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Non-removability of the Sierpiński gasket

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Abstract We prove that the Sierpiński gasket is non-removable for quasiconformal maps, thus answering a question of Bishop (NSF Research Proposal, 2015. http://www.math.stonybrook.edu/~bishop/vita/nsf15.pdf). The proof involves a new technique of constructing an exceptional homeomorphism from $\mathbb{R}^2$ into some non-planar surface S, and then embedding this surface quasisymmetrically back into the plane by using the celebrated Bonk–Kleiner Theorem (Bonk and Kleiner in Invent Math 150(1):127–183, 2002). We also prove that all homeomorphic copies of the Sierpiński gasket are non-removable for continuous Sobolev functions of the class $W^{1,p}$ for $1 \leq p \leq 2$, thus complementing and sharpening the results of the author's previous work (Ntalampekos in A removability theorem for Sobolev functions and detour sets. arXiv:1706.07687).

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Contents

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1 Introduction

1.1 Background and main results

The object of this paper is to prove that the Sierpiński gasket is non-removable for (quasi)conformal maps and Sobolev functions. The problem of (quasi)conformal and Sobolev removability has been studied extensively. Besides earlier results by Besicovitch [3] and Gehring [13], various conditions that guarantee the removability of compact sets have been established by Jones and Smirnov [18, 19], Kaufman and Wu [22, 39], Koskela and Nieminen [26], and recently by the current author [30]. Moreover, removability of Julia sets has been studied by Kahn [20] and also by Graczyk and Smirnov [14]. On the other side of the coin, examples of non-removable sets and constructions of exceptional functions/homeomorphisms have been given by Kaufman and Wu [21,22], Bishop [4,5], and also Koskela et al. [25]. The present paper provides one more result in this direction. Interestingly, according to a conjecture of He and Schramm [17], the problem of removability is also related to the rigidity of circle domains and Koebe's conjecture.

So far, the types of sets which were known to be non-removable were sets of positive Lebesgue measure, some product sets, some Cantor sets, and also "rough" sets which are topologically "simple" (e.g. simple curves). The task of constructing an exceptional homeomorphism becomes very challenging, as the topology of the set deteriorates. To the best of our knowledge, it is the first time that a non-trivial construction is used to prove that a set with infinitely many complementary components, such as the Sierpiński gasket, is non-removable. This hints that a *generic* set with infinitely many complementary components should be non-removable. Such sets are, for example, Sierpiński carpets, the Apollonian gasket, and also SLE_κ for $\kappa \in (4, 8)$; Sheffield [34] has posed the question whether the latter is removable or not.

We include some background of the problem of removability of sets for (quasi)conformal maps in $\mathbb{R}^2$ and Sobolev functions in $\mathbb{R}^n$. We direct the reader [40] for a thorough survey on the topic of (quasi)conformal removability and for proofs of some of the facts that we state here.

Definition 1.1 We say that a compact set $K \subset U \subset \mathbb{R}^2$ is (quasi)conformally removable inside the domain U if any homeomorphism of U, which is (quasi)conformal on $U \setminus K$, is (quasi)conformal on U.

Here we mention some basic facts. A set K is quasiconformally removable inside U if and only if K is conformally removable inside U. Hence, from now on, we will be using the term quasiconformal removability. Furthermore, a set K is quasiconformally removable inside a domain U if and only if K is quasiconformally removable inside the entire plane $\mathbb{R}^2$. Two fundamental open questions are the following:

Question 1 (p. 264, [19]). Is the union of two intersecting compact sets quasiconformally removable, whenever each one of them is removable?

For a partial result in this direction see [41, Theorem 4]. In the same paper [41, p. 1306] the author discusses the problem of local removability. A set K is *locally quasiconformally removable* if for *any* open set U (not necessarily containing K) and for any homeomorphism f of U that is quasiconformal on $U \setminus K$ we have that f is quasiconformal on U.

Question 2 If a set is quasiconformally removable, is it also locally quasiconformally removable?

A stronger notion of removability is the notion of $W^{1,2}$-removability. We give the general definition of $W^{1,p}$-removability in $\mathbb{R}^n$, where $p \in [1, \infty]$. Recall that a function f lies in $W^{1,p}(U)$, where U is an open subset of $\mathbb{R}^n$, if $f \in L^p(U)$ and also f has weak derivatives in U that lie in $L^p(U)$.

Definition 1.2 Let $p \in [1, \infty]$. We say that a compact set $K \subset \mathbb{R}^n$ is $W^{1,p}$-removable if any real-valued function that is continuous in $\mathbb{R}^n$ and lies in $W^{1,p}(\mathbb{R}^n \setminus K)$, also lies $W^{1,p}(\mathbb{R}^n)$.

Using partitions of unity one can show that this definition is local, and thus the answer to the analog of Question 2 is positive in this case. Furthermore, Hölder's inequality implies that if a set of measure zero is $W^{1,p}$-removable, then it is also $W^{1,q}$-removable for $q > p$.

Question 3 (p. 264, [19]). Is $W^{1,2}$-removability in the plane equivalent to quasiconformal removability?

Interestingly, so far the techniques used in the two different notions of removability are the same, but there is no further indication whether the answer to the preceding question should be positive or negative.

If a set $K \subset \mathbb{R}^2$ has measure zero, then $W^{1,2}$-removability of K implies quasiconformal removability. If a set $K \subset \mathbb{R}^2$ has positive measure then it is non-removable for quasiconformal maps. In [30] the author posed the question whether this is true for Sobolev removability. We provide an answer to this question here:

Theorem 1.3 *Let $K \subset \mathbb{R}^n$ be a compact set of positive Lebesgue measure and $1 \leq p < \infty$. Then K is non-removable for $W^{1,p}$.*

However, the statement is not true for $W^{1,\infty}$:

Proposition 1.4 *There exists a compact set $K \subset \mathbb{R}^n$ of positive Lebesgue measure that is $W^{1,\infty}$-removable.*

Classes of $W^{1,2}$- and quasiconformally removable sets include sets of σ-finite Hausdorff 1-measure [3], [36, Section 35], quasicircles, boundaries of John domains, of Hölder domains, and of domains satisfying certain quasihyperbolic conditions [18,19,26]. Also, some novel techniques for the removability of Julia sets of quadratic polynomials appeared in [20].

On the other hand, as already remarked, all sets of positive measure are non-removable for (quasi)conformal maps and $W^{1,2}$ functions. Furthermore, there exist non-removable Jordan curves of Hausdorff dimension 1 [4], and also non-removable graphs of functions [21].

Most of these results refer to compact sets that are the boundary of the union of *finitely many* connected open sets. Until recently, there had been no general result on sets with *infinitely many* complementary components, not falling into the preceding category. The task of proving that such a set is (non)-removable requires the development of different tools. In [30] the author studied this problem and derived a condition that guarantees $W^{1,p}$-removability of a set $K \subset \mathbb{R}^n$ for $p > n$. Roughly speaking, the condition is the following:

1. The complementary components of K are uniform Hölder domains (see e.g. [35] and also [30] for the definition), and
2. for "almost every" line L intersecting K, and for every $\varepsilon > 0$ there exists a "detour path" γ that ε-follows the line L, but intersects only finitely many of the complementary components of K that the line L intersects.

In other words, (2) says that we can "travel" in the direction of the line L using only finitely many components in the complement of K, but still staying arbitrarily close to the line L; see Fig. 1. We call such sets *detour sets*, and the Sierpiński gasket, depicted in Fig. 1, is one such set.

Theorem 1.5 (Corollary 1.4, [30]). *The Sierpiński gasket is $W^{1,p}$-removable for $p > 2$.*

Other sets that fall into the same category are the Apollonian gasket and generalized Sierpiński gasket Julia sets of sub-hyperbolic rational maps; see [30, Section 7].

The *Sierpiński gasket* is constructed as follows. We consider an equilateral triangle in the plane of sidelength 1 and subdivide it into four equilateral triangles of sidelength 1/2. After removing the middle triangle, we proceed inductively with subdividing each of the remaining three triangles into four

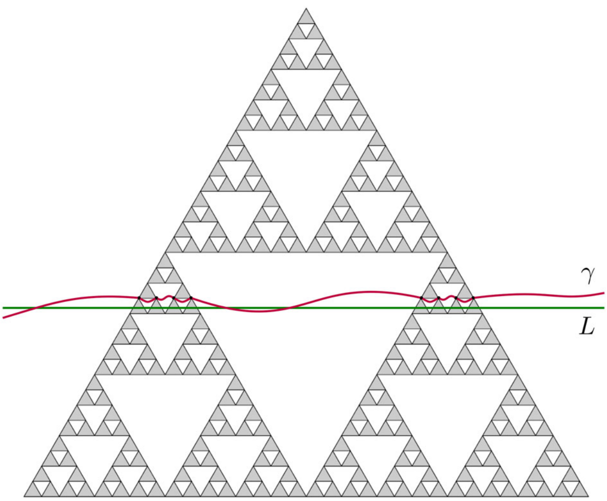

Fig. 1 The Sierpiński gasket, and a detour path γ near the line L

equilateral triangles of sidelength $1/2^2$, and so on. The remaining compact set K is the Sierpiński gasket.

