ON A HÖLDER CONSTANT IN THE THEORY OF QUASICONFORMAL MAPPINGS

ISTVÁN PRAUSE

Dedicated to the memory of F. W. Gehring

ABSTRACT. A K-quasiconformal selfmap of the unit disk with identity boundary values satisfies the Hölder estimate $|f(z)-f(w)| \le 4^{1-1/K}|z-w|^{1/K}$. The constant $4^{1-\frac{1}{K}}$ is sharp.

1. INTRODUCTION

A classical result in the theory of quasiconformal mappings, known as Mori's theorem is the following. If $f\colon \mathbb{D} \to \mathbb{D} = f(\mathbb{D})$, $f(0) = 0$ is K-quasiconformal map of the unit disk $\mathbb{D} = B(0,1)$ then

$$|f(z) - f(w)| \le 16|z-w|^{1/K} \quad z, w \in \mathbb{D}. \tag{1.1}$$

See for instance [4, 1, 3]. Here the constant 16 is optimal as an absolute constant, however it has been conjectured in [3, p. 68] that 16 could be replaced by $16^{1-1/K}$ if we allow dependence on K. We refer to the texts [1, 3, 2] for different definitions and basic properties of quasiconformal mappings.

The purpose of this note is to point out the following sharp counterpart of (1.1). Below, we require identity boundary values, in which case the requirement $f(0) = 0$ may be omitted.

Theorem 1.1. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a K-quasiconformal mapping with (boundary extension) $f(z) = z$ for $|z| = 1$. Then for every $z, w \in \mathbb{D}$ we have*

$$|f(z) - f(w)| \le 4^{1-1/K}|z-w|^{1/K}. \tag{1.2}$$

Moreover, $4^{1-1/K}$ cannot be replaced by any smaller number depending only on K.

2000 *Mathematics Subject Classification.* 30C62.

The author was supported by the Academy of Finland grant 1266182.

A theorem of this type in $\mathbb{R}^n$, $n \geqslant 2$ has recently appeared in [6]. Their constant for $n = 2$ gives the non-optimal value

$$4^{1-1/K} \cdot 2^{1-1/K} K^{\frac{1}{2K}} \left(\frac{K}{K-1}\right)^{\frac{1-1/K}{2}} .$$

Remark 1.2. As it will be clear from the proof, the same bound holds for any K-quasiconformal principal mapping conformal outside the unit disk.

2. Proof

Proof. Let us extend f identically outside the unit disk. This way f becomes a global quasiconformal map of the complex plane with Beltrami coefficient μ, where $\|\mu\|_\infty \leq \frac{K-1}{K+1}$. We embed f in a holomorphic flow of solutions $\{f^\lambda\}_{\lambda\in\mathbb{D}}$ as follows. For $\lambda \in \mathbb{D}$ we solve the uniformly elliptic Beltrami equation

$$f^\lambda_{\bar{z}} = \lambda\mu \cdot \frac{K+1}{K-1} f^\lambda_z, \tag{2.1}$$

under the normalization of the so-called principal solution. As μ vanishes outside the unit disk, f^λ will be conformal outside $\mathbb{D}$ and we require the asymptotics $f(z) = z + o(1)$ as $z \to \infty$. According to a classical Koebe-type distortion theorem [5, Theorem 1.4]

$$|f^\lambda(z)| < 2, \quad \text{for} \quad |z| < 1. \tag{2.2}$$

Fix any two distinct points $z, w \in \mathbb{D}$ and consider the function

$$u(\lambda) = \log \frac{|f^\lambda(z) - f^\lambda(w)|}{4}.$$

In view of holomorphic dependence and injectivity of the flow $\{f^\lambda\}$ [2, Theorem 5.7.3] u is a harmonic function. Moreover, u is negative by (2.2). An application of Harnack's inequality then gives

$$u\left(\frac{K-1}{K+1}\right) \leq \frac{1}{K} u(0).$$

Observe that for $\lambda = (K-1)/(K+1)$ the solution of (2.1) is the original mapping f and for $\lambda = 0$ the solution is the identity map. Thus after exponentiating the previous inequality yields

$$\frac{|f(z) - f(w)|}{4} \leq \left(\frac{|z-w|}{4}\right)^{1/K},$$

as stated in the Theorem.

It remains to show sharpness of the constant $4^{1-1/K}$. The examples will be based on quasiconformal deformation of ellipses. Let $R > 1$ and denote by $\mathcal{E}_R$ the ellipse given by

$$|z-2| + |z+2| < 2\,(R + 1/R).$$

For any $K \geqslant 1$ there is a K-quasiconformal mapping $g\colon \mathcal{E}_R \to \mathcal{E}_{R^{1/K}}$ (onto) which fixes the foci $g(\pm 2) = \pm 2$ and have affine boundary values.

Indeed, it is easy to construct such a mapping explicitly. The conformal mapping $\phi(z) = z + 1/z$ maps the annulus $A_R = \{z\colon 1 < |z| < R\}$ onto $\mathcal{E}_R \setminus [-2, 2]$ and extends as an affine map to $\{z\colon |z| \geqslant R\}$. Now the desired map is given by $g = \phi\circ\rho\circ\phi^{-1}$, where $\rho(z) = z|z|^{1/K-1}$ is a K-quasiconformal radial stretching. Note that g extends over the slit $[-2, 2]$ and thus we have the required K-quasiconformal map $g\colon \mathcal{E}_R \to \mathcal{E}_{R^{1/K}}$.

Let α_R be an affine map from $\mathbb{D}$ onto $\mathcal{E}_R$. Consider now the map $f := \alpha^{-1}_{R^{1/K}} \circ g \circ \alpha_R$. This is a $K \cdot (1 + \varepsilon)^2$-quasiconformal selfmap of the unit disk with identity boundary values (possibly, after a rotation) where $\varepsilon = \varepsilon(R^{1/K}) \to 0$ as $R \to \infty$. Furthermore, f maps a segment of length $\sim \frac{4}{R}$ to a segment of length $\sim \frac{4}{R^{1/K}}$ as $R \to \infty$. Indeed, the segment $\alpha_R^{-1}([-2, 2])$ is mapped onto $\alpha^{-1}_{R^{1/K}}([-2, 2])$. This shows that the constant $4^{1-1/K}$ is best possible in (1.2). □

References

1. L. V. Ahlfors, *Lectures on quasiconformal mappings.* Van Nostrand Mathematical Studies, No. 10, 1966.
2. K. Astala, T. Iwaniec, G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane.* Princeton Mathematical Series, 48, 2009.
3. O. Lehto, K. I. Virtanen, *Quasiconformal mappings in the plane.* Second edition. Translated from the German by K. W. Lucas. Die Grundlehren der mathematischen Wissenschaften, Band 126, 1973.
4. A. Mori, *On an absolute constant in the theory of quasi-conformal mappings.* J. Math. Soc. Japan **8** (1956), 156–166.
5. C. Pommerenke, *Univalent functions.* With a chapter on quadratic differentials by Gerd Jensen. Studia Mathematica, 1975.
6. M. Vuorinen, X. Zhang, *Distortion of quasiconformal mappings with identity boundary values*, preprint arXiv:1203.0427, 2012

University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68 , FIN-00014 University of Helsinki, Finland

E-mail address: istvan.prause@helsinki.fi