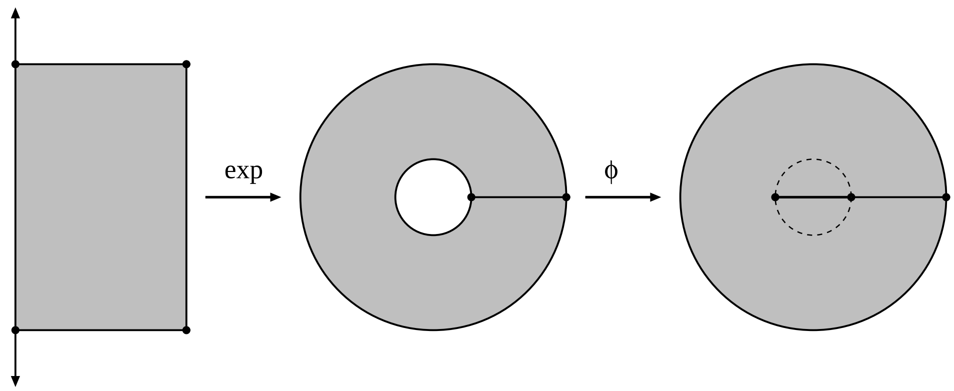


FIGURE 8.1. The exponential function maps the rectangle $[1,2] \times J$ conformally to the slit annulus $\{e < |z| < e^2\} \setminus [e, e^2]$. The map ϕ is chosen to map the annulus A=$\{e < |z| < e^2\}$ to the slit disk $\{|z| < e^2\} \setminus [-e, e]$ so that it equals the identity on $\{|z| = e^2\}$ and equals $\frac{1}{2}(z + \frac{e^2}{z})$ on $\{|z| = e\}$.

Actually, the cosh function in the lemma can be replaced by any function $h : J \to [-1,1]$ that has the property that $h(z)$ only depends on the distance from z to the endpoint of J. This will ensure that after applying a folding map, points that started on opposite sides of some slit X_k will end up being identified by h, which is all we need.

This completes the proof of Theorem **??**.

CHAPTER 6

Examples in the Eremenko-Lyubich class

CHAPTER 7

Wandering domains

1. Non-wandering in the Speiser class

In 1885 Dennis Sullivan proved that a rational map has no wandering domains, a famous open problem dating back to the origins of the subject. Two sets of researchers, Alex Eremenko and Misha Lyubich [**?**], and Lisa Goldberg and Linda Keen [**?**], soon generalized Sullivan's proof from rational functions to the Speiser class:

THEOREM 1.1. *If f is an entire function that has a finite singular set, then f has no wandering domains.*

In this section, we will give a proof of this result, first for polynomials and then for the Speiser class (and with minor modifications, the proof would also cover the original case of rational functions).

The main idea is fairly easy to state. If there were a wandering domain U for f, then any dilatation μ on U could be extended to a dilatation on the grand orbit of U so that the corresponding quasiconformal mapping has the property that $g = h \circ f \circ h^{-1}$ is also entire. This gives a continuous map from dilatations on U to entire functions that are quasiconformally conjugate to f, a finite dimensional space by Theorem **??**. By an explicit construction, we can choose a subspace of dilatations with larger dimension on which the map must be 1-to-1 and this violates Brouwer's invariance of domain theorem (you can't map an open subset of $\mathbb{R}^n$ continuously and 1-1 into $\mathbb{R}^{n-1}$). You can avoid the use of Brouwer's theorem by proving that the map from dilatations to entire functions is continuously differentiable and then using the rank theorem instead.

t is interesting to note that the proof is mostly topological; the main steps will be presented as lemmas that are applications of the following facts:

(1) the lifting lemma for covering spaces,
(2) the Jordan curve theorem,
(3) continuous images of connected sets are connected,
(4) Brouwer's invariance of domain theorem.

The last is the only one that is not simple enough to be included in a first course on topology, e.g., as in Munkres' book [**?**]. It states that a continuous map f from an open set in $\mathbb{R}^{n+1}$ into $\mathbb{R}^n$ cannot be 1-to-1. In fact, we will use a strengthening of this that there must be a point z in the image so that $f^{-1}(z)$ contains a non-trivial connected subset. Since this material is not standard, we include a brief description of the relevant definitions and results in Appendix **??**.