In this work we first prove that Theorem 1.5 is sharp:

Theorem 1.6 *The Sierpiński gasket is non-removable for $W^{1,p}$, for $1 \leq p \leq 2$.*

In fact, by the monotonicity discussed after Definition 1.2, it suffices to prove that the gasket is non-removable for $W^{1,2}$.

The method used is very flexible and we obtain the following result:

Theorem 1.7 *Let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and K be the Sierpiński gasket. Then $h(K)$ is non-removable for $W^{1,2}$.*

Of course, if $h(K)$ has positive Lebesgue measure, then the conclusion follows immediately from Theorem 1.3, so the measure zero case is the interesting one.

Finally, we boost the proof of Theorem 1.6 and add a new ingredient to obtain the main result:

Theorem 1.8 *The Sierpiński gasket is non-removable for quasiconformal maps.*

In other words, there exists a homeomorphism of $\mathbb{R}^2$ that is quasiconformal in the complement of the gasket, but not globally quasiconformal. This answers a question raised by Bishop [6, Question 13]. We were not able to show the analog of Theorem 1.7 in this case, i.e., that all homeomorphic copies of the gasket are non-removable for quasiconformal maps, but we believe that a modification of the techniques used here can provide the answer.

We now discuss a natural approach to the problem, which, however, seems extremely hard to pursue; then, in Sect. 1.2 we give a brief outline of the proof of Theorem 1.8 and explain what this "new ingredient" that we use in our approach is.

As explained in [6, p. 15], one may try to construct an exceptional homeomorphism as in Theorem 1.8 as follows. Note that between any two triangles V_1 and V_2 there exists a conformal map, and this map is unique if one requires that each vertex of V_1 is mapped to a prescribed vertex of V_2, and the vertex correspondence is orientation-preserving. Here, we allow V_i, $i = 1, 2$, to be an "unbounded" triangle, i.e., the unbounded complementary component of a triangle. Hence, one can first map the unbounded complementary component of the Sierpiński gasket onto an unbounded non-equilateral triangle, and then (inductively) require that every bounded complementary equilateral triangle of the gasket is mapped uniquely by a conformal map to a triangle. Note that at each level of this construction the image of the vertices of a triangle is prescribed by the map of the previous level. This process will uniquely determine a map that is conformal on each complementary component of the gasket. If this map extends to a homeomorphism of $\mathbb{R}^2$, then it cannot be globally conformal, since it changes angles. However, it is not clear at all whether this map can be extended to a continuous map on $\mathbb{R}^2$, and thus to a homeomorphism of $\mathbb{R}^2$.

Before discussing the outline of the proof of Theorem 1.8 in Sect. 1.2, we conclude this section with some remarks on the (non)-removability of another related type of fractals whose complement has infinitely many components, namely, Sierpiński carpets.

The *standard Sierpiński carpet* S_3 is constructed by subdividing the unit square $[0, 1]^2$ into nine squares of sidelength $1/3$ and removing the middle square, and then proceeding inductively in each of the remaining eight squares. It is easy to see that the standard Sierpiński carpet is non-removable for quasiconformal and $W^{1,p}$ maps for $1 \leq p \leq \infty$. We sketch the proof for quasiconformal non-removability. Note that S_3 contains a copy of $C \times [0, 1]$, where C is the middle-thirds Cantor set. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the Cantor staircase function and let $\psi\colon \mathbb{R} \to [0, 1]$ be a smooth function with $\psi \equiv 0$ outside $[0, 1]$ and $\psi \equiv 1$ in $[1/9, 8/9]$. Then $f(x, y) := (x + h(x)\psi(y), y)$ is a homeomorphism of $\mathbb{R}^2$ that is quasiconformal on $\mathbb{R}^2 \setminus S_3$, but not globally quasiconformal.

A *(generalized) Sierpiński carpet* is a planar set $S \subset \mathbb{R}^2$ that is homeomorphic to S_3; the homeomorphism need not be defined on all of $\mathbb{R}^2$. These sets can be characterized as the compact sets of the plane arising by removing from the interior of a Jordan region Ω countably many Jordan regions Q_i for $i \in \mathbb{N}$, whose closures are disjoint and contained in Ω, such that $\mathrm{diam}(Q_i) \to 0$ as $i \to \infty$ and $S := \overline{\Omega} \setminus \bigcup_{i\in\mathbb{N}} Q_i$ has empty interior; see [37].

It is not known, in general, whether these sets are removable for quasiconformal maps or Sobolev functions, but we conjecture the following:

Conjecture 1 *All Sierpiński carpets are non-removable for quasiconformal maps and for $W^{1,p}$ functions, for $1 \leq p \leq \infty$.*

1.2 Sketch of the proof of Theorem 1.8

Let K be the Sierpiński gasket. We quickly sketch the strategy of constructing a homeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is quasiconformal on $\mathbb{R}^2 \setminus K$, but not globally quasiconformal. The lemmas introduced here are informal and all details can be found in Sect. 5. We suggest that the reader browse through the colored figures (see online version for colored figures) (located in Sect. 5) while reading the sketch.

First, we will define a continuous map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is the identity on the unbounded component of $\mathbb{R}^2 \setminus K$, it is injective outside the bounded equilateral triangle components of $\mathbb{R}^2 \setminus K$, and collapses each equilateral triangle component to a tripod; see Fig. 7. This map is defined inductively. More precisely, the map f is the identity on the boundary of the unbounded complementary triangle. Then, one collapses in a continuous way the "middle" complementary equilateral triangle W of sidelength $1/2$ to a tripod, whose barycenter is precisely the barycenter of the vertices of W and whose vertices are the vertices of W. We also require that the midpoints of the three edges of W are collapsed to the barycenter of the tripod G. Note that on the vertices of W the function f has already been defined to be the identity, by the previous step. Inductively, all complementary equilateral triangles of the gasket will be collapsed to tripods. The proof of the uniform continuity of such a map is a significant hurdle that we have to deal with. In fact, one has to choose very carefully the collapsing maps in each step, so that they satisfy a certain modulus of continuity. Note that after this procedure the gasket K is blown up by f to a set of full measure, since the tripods have measure zero. Summarizing, we have:

Lemma 1.9 *There exists a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is injective outside the complementary equilateral triangles of the gasket, and collapses each (bounded) complementary equilateral triangle to a tripod. Furthermore, $f(K)$ has positive Lebesgue measure.*

Of course, this map is not a homeomorphism on the complementary equilateral triangles, so we have to correct it there, but otherwise keep the existing definition. Unfortunately, there is no way to correct our function in this way, if the target is $\mathbb{R}^2$. What we do instead, is change the target to a non-Euclidean metric surface S and correct the map f inside the complementary equilateral triangles in order to obtain a homeomorphism $\Phi : \mathbb{R}^2 \to S$ that is quasiconformal in $\mathbb{R}^2 \setminus K$.

The "correction" that we do in each equilateral triangle W is the following. We "fold" W on top of the tripod $f(W)$; see Fig. 8. The folding map will be M-quasiconformal for a universal $M > 0$. In fact, the folding map will just be piecewise linear. More precisely, we prove that for an arbitrary equilateral triangle W and an arbitrary tripod G we can find an M-quasiconformal map that folds W onto six rectangles that are attached on top of the edges of G, with appropriate identifications. We call *flap* the metric space arising by folding a single equilateral triangle over a tripod. In this folding procedure, one can choose the "height" of the flap to be arbitrarily small, without affecting the constant M. Furthermore, a crucial property is that the folding map has to be compatible, in a sense, with f on ∂W, because we we wish to paste the two maps. In particular, the folding map has to have a certain modulus of continuity (the one that ensures the uniform continuity of f) and it must map the midpoints of the edges of W to "lifts" of the barycenter of the tripod G; see Fig. 8. We now summarize:

Lemma 1.10 *There exists a universal $M > 0$ such that for each equilateral triangle W and for each tripod G there exists a folding map ϕ_W from W onto the flap space corresponding to G. The height of the flap can be chosen to be arbitrary small. Moreover, the maps f and ϕ_W can be chosen to be compatible for all complementary equilateral triangles W of the gasket K.*

By folding all complementary equilateral triangles over their corresponding tripods, one obtains a *flap-plane* S, which is a non-Euclidean surface, and a homeomorphism $\Phi\colon \mathbb{R}^2 \to S$ that is M-quasiconformal on $\mathbb{R}^2\setminus K$ and "agrees" with f outside the triangles; see Fig. 7 (we remark here that in the figure the edges of the green rectangles are not glued to the red rectangles (see online version for colored figures), except possibly at one point; see also Fig. 4 for the gluing pattern). The map Φ is the result of pasting the map f with all the folding maps ϕ_W. The map Φ maps the gasket to a subset of S that has positive measure. In brief:

Lemma 1.11 *There exists a homeomorphism Φ from $\mathbb{R}^2$ onto a metric surface S that is M-quasiconformal on $\mathbb{R}^2\setminus K$ and maps K to a subset of S that has positive Hausdorff 2-measure.*

If the target of Φ were not S but it were $\mathbb{R}^2$, then the proof of non-removability would be finished. Hence, we have to find a way to change the target to $\mathbb{R}^2$. This is facilitated by the Bonk–Kleiner Theorem [7], which allows us to embed S into $\mathbb{R}^2$ with a quasisymmetric map. The Bonk–Kleiner Theorem asserts that a metric sphere that is Ahlfors 2-regular and linearly locally connected is quasisymmetrically equivalent to the standard Euclidean sphere. We develop a theory of flap-planes constructed similarly to S. These are just spaces arising by gluing to the plane an infinite collection of rectangles (or flaps), which are "perpendicular" to the plane. Using the Bonk–Kleiner Theorem we will show that flap-planes can be quasisymmetrically embedded to the plane, provided that the heights of the flaps are sufficiently small. In our case, we can obtain a quasisymmetry $\Psi\colon S \to \mathbb{R}^2$. Note that Ψ is a quasisymmetry on *all* of S.