This result from topological dimension theory can be replaced by a more standard result, such as the rank theorem, that also proves the existence of a non-trivial, connected preimage under the additional assumption that the map f is continuously differentiable. It is a standard fact (e.g., see Ahlfors' book [**?**]) that quasiconformal maps depend analytically on the dilatation and from this we can deduce the necessary smoothness to apply the rank theorem. See, e.g., the presentations of Sullivan's theorem in [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]. We avoid the use of differentiability here only to offer an alternate approach which trades topological technicalities for analytic ones, but seems closer (at least to the author) to the heart of the matter.

We say that two entire functions f,g are **topologically equivalent** if there are homeomorphisms $\varphi,\psi:\mathbb{C}\to\mathbb{C}$ such that

$$\psi\circ g = f\circ\varphi.$$

The maps are **quasiconformally conjugate** if ψ,φ can be taken to be quasiconformal homeomorphisms.

We say f and g are **topologically conjugate** if there is a homeomorphism $\varphi:\mathbb{C}\to\mathbb{C}$ such $\varphi\circ g = f\circ\varphi$. and call the maps **quasiconformally conjugate** if φ can be taken to be quasiconformal. Note that this a stronger condition than equivalence and if f and g are conjugate then

$$f^n = (\varphi\circ g\circ\varphi^{-1})^n = \varphi\circ g^n\circ\varphi^{-1},$$

so that the dynamical behavior of f and g is essentially identical. Obviously the identity map conjugates a map f to itself. The next result says that in some situations, this is the only possible such conjugation.

LEMMA 1.2. *Suppose $\{\varphi_t\}$ is a family of quasiconformal maps on $\mathbb{C}$ so that $\varphi_t(z):X\to\mathbb{C}$ is continuous each fixed z as a function of $t\in X$, X a connected space. Suppose that φ_{t_0} is the identity for some $t_o\in X$. Suppose that $f\in\mathscr{S}$ has the property that $\varphi_t\circ f = f\circ\varphi_t$ for all $t\in X$. Then $\varphi_t(z)=z$ for all $t\in X$ and all $z\in\mathscr{J}(f)$, i.e., every φ_t is the identity when restricted to the Julia set of f.*

PROOF. Because φ_t conjugates the action of f to itself, periodic points are mapped to periodic points with the same period. Since there only countable many such points, they form a discrete set and so $\{\varphi_t(z): t\in X\}$ must be a single point, since X is connected. Since one of these maps is the identity, every map must fix every periodic point. Finally, since periodic points are dense in the Julia set (Theorem **??**), and quasiconformal maps are continuous, each map φ_t must fix every point in $\mathscr{J}(f)$. $\square$

LEMMA 1.3. *If $f : \mathbb{D} \to \Omega \subset \mathbb{C}$ is conformal and $\varphi : \Omega \to \Omega$ is a quasiconformal map that extends continuously to the identity on $\partial\Omega$, then $\Phi = f^{-1} \circ \varphi \circ f$ is a quasiconformal map of the disk to itself that extends to the identity on $\partial\mathbb{D}$.*

PROOF. Clearly $\Phi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and hence extends continuously to a homeomorphism of the unit circle (see Theorem **??**). If the extension of Φ to $\partial\mathbb{D}$ is not the identity, then there is an arc $I \subset \mathbb{T}$ such that $I \cap \Phi(I) = \emptyset$. Choose a point $w \in I$ so that f has a finite radial limit at both z and $\Phi(z)$; we can do this because (1) conformal maps have finite radial limits except on a set of zero capacity (Corollary 4.5), and (2) sets of zero capacity map to zero capacity under quasiconformal maps (immediate from Pfluger's theorem).

Take the union of the two radial line segments $[0,w]$ and $[0,\Phi(w)]$. Because φ extends as the identity to $\partial\Omega$, the images of these radial segments under f have the same endpoint on $\partial\Omega$ and hence their union is a a closed Jordan curve γ_w. Now, choose a distinct point $z \in I$ with the same properties and form the closed Jordan curve γ_z. Choose z so that the intersection of γ_z with $\partial\Omega$ is different that the intersection of γ_w with $\partial\Omega$; we can do this because only a set of logarithmic capacity zero on the circle can have the same radial limit. Then $\gamma_z \cap \gamma_w = f(0)$ and γ_z hits both sides of γ_w (since z and $\Phi(z)$ are in different components of $\mathbb{T} \setminus ([0,w] \cup [0,\Phi(w)])$. See Figure **??**. This contradicts the Jordan curve theorem, and thus Φ must extend to the identity on the boundary. $\square$