Lemma 1.12 *There exists a quasisymmetry $\Psi\colon S \to \mathbb{R}^2$. Moreover, Ψ maps sets of positive Hausdorff 2-measure to sets of positive Lebesgue measure.*

The composition $F = \Psi \circ \Phi$ will be a homeomorphism of $\mathbb{R}^2$ that is M'-quasiconformal on $\mathbb{R}^2\setminus K$ for some uniform $M' > 0$, but it cannot be globally quasiconformal, because it has to blow the gasket K to a set of positive area. This is because $\Phi(K)$ had positive measure in S and in our setting the quasisymmetry Ψ has to map sets of positive measure to sets of positive measure.

1.3 Organization of the paper

In Sect. 2 we introduce our notation and discuss some preliminaries regarding quasiconformal and quasisymmetric maps, and also convergence of metric spaces in the pointed Gromov–Hausdorff sense.

In Sect. 3 we develop a theory of *flap-planes*, which are surfaces arising by gluing rectangles, or else, *flaps*, on top of the Euclidean plane. Under some assumptions, our goal is to apply the Bonk–Kleiner Theorem to show that these surfaces can be quasisymmetrically embedded into the plane; see Theorem 3.7. Hence, we focus on proving that they are Ahlfors 2-regular and linearly locally connected. These proofs occupy most of the section. Also, this section is independent of the other sections, and can be mostly skipped in a first reading of the paper. We will only need the definition and some general properties of flap-planes from Sect. 3.1 and also we will use the embedding Theorem 3.7.

The main content of Sect. 4 is the proof of Theorem 1.6, i.e., the non-removability of the gasket for continuous functions of the class $W^{1,2}$. The proof consists of several steps, which are organized in the subsections. The heart of the argument is Lemma 4.7. The proof of Theorem 1.7 is contained in Sect. 4.6. There, we also include the proofs of the general statements in Theorem 1.3 and Proposition 1.4. Moreover, in Sect. 4.1 we include basic terminology and geometric properties of the gasket that we use repeatedly throughout the paper.

Finally, in Sect. 5 we prove the quasiconformal non-removability in Theorem 1.8. First, in Sect. 5.1 we show how to collapse the complementary equilateral triangles to tripods in a continuous way with a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, as described in Sect. 1.2. The proof of the continuity of f is the same as the proof of continuity for the Sobolev non-removability in Sect. 4, so we recommend the reader to read first that proof.

Then, in Sect. 5.2 we explain how to fold a single equilateral triangle on top of a tripod with a quasiconformal map. In Sect. 5.3 these folding maps are pieced together with f to obtain a homeomorphism Φ from $\mathbb{R}^2$ onto a flap-plane S. Finally, in Sect. 5.4 we finish the proof of non-removability by embedding S into the plane and obtaining the desired exceptional homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$.

Update

Since the completion and distribution of the first version of this paper, there has been some further progress. The current author in [31] has proved that all Sierpiński carpets are non-removable for quasiconformal maps, providing an answer to Conjecture 1.

Moreover, further connections between the problems of rigidity of circle domains and removability of their boundary have been established in [32], in the spirit of the conjecture of He and Schramm [17]. In particular, it is proved that all circle domains satisfying the quasihyperbolic condition of [19] are rigid.

2 Preliminaries

2.1 Notation

We will say that a statement (A) implies a statement (B) *quantitatively* if the implicit constants or functions in statement (B) depend only on the implicit constants or functions in statement (A).

We use the notation $a \lesssim b$ if there exists an implicit constant $C > 0$ such that $a \leq Cb$, and $a \simeq b$ if there exists a constant $C > 0$ such that $C^{-1}b \leq a \leq Cb$. We will be mentioning the parameters on which the constant C depends, unless C is a universal constant.

The Lebesgue measure in $\mathbb{R}^n$ is denoted by m_n. If a function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable, then we will denote its integral against Lebesgue measure by $\int f$. The open ball around $x \in \mathbb{R}^n$ of radius $r > 0$ is denoted by $B(x, r)$. For visual purposes, the closure of a set U_1 is denoted by $\overline{U}_1$, instead of $\overline{U_1}$ and the closure of a Euclidean ball $B(x, r)$ is denoted by $\overline{B}(x, r)$. We use the notation $A(x; r, R)$ for the annulus $B(x, R) \setminus \overline{B}(x, r)$, where $0 < r < R$.

If (X, d) is a metric space and $Q > 0$, then we denote the Hausdorff Q-measure by $\mathcal{H}^Q_d$; see e.g. [12, Section 11.2] for the definition. Also, we use the notation $B_d(x, r)$ for the open ball of radius $r > 0$, centered at $x \in X$. In Sect. 3 we will be endowing planar sets with different metrics. If there is no subscript d in the ball notation, then the ball will always refer to the Euclidean metric.

We normalize the Hausdorff 2-measure, so that it agrees with the Lebesgue measure in $\mathbb{R}^2$. This $\mathbb{R}^2$-normalization will also be used when we study the Hausdorff 2-measure of an arbitrary metric space. For example, if X is a metric space and $U \subset X$ is a set that is isometric to a measurable subset of $\mathbb{R}^2$, then its Hausdorff 2-measure agrees with the Lebesgue measure of its isometric image in $\mathbb{R}^2$.

2.2 Quasiconformal and quasisymmetric maps

We first recall the definition of a quasiconformal map; we direct the reader to [2,36] for background on quasiconformal maps.

Definition 2.1 Let $U, V \subset \mathbb{R}^2$ be open sets. An orientation-preserving homeomorphism $f: U \to V$ is M-quasiconformal for some $M > 0$ if $f \in W^{1,2}_{\mathrm{loc}}(U)$ and

$$\|Df(z)\|^2 \leq M J_f(z)$$

for a.e. $z \in U$, where $\|Df(z)\|$ denotes the operator norm of the differential of f at z, and J_f is the Jacobian of f. We also say that f is quasiconformal if it is M-quasiconformal for some $M > 0$. The number $M > 0$ is called the quasiconformal distortion of f.

Remark 2.2 A priori, if a set $K \subset U$ is removable in the sense of Definition 1.1, it could be the case that the quasiconformal distortion of a map f in U is larger than the distortion in $U \setminus K$. However, as remarked in the Introduction, removable sets necessarily have measure zero. Since the inequality $\|Df(z)\|^2 \leq M J_f(z)$ is required to hold a.e., it follows that the quasiconformal distortion of f on U is the same as the distortion on $U \setminus K$.

Quasiconformal maps have the important property that they preserve null sets:

Lemma 2.3 (Theorem 33.2, [36]). *Let $U, V \subset \mathbb{R}^2$ be open sets and let $f: U \to V$ be a quasiconformal map. A Borel set $A \subset U$ is mapped by f to a set of measure zero if and only if A has measure zero.*

Definition 2.4 If two metric spaces (X, d_X) and (Y, d_Y) are locally isometric to open subsets of $\mathbb{R}^2$, then we say that a homeomorphism $f: X \to Y$ is M-quasiconformal if the following holds. For each $x \in X$ there exist open neighborhoods U_x of x and $V_{f(x)}$ of $f(x)$ and there exist isometries $\phi: U_x \to U$ and $\psi: V_{f(x)} \to V$, where U and V are open subsets of $\mathbb{R}^2$ such that $\psi \circ f \circ \phi^{-1}: U \to V$ is M-quasiconformal, in the preceding sense.

There exists already a theory of quasiconformal maps between metric spaces that is compatible with the definition that we gave. Nevertheless, we will not need any deep results from that theory, and we wish to keep our approach as simple as possible, so we do not give the general definition. See, for example, [16] for more background.

Now, we define the notion of a quasisymmetry between two metric spaces (X, d_X) and (Y, d_Y); see also [15, Chapters 10–11].

Definition 2.5 A homeomorphism $f: X \to Y$ is η-quasisymmetric if there exists a homeomorphism $\eta: [0, \infty) \to [0, \infty)$ such that for every triple of distinct points $x, y, z \in X$ and for their images $x' = f(x)$, $y' = f(y)$, $z' = f(z)$ we have

$$\frac{d_Y(x', y')}{d_Y(x', z')} \le \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right).$$

The function η is called the distortion function associated to f.

If $X = U$ and $Y = V$ are open subsets of the plane, then an η-quasisymmetric orientation-preserving homeomorphism $f: U \to V$ is M-quasiconformal, where M depends only on η; this is proved in Chapter 4 of [36], and in particular in Theorem 34.1. The converse does not hold without extra assumptions. More generally, we have:

Lemma 2.6 *If X and Y are locally isometric to open subsets of the plane, then an η-quasisymmetric map $f: X \to Y$ is M-quasiconformal, in the sense of Definition 2.4. The constant $M > 0$ depends only on the distortion function η.*

Lemma 2.7 *Let X, Y, Z be metric spaces that are locally isometric to open subsets of the plane. Also, consider homeomorphisms $f: X \to Y$ and $g: Y \to Z$ such that f is M-quasiconformal and g is M'-quasiconformal. Then the composition $g \circ f: X \to Z$ is $M \cdot M'$- quasiconformal.*

See [36, Theorem 13.2] for the preceding fact, in case the spaces X, Y, Z are Euclidean. We also need the following removability lemma:

Lemma 2.8 (Theorem 35.1, [36]). *Let $f: U \to V$ be an orientation-preserving homeomorphism between open subsets of the plane. Let $A \subset \mathbb{R}^2$ be a closed set, and assume that $f\big|_{U\setminus A}$ is M-quasiconformal. If A has σ-finite Hausdorff 1-measure, then f is M-quasiconformal on U.*

Note that this lemma implies that sets of σ-finite Hausdorff 1-measure are locally removable, in the sense of Question 2 of the Introduction.

Finally, we need a lemma for quasisymmetric maps from a metric space onto the plane. A metric space (X, d) is *Ahlfors Q-regular* for some $Q > 0$ if there exists a constant $C \ge 1$ such that for all $x \in X$ and $0 < r < \operatorname{diam}(X)$ we have

$$\frac{1}{C} r^Q \le \mathcal{H}^Q_d(B_d(x, r)) \le C r^Q.$$

Lemma 2.9 *Let (X, d) be an Ahlfors 2-regular metric space and assume that there exists a quasisymmetric map f from X onto $\mathbb{R}^2$. Then the pushforward measure $f_*(\mathcal{H}^2_d)$ and the Lebesgue measure on $\mathbb{R}^2$ are mutually absolutely continuous.*

For the proof see [8, Proposition 4.3] and the references therein. The authors prove the above statement for a quasisymmetry from X onto the sphere $\widehat{\mathbb{C}}$, but the same proof applies in our case.

2.3 Convergence of metric spaces

Here we discuss the notion of pointed Gromov–Hausdorff convergence of metric spaces and the relevant properties.

We follow the approach of [9, Chapter 8]. For a map $f\colon (X,d_X)\to(Y,d_Y)$ between two metric spaces we define its *distortion* by

$$\mathrm{dis}(f)=\sup\{|d_X(x,x')-d_Y(f(x),f(x'))|:x,x'\in X\}.$$

A *pointed metric space* is a triple (X,d,p), where d is the metric of X and $p\in X$ is a point.

Definition 2.10 A sequence (X_n,d_n,p_n) of pointed metric spaces converges to a pointed metric space (X,d,p) in the Gromov–Hausdorff sense if the following holds. For every $r>0$ and $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for every $n>n_0$ there exists a (not necessarily continuous) map $f\colon B_{d_n}(p_n,r)\to X$ such that the following hold:

1. $f(p_n)=p$,
2. $\mathrm{dis}(f)<\varepsilon$, and
3. the ε-neighborhood of the set $f(B_{d_n}(p_n,r))$ contains the ball $B_d(p,r-\varepsilon)$.

A metric space (X,d) is *doubling* if there is a constant $N\in\mathbb{N}$ such that each ball of radius r can be covered by at most N balls of radius $r/2$. It is easy to see that if (X,d) is Ahlfors Q-regular for some $Q>0$ then it is also doubling, and the implicit constant depends only on Q and the Ahlfors regularity constant. A family of spaces is *uniformly* Ahlfors Q-regular, if the implicit constants are the same for all spaces in the family. Similarly, one defines a *uniformly* doubling family of spaces. We will need the following lemma, regarding the convergence of Ahlfors regular spaces:

Lemma 2.11 *Let (X_n,d_n,p_n) be a sequence of uniformly Ahlfors Q-regular pointed metric spaces. Suppose that (X_n,d_n,p_n) converges to a space (X,d,p) in the Gromov–Hausdorff sense. Then the metric space (X,d,p) is Ahlfors Q-regular, with implicit constants depending only on the constants of the spaces (X_n,d_n,p_n).*

This lemma appears in [10, Lemma 8.29]. The authors use a different definition for the convergence of metric spaces; see [10, Definition 8.9]. This definition is not very handy in practice and we will not use it. However, their definition agrees with our Definition 2.10, in case the spaces involved are uniformly doubling. Finally, we need the following lemma regarding the convergence of a *mapping package*:

Lemma 2.12 *Let* (X_n, d_n, p_n) *and* (Y_n, ρ_n, q_n) *be pointed metric spaces for* $n \in \mathbb{N}$*, which are complete and uniformly doubling. Moreover, suppose that there exists a sequence of* η*-quasisymmetric homeomorphisms* $f_n : X_n \to Y_n$ *such that* $f_n(p_n) = q_n$ *for all* $n \in \mathbb{N}$ *and there exists a constant* $C > 0$ *and points* $x_n \in X_n$ *such that*

$$\frac{1}{C} \le d_n(p_n, x_n) \le C \quad \textit{and} \quad \frac{1}{C} \le \rho_n(q_n, f(x_n)) \le C,$$

for each $n \in \mathbb{N}$*. Then there exist subsequential Gromov–Hausdorff limits* (X, d, p) *and* (Y, ρ, q) *of* (X_n, d_n, p_n) *and* (Y_n, ρ_n, q_n)*, respectively, and there exists a limiting* η*-quasisymmetric homeomorphism* $f : X \to Y$ *with* $f(p) = q$*.*

This lemma follows from [10, Lemma 8.22], since our assumptions guarantee equicontinuity and uniform boundedness; see also [15, Corollary 10.30]. This lemma has also appeared in [24, Lemma 2.4.7].

3 Flap-planes

3.1 Definition and general properties

3.1.1 Constructing a flap-plane out of a single tripod

A *tripod* G is the union of three line segments (also called edges) in the plane, which have a common endpoint, but otherwise they are disjoint; note that their length need not be the same and their angles could vary. The common endpoint of the edges is called the *center* or *central vertex* of the tripod.

We cut the plane along a tripod G, and then glue two rectangles (or else a rectangular pillow) on each slit that arises from cutting an edge e of G with the identifications shown in Fig. 2, so that we obtain a space homeomorphic to the plane. We write $E \sim G$ to denote that a rectangle E is glued to an edge of G. The width of each of these rectangles is equal to the the length of the corresponding edge e of G, and the height is a prescribed constant $h > 0$. Whenever two rectangles are glued along one of their edges, or a rectangle is glued to a slitted edge of G, then the gluing map is taken to be the "identity", namely an isometry. We direct the reader to [9, Chapters 3.1–3.2] for details on gluing length spaces and constructing polyhedral spaces.

The resulting space $S = S(G)$ (the height h is suppressed in the notation) is equipped with its internal metric d and it is homeomorphic to the plane by construction, while the subset of S consisting of the six rectangles attached to G is homeomorphic to a closed Jordan region $\overline{\Omega}$. Topologically, one can think of this construction as cutting the plane along G and "inserting" a Jordan

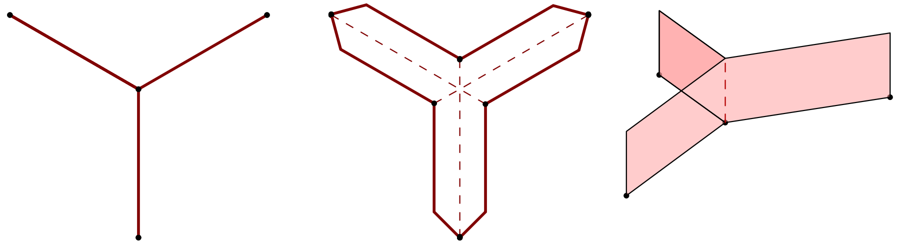

Fig. 2 The tripod G on the left is first slitted and its complement is homeomorphic to the complement of a closed Jordan region $\overline{\Omega}$, as the one bounded by the solid red curve in the middle figure; note that the central vertex of G corresponds to three points in $\partial\Omega$. Then we consider, as depicted, six topological rectangles in this Jordan region Ω, each bounded by the dashed and solid lines. The middle figure shows the (topological) gluing pattern of the six Euclidean rectangles in the figure on the right, which are glued to the slitted edges of the tripod G, giving rise to the flap-plane. Although in the figure on the right we only see three rectangles, in fact each of them represents a rectangular pillow having two distinct faces and two slits, along which these two faces are glued to the slitted edge of the tripod and to the other two pillows (color figure online)

region Ω in the plane, whose boundary consists of the six edges of the slitted G; see Fig. 2. We identify S with the union of $\mathbb{R}^2\setminus G$ and the rectangles $E \sim G$, after proper identifications.