FIGURE JordanContradiction

LEMMA 1.4. *A wandering domain for a polynomial must be simply connected.*

PROOF. The basin of ∞ is periodic, not wandering, so any wandering domain must be bounded and have a bounded orbit. By the maximum principle, the iterates of f are bounded in the interior of any closed curve in the component and hence form a normal family inside the curve. Thus the curve does not surround any Julia points and the component must be simply connected. $\square$

POLYNOMIALS HAVE NO WANDERING DOMAINS. Choose a smooth, non-negative function h on $\mathbb{C}$ supported in $\mathbb{D}$ with gradient bounded by 1 and such that $h(0) > 0$. Define a family of mappings of the upper half-plane to itself by

$$\Phi_t(z)z + th(z).$$

It is easy to check that these are quasiconformal self-maps of $\mathbb{H}_u$ if we restrict t to a small enough interval $[0,\varepsilon]$ and that Φ_0 is the identity. If $t>0$, then the mapping is definitely not the identity since the cross ratio of the points $-1,0,1,\infty$ changes.

Now choose N disjoint intervals $I_k=\{[2k-3,2k-1]\}_1^N$ and define an N-dimensional family of maps by $\mathbf{t}=(t_1,\dots,t_N)$, and

$$\Phi_{\mathbf{t}}(z)=z+\sum_{k=1}^{N}t_k h(z-(2k-2))t_k h(z-(2k-2)).$$

Suppose Ω were a wandering domain for f. Since f has only finitely many critical values, we can replace Ω, if necessary, by an iterate of itself so that neither it nor any iterate contains a critical point. Therefore we may assume f is univalent on Ω and on all forward orbits.

By Lemma **??** Ω is simply connected, so we can map it conformally by f to $\mathbb{H}_u$ and define a quasiconformal map $\varphi_{\mathbf{t}}=f^{-1}\circ\Phi_{\mathbf{t}}\circ f$. This defines a dilatation μ on Ω that we extend to the grand orbit of Ω using the composition rule for dilatation so that the corresponding quasiconformal map $\Psi_{\mathbf{t}}$ given by the measurable Riemann mapping theorem has the property that

$$g_{\mathbf{t}}=\Psi_{\mathbf{t}}^{-1}\circ f\circ\Psi_{\mathbf{t}},$$

is entire. Doing the extension backwards is always possible; extending to the forward iterates uses the assumption that f and all its iterates are univalent on Ω.

A consequence of Brouwer's invariance of domains theorem is that any continuous map of an open set in $\mathbb{R}^n$ into $\mathbb{R}^k$ for $k<n$ there is a point $z\in\mathbb{R}^k$ whose preimage has topological dimension ≥ 1 and hence contains a connected set X. Choose some $\mathbf{s}\in X$ and consider the maps $\Psi_{\mathbf{t}}\circ\Psi_{\mathbf{s}}^{-1}$. These conjugate f to itself and one of them is the identity, so by Lemma **??**, they are all the identity on $\mathscr{J}(f)$, hence on $\partial\Omega$ and hence the corresponding maps $\Phi_{\mathbf{t}}\circ\Phi_{\mathbf{s}}^{-1}$ are extend to the identity on $\mathbb{R}$. However, this is manifestly false by construction; the boundary maps are not the identity unless $\mathbf{s}=\mathbf{t}$. Therefore there a polynomial has no wandering domains. □

Next we show how the proof given above for polynomials adapts to entire functions with finite singular sets. By Lemma **??** Ω is simply connected. for Eremenko-Lyubich functions and hence Speiser class functions.

The only non-trivial new step is to prove that the collection of entire functions with with a given finite singular set is finite dimensional.

By Lemma **??** Ω is simply connected.

Let M_g denote the collection of all entire functions f that are topologically equivalent to g. An important result of Eremenko and Lyubich [**?**] says that for $g\in\mathscr{S}$, the collection M_g of all f that are topologically equivalent to

g form a finite dimensional, complex analytic manifold. We shall just prove a part of this, showing that M_g is finite dimensional in the following sense.

LEMMA 1.5. *If $f,g \in \mathscr{S}$ have the same singular values then there is an $\varepsilon > 0$ so that the following holds. If*

$$\psi \circ g = f \circ \varphi,$$

where ψ, φ are $(1+\varepsilon)$-quasiconformal, then $g(z) = f(az+b)$ for some $a,b \in \mathbb{C}, a \neq 0$.