Let $P\colon S \to \mathbb{R}^2$ denote the "orthogonal" projection map. This collapses each point of a rectangle E that is glued on top of a slitted edge e to the corresponding point of the edge $e \subset \mathbb{R}^2$. For instance, if a rectangle $E = [a, b] \times [c, d]$ is glued to the edge $e = [a, b] \times \{0\} \subset G$ along its side $[a, b] \times \{0\} \subset E$ with the identity map, then the projection of a point $x = (s, t) \in E$ is $P((s, t)) = (s, 0) \in e \subset \mathbb{R}^2$. Outside the rectangles $E \sim G$, the map P is the "identity". The projection of a point $x \in S$ to the plane will be denoted by $\tilde{x} = P(x)$. Some immediate properties of P are the following:

(i) P is 1-Lipschitz, i.e.,

$$|\tilde{x} - \tilde{y}| \leq d(x, y)$$

for all $x, y \in S$, where $|\tilde{x} - \tilde{y}|$ denotes the Euclidean distance between $\tilde{x}$ and $\tilde{y}$.

(ii) For all $x, y \in S$ we have

$$d(x, y) \leq |\tilde{x} - \tilde{y}| + 6h.$$

This is because the line segment $[\tilde{x}, \tilde{y}] \subset \mathbb{R}^2$ has a lift under P that is a continuum γ connecting x and y, and whose length inside each glued rectangle is either h or 0. On the other hand, the number of rectangles that we glue is six (or three two-sided rectangular pillows).

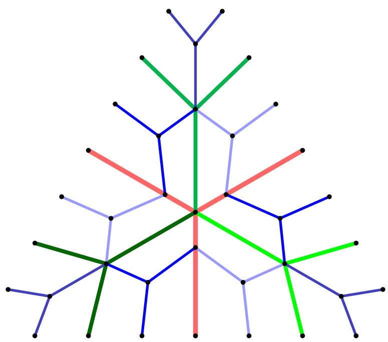

Fig. 3 A family of tripods possessing property (G). Different tripods meeting at a point are denoted by different colors (color figure online)

(iii) Let $\tilde{\gamma} \subset \mathbb{R}^2$ be a polygonal path that connects $\tilde{x}$ and $\tilde{y}$. If $x \in P^{-1}(\tilde{x})$ and $y \in P^{-1}(\tilde{y})$, then $P^{-1}(\tilde{\gamma})$ is a continuum that connects x and y. In fact, $P^{-1}(\tilde{\gamma})$ contains a polygonal path γ with the same property. Here, $\gamma \subset S$ is a polygonal path in the sense that it is the union of finitely many isometric copies of compact intervals, whose endpoints are glued appropriately.

3.1.2 Constructing a flap-plane with multiple tripods

Now, assume that we are given a sequence of tripods in the plane G_i, $i \in \mathbb{N}$, such that if the tripods G_i and G_j intersect for $i \neq j$, then $G_i \cap G_j$ is a singleton and more specifically it is a non-central vertex of one of G_i or G_j. There are essentially three ways this can occur:

(i) a non-central vertex of G_j lies on the central vertex of G_i, as in the left of Fig. 4,
(ii) a non-central vertex of G_j lies on an open edge of G_i,
(iii) a non-central vertex of G_j lies on a non-central vertex of G_i,

or the above occur with the roles of i and j reversed. More generally, if any collection of planar tripods $\{G_i\}_{i\in I}$ has this property, we say that

$$\{G_i\}_{i\in I} \text{ possesses } \textit{property}(G).$$

See Fig. 3 for a family of tripods with this property.

We wish to use the tripods $G_1, \ldots, G_n$ in order to construct a flap-plane $S_n = S(G_1, \ldots, G_n)$ that "distinguishes" between the tripods $G_1, \ldots, G_n$, in the sense that natural projections can be defined from S_n onto flap-planes S_k, $k < n$. To achieve this, we do not glue the rectangles corresponding to the flap-planes of different tripods, even if the tripods intersect each other (the intersection can contain at most one point).

More precisely, we can construct a flap plane $S_2 = S(G_1, G_2)$ as follows. If $G_2 \cap G_1 = \emptyset$, then we can construct the flap-plane S_2 by cutting the plane along

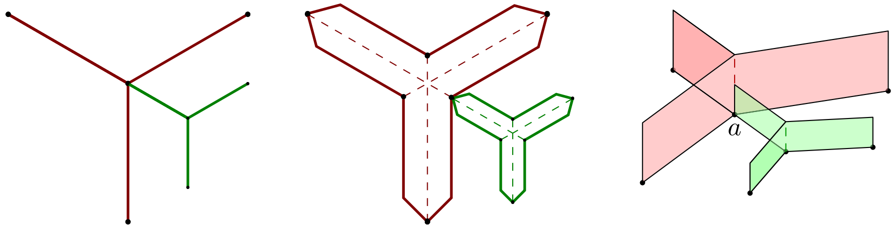

Fig. 4 The flap-plane (right) corresponding to two tripods G_1, G_2 (left) that intersect at a point. The rectangles attached to G_1 are not glued to any of the rectangles attached to G_2, except at a single point a; see the middle figure for the gluing pattern

G_2 and attaching rectangles of height h_2, as before. In this case, the rectangles corresponding to G_1 and G_2 do not intersect. If, however, $a \in G_2 \cap G_1$, then we cut the plane along G_2 and attach rectangles to G_2, but the rectangles of G_2 containing a are not attached to any of the rectangles corresponding to G_1, except at the point a; see Fig. 4. We also provide an alternative way to construct S_2, which shows that S_2 is homeomorphic to the plane. First consider the flap-plane $S_1 = S(G_1)$ corresponding to G_1, with associated height h_1. Using the property (G) of the tripods G_1 and G_2 and studying their relative positions we see that $S(G_1)$ contains a (unique) "distinguished" homeomorphic copy $\widetilde{G}_2$ of G_2 that projects homeomorphically onto $G_2 \subset \mathbb{R}^2$ under the restriction of the projection $P_1 {:=} P$; namely, if a non-central vertex of G_2 lies on G_1, then this copy is the closure in $S(G_1)$ of the preimages under P_1 of the open edges of G_2. We first glue the rectangles of height h_2 to G_2, disregarding the presence of G_1, and obtain a flap-plane $S(G_2)$ in this way. Then one uses the projection $P_1 \colon S(G_1) \to \mathbb{R}^2 \supset G_2$ in order to glue the rectangles attached to G_2 to the "distinguished" homeomorphic copy of G_2 in $S(G_1)$. Topologically, we are just cutting the tripod $\widetilde{G}_2 \subset S(G_1) \simeq \mathbb{R}^2$ and we are "inserting" a Jordan region, so the resulting space S_2 is also homeomorphic to $\mathbb{R}^2$.

We identify $S(G_1, G_2)$ with the union of $\mathbb{R}^2 \setminus (G_1 \cup G_2)$ and the rectangles attached to G_1 and G_2, after proper identifications as explained above. One can define natural projections $P_2 \colon S_2 \to S_1$ and $P_{2,0} \colon S_2 \to \mathbb{R}^2$. Suppose that $E_2 \sim G_2$ is a rectangle $E_2 = [a, b] \times [c, d]$ and it is attached to the edge $e_2 = [a, b] \times \{0\} \subset G_2$. Then for $(s, t) \in E_2$ with $(s, 0) \in \mathbb{R}^2 \setminus G_1$ we have $P_2((s, t)) = (s, 0) \in S_1$ (after identifying points of S_1 lying outside the rectangles $E \sim G_1$ with $\mathbb{R}^2 \setminus G_1$). If $(s, 0) \in G_1$, then $P_2((s, t))$ is the point of S_1 lying on the base of a rectangle $E_1 \sim G_1$, which is in fact the "intersection" point of E_2 and E_1 in S_2; this would be the point a in Fig. 4. For points of S_2 that do not lie on rectangles $E \sim G_2$ the map P_2 is defined to be the "identity". The map $P_{2,0}$ is defined to be the "identity" on $\mathbb{R}^2 \setminus (G_1 \cup G_2)$ and on the rectangles attached to G_1 and G_2 it is defined to be the "orthogonal" projection as above. One sees that $P_{2,0} = P_1 \circ P_2$.

Using the property (G) and induction on the number of tripods, for each $n \in \mathbb{N}$ we can define the flap-plane $S_n = S(G_1, \dots, G_n)$, with associated heights $h_1, \dots, h_n$ for the rectangles corresponding to $G_1, \dots, G_n$, respectively, and also define natural projections $P_n : S_n \to S_{n-1}$, $P_{n,0} : S_n \to \mathbb{R}^2$ so that:

- S_n is homeomorphic to the plane and
- if $G_0 \subset \mathbb{R}^2$ is a tripod such that the family $\{G_0\} \cup \{G_i : i \in \{1, \dots, n\}\}$ possesses property (G), then S_n contains a "distinguished" homeomorphic copy $\widetilde{G}_0$ of G_0 that projects homeomorphically onto $G_0 \subset \mathbb{R}^2$ under the restriction of the projection $P_{n,0} : S_n \to \mathbb{R}^2$.

We provide a sketch of the proof. Suppose that S_{n-1} has the above two properties and let $G_0 \subset \mathbb{R}^2$ be a tripod such that the family $\{G_0\} \cup \{G_i : i \in \{1, \dots, n\}\}$ possesses property (G). Since both families $\{G_i : i \in \{1, \dots, n\}\}$ and $\{G_0\} \cup \{G_i : i \in \{1, \dots, n-1\}\}$ have property (G), it follows that S_{n-1} contains "distinguished" homeomorphic copies $\widetilde{G}_n$ and $\widetilde{G}_0$ of G_n and G_0, respectively. Since S_{n-1} is homeomorphic to $\mathbb{R}^2$, we may think of $\widetilde{G}_n$ and $\widetilde{G}_0$ as (topological) tripods in the plane with non-straight edges. Property (G) imposes the same restrictions on the relative positions of $\widetilde{G}_n$ and $\widetilde{G}_0$. For the construction of S_n, one cuts the plane along this "distinguished" tripod $\widetilde{G}_n$ and attaches the rectangles, or else, "inserts" a Jordan region whose boundary consists of the six edges of the slitted $\widetilde{G}_n$. This implies that S_n is homeomorphic to the plane. Then the restrictions on the relative positions of $\widetilde{G}_n$ and $\widetilde{G}_0$ imply that S_n contains the desired "distinguished" homeomorphic copy of $\widetilde{G}_0$ and thus of G_0.

We now record some important properties of the flap-planes S_n. We endow the space S_n with its internal path metric d_n. With this metric (S_n, d_n) is homeomorphic to the plane, and it is also complete and locally compact. We also set $S_0 = \mathbb{R}^2$. The notation $E \sim G_i$ is used to denote that a rectangle $E \subset S_n$ is glued to an edge of G_i, $i \in \{1, \dots, n\}$. The space S_n is regarded as the union of $\mathbb{R}^2 \setminus \bigcup_{i=1}^n G_i$ and the rectangles $E \sim G_i$, $i \in \{1, \dots, n\}$, with proper identifications. The metric d_n has the property that is locally isometric to d_{n-1}, away from the rectangles $E \sim G_n$. Furthermore, d_n restricted to a rectangle $E \sim G_i$ is locally isometric to the planar metric on the rectangle E. For any two points $x, y \in S_n$ there exists a (not necessarily unique) geodesic γ that connects them, with $d_n(x, y) = \text{length}_{d_n}(\gamma)$; see [9, Section 2.5.2]. We will use the notation $B_n(x, r)$ for a ball in S_n, whenever it is more convenient than the notation $B_{d_n}(x, r)$.

Consider now the natural projections $P_k : S_k \to S_{k-1}$ for $k \geq 1$. These natural projections are defined similarly to P_1 and P_2 as above. We also define projections $P_{k,l} : S_k \to S_l$ by $P_{k,l} = P_k \circ \cdots \circ P_{l+1}$ for $0 \leq l \leq k-1$ and let $P_{k,k}$ be the identity map on S_k. These projections satisfy the following:

$(G1)$ P_k and $P_{k,l}$ are 1-Lipschitz.
$(G2)$ For all $x, y \in S_k$ we have

$$d_k(x, y) \leq d_{k-1}(P_k(x), P_k(y)) + 6h_k.$$

$(G3)$ If $\gamma^* \subset S_{k-1}$ is a polygonal path connecting x^* and y^*, then $P_k^{-1}(\gamma^*)$ contains a polygonal path that connects $P_k^{-1}(x^*)$ to $P_k^{-1}(y^*)$.
$(G4)$ The projections are compatible, i.e., for $0 \leq m \leq l \leq k$ we have

$$P_{k,m} = P_{l,m} \circ P_{k,l}.$$

$(G5)$ If γ^* is a geodesic in S_{k-1} that does not intersect the tripod G_k (or rather the lift $P_{k-1,0}^{-1}(G_k)$), then γ^* lifts isometrically under P_k to a unique geodesic $\gamma \subset S_k$. Conversely, if $\gamma \subset S_k$ is a geodesic that does not intersect the rectangles $E \sim G_k$, then its projection γ^* is also a geodesic in S_{k-1} and is isometric to γ.

The latter property, implies the next important property:

$(G6)$ If a ball $B_{k-1}(x^*, r) \subset S_{k-1}$ does not intersect the tripod G_k, then x^* has a unique preimage x under P_k, and $P_k^{-1}(B_{k-1}(x^*, r)) = B_k(x, r)$. Conversely, if $B_k(x, r) \subset S_k$ does not intersect the rectangles $E \sim G_k$, then it projects onto a ball $B_{k-1}(x^*, r)$.

In the same spirit, we have the following property for the Hausdorff 2-measure μ_k on S_k:

$(G7)$ If $A \subset S_k$ is a Borel set, then

$$\mu_k(A) \geq \mu_{k-1}(P_k(A)).$$

Moreover, if the set A does not intersect G_k, then we obtain equality.

We finally record a property of the metric d_k on the rectangles E glued to the tripod G_i:

$(G8)$ The metric d_k restricted on a rectangle $E \sim G_i$, $i \in \{1, \dots, k\}$, is isometric, not only locally but also globally, to the Euclidean metric on E.

3.1.3 The inverse limit S_∞ of S_n

We consider the set $T_n = \bigcup_{i=1}^n G_i$ as a (possibly disconnected) planar graph, whose vertices are the vertices of $G_1, \dots, G_n$. Note that an edge e of G_j might be "cut" into two (or more) edges, if a vertex of G_i, $i \neq j$, lies in the interior of the edge e. See Fig. 3 for such a graph T_n. We also consider the "graph" $T_\infty = \bigcup_{i=1}^\infty G_i$. A point $x \in \mathbb{R}^2$ is a vertex of T_∞ (by definition) if x is a vertex of T_n for all sufficiently large $n \in \mathbb{N}$. We suppose that T_∞ has *uniformly bounded degree*, that is, there exists $N_0 > 0$ such that the degree of each vertex x of T_n is bounded by N_0 for all $n \in \mathbb{N}$.

Under these assumptions, we can identify the *inverse limit* S_∞ of the sequence of spaces S_n. The set S_∞ is the subset of $\prod_{n=0}^{\infty} S_n$ consisting of points $z = (z_0, z_1, \dots)$ with the property that $P_n(z_n) = z_{n-1}$ for all $n = 1, 2, \dots$. We will find a simpler representation for S_∞.

Note that if $z_0 \in \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} G_i$ then $z_0 = z_1 = \dots$, recalling that we have identified points of S_n not lying on any tripod with $\mathbb{R}^2 \setminus \bigcup_{i=1}^{n} G_i$. Hence, in this case we may identify the point $z \in S_\infty$ with $z_0 \in \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} G_i$ and say that $z \in S_n$ for all $n = 0, 1, \dots$.

Now, if z_0 lies on a tripod G_i and is not a vertex of T_∞, then $z_i \in S_i$ lies on a rectangle attached to G_i and z_k has a unique preimage under P_{k+1} for all $k \geq i$, since no rectangle is attached to z_i in order to obtain the spaces S_k, $k > i$. Hence, we may write that $z_i \in S_k$ for all $k \geq i$ and we may identify the point $z \in S_\infty$ with z_i, thus saying that z lies in S_n for all sufficiently large n.

Finally, suppose that z_0 lies on a tripod G_i and it is a vertex of T_∞. Since T_∞ has uniformly bounded degree, it follows that there are finitely many tripods containing z_0, and we may assume that i is the largest index with $z_0 \in G_i$. It follows that $z_i \in S_i$, and as in the previous case, no rectangle is attached to z_i in order to obtain the spaces S_k, $k > i$. Hence, we may write again that $z_i \in S_k$ for all $k \geq i$ and we may identify the point $z \in S_\infty$ with z_i, thus saying that z lies in S_n for all sufficiently large n.

Summarizing, if $z \in S_\infty$, then z lies in (or rather projects to) S_n for all sufficiently large n. With this in mind, we may also identify S_∞ with the union of $\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} G_i$ and the rectangles attached to each G_i, after proper identifications.

Remark 3.1 If we had not assumed that the degree of T_∞ is uniformly bounded, then we would not be able to represent the inverse limit S_∞ as above, since it could contain points that do not lie on $\mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} G_i$ or in any rectangle attached to the tripods.