PROOF. The proof is essential an exercise about covering spaces, and we will need the following lifting lemma that is Theorem 14.3 of Munkres' book [**?**]:

THEOREM 1.6 (The general lifting lemma). *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to a map $F : Y \to E$ such that $F(y_0) = e_0$ if and only if*

$$f_*(\pi_1(Y,y_0)) \subset p_*(\pi_1(E,e_0)).$$

Here π_1 denotes the fundamental group and f_* is the map between fundamental groups induced by the continuous map f.

In our application, we let $X = \mathbb{C} \setminus S(f) = \mathbb{C} \setminus S(g)$ and let $Y_f = \mathbb{C} \setminus f^{-1}(S(f)), Y_g = \mathbb{C} \setminus g^{-1}(S(g))$. Choose some point $z_0 \in Y_g$. By Lemma **??** $f : Y_f \to X$ and $g : Y_g \to X$ are covering maps.

Since $S(g)$ is a finite set, there is a positive lower bound $\delta > 0$ between any two points in $S(g)$. Since $S(g)$ is bounded, there is an $\varepsilon > 0$ so that any $(1+\varepsilon)$-quasiconformal map fixing $0, 1, \infty$ moves each point of $S(f)$ by less than $\delta/10$. Thus if φ is $(1+\varepsilon)$-quasiconformal, it is isotopic to the identity via a path of quasiconformal maps that fix each point of $S(g)$. Thus for any closed loop γ in Y_g, the image loop $g(\gamma) = \psi^{-1} \circ f \circ \varphi(\gamma)$ is homotopic to $f \circ \varphi(\gamma)$. Thus $g_*(\pi_1(Y_g, z_0)) \subset f_*(\pi_1(Y_f, \varphi(z_0)))$. In fact, we have equality, since $\pi_1(Y_f)$ is isometric to $\pi_1(Y_g)$ via the homeomorphism φ. By the general lifting lemma we get a homeomorphism $h : F : Y_g \to Y_f$ and this map is locally a composition of g and a branch of f^{-1} and hence is holomorphic. Thus it must be conformal linear, i.e., $h(z) = az + b, a \neq 0$, as claimed. □

alues of f and suppose

$$\psi_0 \circ g = f_0 \circ \varphi_0,$$

$$\psi_1 \circ g = f_1 \circ \varphi_1,$$

and that ψ_0, ψ_1 are the endpoints of an isotopy of the plane that fixes each point of $\mathscr{S}(g)$ (i.e., ψ_t is a homeomorphism of the plane for each t and varies continuously as a function of t). We want to show that f_0 and f_1 are the same up to a conformal linear change of variable.

Let $X = \mathbb{C} \setminus \mathscr{S}(g)$ and $Y = \mathbb{C} \setminus \mathscr{CV}(g)$ and recall (Lemma 1.23) that g is a covering map from Y to X. The Covering Homotopy Theorem says that given any homotopy $\psi_t \circ g : Y \to X$ and any map $\Psi : Y \to Y$ so that $\psi_0 \circ g = f_0 \circ \Psi_0$, there exists a homotopy $\Phi_t : Y \times [0,1] \to Y$ so that $\Phi_0 = \varphi_0$ and $f_0 \Phi_t = \psi_t \circ g$. Taking $t = 1$ gives

$$f_0 \circ \Phi_1 = \psi_1 \circ g = f_1 \circ \varphi_1,$$

or

$$f_0 = f_1 \circ \varphi_1 \circ \Phi_1^{-1}.$$

Note that $\varphi_1 \circ \Phi_1^{-1}$ is a homeomorphism of plane (since each part is) and is holomorphic except on the (finite) singular set of f, since it is locally equal to a branch of $f_0 \circ f_1^{-1}$. Thus it must be of the form $ax+b$, so we have $f_0(z) = f_1(az+b)$.

Therefore, if we fix g, and two values z_1, z_2 where g takes distinct values, then all the entire functions f that are topologically equivalent by homeomorphisms that are sufficiently close to the identity are uniquely determined by the values of $\mathscr{S}(f)$, $f(z_1)$ and $f(z_2)$.

CHAPTER 8

The folding theorem

CHAPTER 9

Speiser class examples

Index