Proposition 3.2 *If the heights $\{h_n\}_{n\in\mathbb{N}}$ of the rectangles attached to the tripods are chosen so that*

$$\sum_{i=1}^{\infty} h_i < \infty$$

then the inverse limit (S_∞, d_∞) of (S_n, d_n) is a complete metric space.

Proof The metric of d_∞ is defined as the limit of d_n. To ensure that this exists, we fix $x, y \in S_\infty$, so $x, y \in S_n$ for all sufficiently large n, and note that

$$d_{n-1}(x, y) = d_{n-1}(P_n(x), P_n(y)) \leq d_n(x, y)$$

if n is sufficiently large, by the Lipschitz property $(G1)$ of the projections; here we have considered the identifications of x, y with $P_n(x), P_n(y)$, respectively, for large n. Hence, the sequence $d_n(x, y)$ converges, possibly to ∞. To exclude this possibility, we apply repeatedly property $(G2)$ together with the compatibility $(G4)$ and obtain

$$d_n(x, y) \leq d_{n-1}(P_n(x), P_n(y)) + 6h_n \leq |\tilde{x} - \tilde{y}| + 6\sum_{i=1}^{n} h_i,$$

where $\tilde{x}, \tilde{y}$ are the projections of x, y, respectively, to the plane. By assumption, the last sum is convergent, so our claim is proved.

Now, we show that (S_∞, d_∞) is complete. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in S_∞. If x_k lies infinitely often in a given rectangle $E \subset S_i$, then x_k has a convergent subsequence, since the metric d_∞, restricted on E, is isometric to the Euclidean metric, by the limiting version of $(G8)$. Hence, we may assume that x_k either lies in $\mathbb{R}^2 \setminus \bigcup_{i=1}^\infty G_i$ infinitely often, or it has a subsequence, still denoted by x_k, that lies in distinct rectangles $E_k \subset S_{i_k}$.

In the first case, x_k (or rather its projection to the plane) converges in the Euclidean metric to a point $x \in \mathbb{R}^2$, because the projection of S_∞ to the plane is 1-Lipschitz. If $x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^\infty G_i$, then the line segment $[x_k, x] \subset \mathbb{R}^2$ does not intersect any given tripod G_i for sufficiently large k. Hence, for each $i_0 \in \mathbb{N}$ there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ the segment $[x_k, x]$ does not intersect $G_1, \dots, G_{i_0}$. For $n \geq i_0 + 1$, by a repeated application of $(G2)$, we have

$$\begin{aligned} d_n(x, x_k) &\leq d_{n-1}(x, x_k) + 6h_n \\ &\leq d_{i_0}(x, x_k) + 6\sum_{i=i_0+1}^{n} h_i. \end{aligned}$$

Now, applying repeatedly $(G5)$, we note that $d_{i_0}(x, x_k) = |x - x_k|$, which is the length of the geodesic $[x_k, x]$ in the plane. Hence,

$$d_n(x, x_k) \leq |x - x_k| + 6\sum_{i=i_0+1}^{\infty} h_i,$$

and passing to the limit we have

$$d_\infty(x, x_k) \le |x - x_k| + 6 \sum_{i=i_0+1}^{\infty} h_i$$

for all $k \ge k_0$. This implies that $d_\infty(x, x_k) \to 0$, as desired.

On the other hand, if x lies on a tripod, and x is a vertex of T_∞, then there exists $N_x \in \mathbb{N}$ such that the degree of x in T_n is equal to $N_x \le N_0$ for all sufficiently large $n \in \mathbb{N}$; recall that N_0 is a uniform bound on the degree of the graphs T_n. For a small $r > 0$ the edges of tripods G_n that meet at x split the ball $B(x, r) \subset \mathbb{R}^2$ into N_x components. Each of these components contains x in its boundary and is a circular sector if r is sufficiently small, so in particular, it is convex. One of these components, say V, must contain infinitely many points x_k, and thus all, after passing to a subsequence. We let G_{n_0} be a tripod such that $x \in G_{n_0}$, and also G_{n_0} contains one of the edges that bounds the sector V. We let x_0 be the point of S_{n_0} that projects to x, is accessible from V, and lies on the boundary of a rectangle $E \sim G_{n_0}$. We claim that $x_k \to x_0$ in d_∞. We look at the open segments (x_k, x) and note that they do not intersect G_{n_0} or any other tripod G_i infinitely often. Arguing as before, we have that for all $i_0 \in \mathbb{N}$ there exists $k_0 \in \mathbb{N}$ such that

$$d_\infty(x_0, x_k) \le |x - x_k| + 6 \sum_{i=i_0+1}^{\infty} h_i,$$

for $k \ge k_0$. This shows convergence. The case that $x \in G_{n_0}$ but x is not a vertex of T_∞ is treated in the same way, and here $B(x, r) \setminus G_{n_0}$ contains only two components, provided that r is small.

The last case is to assume that $x_k \in E_k \sim G_{i_k}$, where E_k are rectangles of height h_{i_k}, and $h_{i_k} \to 0$, since the rectangles E_k are distinct. In this case, we can find a point y_k in the "base" of the rectangle E_k that is glued to $\mathbb{R}^2$ such that

$$d_\infty(x_k, y_k) \le h_{i_k},$$

by $(G8)$. Hence, y_k is also Cauchy in d_∞, and it suffices to show that it converges, because $h_{i_k} \to 0$. Arbitrarily close to each y_k we can find a point $z_k \in \mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} G_i$, with $d_\infty(y_k, z_k) \le 1/k$. This is justified as in the previous paragraph and using the observation that $\bigcup_{i=1}^{\infty} G_i$ has empty interior (e.g., using the Baire category theorem). Now, the convergence of z_k is obtained by the previous case. □

Remark 3.3 The preceding proof, together with the Lipschitz property ($G1$), show that if we restrict the metric d_∞ of S_∞ to $\mathbb{R}^2 \setminus \bigcup_{i=1}^\infty G_i$, then it is topologically equivalent to the Euclidean metric, in the sense that a sequence $x_k \in \mathbb{R}^2 \setminus \bigcup_{i=1}^\infty G_i$ converges to a point $x_0 \in S_\infty$ if and only if the projections x_k converge to the projection of x_0 in the Euclidean metric.

Remark 3.4 There exist natural projections from S_∞ onto S_n and $\mathbb{R}^2$ for all $n \in \mathbb{N}$. Analogs of properties ($G1$), ($G4$) and ($G7$) also hold for these projections. Moreover, property ($G8$) is true for S_∞, so the metric d_∞ restricted on a rectangle $E \sim G_i$ for $i \in \mathbb{N}$ is isometric to the Euclidean metric on E.

Proposition 3.5 *If the heights $\{h_n\}_{n\in\mathbb{N}}$ are chosen as in Proposition 3.2, then for each $p \in S_\infty$ there exist points $p_n \in S_n$ such that the sequence of spaces (S_n, d_n, p_n) converges to (S_∞, d_∞, p) in the pointed Gromov–Hausdorff sense of Definition 2.10.*

Proof Let $p \in S_\infty$. Then, by the representation of S_∞ that we gave, p lies in S_n for all sufficiently large n, so we set $p_n = p \in S_n$ for, say, $n \geq n_0$. We fix $n \geq n_0$ and $r, \varepsilon > 0$ and define $f : S_n \to S_\infty$ to be any right inverse of the projection from S_∞ onto S_n, which is surjective. Clearly, $f(p_n) = p$.

Let $\tilde{x}, \tilde{y} \in S_n$ be arbitrary. Also, consider arbitrary lifts $x, y \in S_\infty$ of $\tilde{x}, \tilde{y}$, respectively. Then $x, y \in S_m$ for sufficiently large $m \in \mathbb{N}$. By ($G1$) and ($G2$), for sufficiently large $m \geq n$ we have

$$0 \leq d_m(x, y) - d_n(\tilde{x}, \tilde{y}) \leq 6 \sum_{i=n+1}^{\infty} h_i.$$

Letting $m \to \infty$ yields

$$0 \leq d_\infty(x, y) - d_n(\tilde{x}, \tilde{y}) \leq 6 \sum_{i=n+1}^{\infty} h_i. \tag{3.1}$$

Since $\sum_{i=n+1}^\infty h_i \to 0$ as $n \to \infty$, this shows that $\mathrm{dis}(f)$ can be made less than ε, if n is sufficiently large (independent of $\tilde{x}, \tilde{y}$). Hence, condition (2) of Definition 2.10 holds.

Finally, we check condition (3). If $d_\infty(p, x) < r - \varepsilon$, then the projection $\tilde{x}$ of x to S_n satisfies $d_n(p_n, \tilde{x}) \leq d_\infty(p, x) < r - \varepsilon < r$, for $n \geq n_0$; see property ($G1$) and Remark 3.4. This implies that $\tilde{x} \in B_{d_n}(p_n, r)$. On the other hand, $f(\tilde{x})$ is a lift of $\tilde{x}$ and the points $f(\tilde{x})$ and x project to the same point of S_n. Thus, by (3.1) we obtain

$$d_\infty(f(\tilde{x}), x) \leq 6 \sum_{i=n+1}^{\infty} h_i.$$

If n is sufficiently large (independent of x), then the above is less than ε, so x is contained in the ε-neighborhood of $f(B_{d_n}(p_n, r))$, as desired. □

Remark 3.6 The preceding results, Propositions 3.2 and 3.5, do not depend on the geometry of the tripods G_n. Hence, they also hold if the tripods G_n, $n \in \mathbb{N}$, are not a priori given to us, but they are constructed inductively, based on the previous tripods. In fact, once we have the tripods $G_1, \dots, G_{n-1}$ and the corresponding heights $h_1, \dots, h_{n-1}$, then the tripod G_n can depend both on $G_1, \dots, G_{n-1}$ **and** the heights $h_1, \dots, h_{n-1}$! In particular, in this more general setting, one can still obtain a limiting space (S_∞, d_∞), as long as $\sum_{i=1}^{\infty} h_i < \infty$, and the degree of T_∞ is uniformly bounded. A different way to think of that is as follows. Suppose we have a "machine" or an algorithm that produces the tripod G_n, based on $G_1, \dots, G_{n-1}$ and $h_1, \dots, h_{n-1}$. Then we can choose h_n to be sufficiently small and feed this back into the "machine" to obtain the next tripod G_{n+1}. The point of this remark will be evident in Sect. 5.3, where a sequence of tripods is constructed inductively and is used to build a flap-plane.

3.1.4 Quasisymmetric embedding of S_∞ into the plane

The main result in Sect. 3 is the next theorem.

Theorem 3.7 *If the heights $\{h_n\}_{n\in\mathbb{N}}$ are chosen to be sufficiently small (depending on the tripods G_n), then there exists a quasisymmetry from (S_∞, d_∞) onto $\mathbb{R}^2$ with the Euclidean metric. Furthermore, (S_∞, d_∞) is Ahlfors 2-regular.*

Recall that a standing assumption is that the degree of the "graph" T_∞ is uniformly bounded. Again, the heights h_n have to satisfy the condition in Proposition 3.2, but this time we have more restrictions as it will be evident from the proof. Moreover, as in Remark 3.6, the choice of the heights is to be interpreted as follows: for each $n \in \mathbb{N}$ the height h_n has to be chosen to be sufficiently small, but only depending on the tripods $G_1, \dots, G_n$. Also, the tripod G_n is not necessarily a priori given to us, but it can depend on $G_1, \dots, G_{n-1}$ and the already chosen heights $h_1, \dots, h_{n-1}$. In the latter scenario, it is necessary that G_n is chosen so that the family $\{G_1, \dots, G_n\}$ has the property (G) for all $n \in \mathbb{N}$.

The proof of Theorem 3.7 is based on the Bonk–Kleiner Theorem:

Theorem 3.8 (Theorem 1.1, [7]). *Let (X, d) be an Ahlfors 2-regular metric space homeomorphic to the sphere S^2, equipped with the spherical metric. Then (X, d) is quasisymmetric to S^2 if and only if (X, d) is linearly locally connected.*

See Sect. 3.3 for the definition of linear local connectivity. This theorem gave an excellent criterion for quasisymmetric parametrizability of 2-dimensional surfaces, and a necessary and sufficient condition is still to be found. Since its publication, there have been some improvements and generalizations [28,38], and very recently two new proofs of the theorem were published [27,33], which give an alternative perspective to the problem of finding parametrizations of 2-dimensional surfaces. We direct the reader to [33] for more references and background on the problem of uniformization of 2-dimensional surfaces.

In fact, we will need a plane version of this theorem, which was proved by Wildrick:

Theorem 3.9 (Theorem 1.2, [38]). *Let (X, d) be an Ahlfors 2-regular and linearly locally connected metric space that is homeomorphic to the plane $\mathbb{R}^2$. If (X, d) is unbounded and complete then (X, d) is quasisymmetrically equivalent to $\mathbb{R}^2$ with the Euclidean metric.*

The statement is quantitative, in the sense that the distortion function of the quasisymmetry can be chosen to depend only on the constants associated to the Ahlfors 2-regularity and linear local connectivity.

In the next two sections we prove that the spaces (S_n, d_n) are Ahlfors 2-regular and linearly locally connected with uniform constants, under the assumptions of Theorem 3.7. The proof of Theorem 3.7 is completed in Sect. 3.4.

3.2 Ahlfors regularity

Recall that a metric space (X, d) is Ahlfors Q-regular for some $Q > 0$ if there exists a constant $C \geq 1$ such that for each $x \in X$ and for all $0 < r < \operatorname{diam}(X)$ we have

$$\frac{1}{C} r^Q \leq \mathcal{H}_d^Q(B_d(x, r)) \leq C r^Q, \tag{3.2}$$

where $\mathcal{H}_d^Q$ denotes the Hausdorff Q-measure.

In this section we prove, under the assumptions of Theorem 3.7, that the spaces (S_n, d_n) are Ahlfors 2-regular with uniform constants. Remark 3.6 also applies here, i.e., at the n-the stage the tripod G_n need not be given to us, but it can depend on the tripods $G_1, \dots, G_{n-1}$ and the heights $h_1, \dots, h_{n-1}$.

In fact, we only need to show the right inequality in (3.2). The left inequality will follow from right inequality and the linear local connectivity of (S_n, d_n) that is discussed in the next Sect. 3.3; see Proposition 3.12 and Proposition 3.13.

Proposition 3.10 *Assume that the degree of T_n is bounded by $N_0 > 0$, for all $n \in \mathbb{N}$. If the heights $\{h_n\}_{n\in\mathbb{N}}$ are chosen to be sufficiently small, then the spaces (S_n, d_n) are Ahlfors 2-regular, with uniform constants, depending only on N_0.*

As we remarked, we will only show the upper bound in (3.2). We denote the Hausdorff 2-measure of (S_n, d_n) by μ_n, and we use the ball notation $B_n(x, r)$, instead of $B_{d_n}(x, r)$.

Lemma 3.11 *Assume that the degree of T_n is bounded by $N_0 > 0$, for all $n \in \mathbb{N}$. If the heights $\{h_n\}_{n\in\mathbb{N}}$ are chosen to be sufficiently small, then there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$, $x \in S_n$, and $r > 0$ we have*

$$\mu_n(B_n(x, r)) \le cr^2.$$

The constant c depends only on N_0.

Proof Using $(G7)$, we write

$$\mu_n(B_n(x,r)) = \mu_0(P_{n,0}(B_n(x,r))) + \sum_{k=1}^{n} \mu_n\left(\bigcup_{E\sim G_k} B_n(x,r)\cap E\right), \tag{3.3}$$

where μ_0 is the Lebesgue measure in $\mathbb{R}^2$ and $P_{n,0}$ is the projection from S_n to $\mathbb{R}^2$. By the Lipschitz property $(G1)$ we have

$$\mu_0(P_{n,0}(B_n(x,r))) \le \mu_0(B(P_{n,0}(x), r)) = \pi r^2.$$

Also, if $E \sim G_k$ and $y \in B_n(x, r) \cap E$, then $B_n(x, r) \cap E \subset B_n(y, 2r) \cap E$. On the other hand, $B_n(y, 2r) \cap E$ is isometric to the intersection of a Euclidean ball of radius $2r$ with the Euclidean subset E by $(G8)$, therefore

$$\mu_n\left(\bigcup_{E\sim G_k} B_n(x,r)\cap E\right) \le 6\cdot 4\pi r^2. \tag{3.4}$$

Recall at this point that 6 rectangles are attached to a given tripod G_k.

If the projection of $B_n(x, r)$ to $\mathbb{R}^2$ intersects at most N_0 tripods $G_{i_1}, \dots, G_{i_N}$, $N \le N_0$, then by (3.3) we have

$$\mu_n(B_n(x,r)) \le \pi r^2 + N_0(6\cdot 4\pi r^2) = (\pi + 24N_0\pi)r^2.$$

This bound is independent of the choice of heights h_i and tripods G_i (cf. Remark 3.6).

We claim that we can choose inductively the height h_n of the rectangles $E \sim G_n$ depending only on the tripods $G_1, \dots, G_n$ such that whenever the projection of a ball $B_n(x,r)$ to the plane intersects some tripods $G_{i_1}, \dots, G_{i_{N_0}}, G_{i_{N_0+1}}$ with $i_1 < \dots < i_{N_0+1} \le n$, we have

$$\mu_n\left(\bigcup_{E \sim G_{i_{N_0+1}}} E\right) \le r^2, \tag{3.5}$$

and also

$$\mu_n\left(\bigcup_{E \sim G_{i+1}} E\right) \le \frac{1}{2}\mu_n\left(\bigcup_{E \sim G_i} E\right), \tag{3.6}$$

for all $i = 1, \dots, n-1$.

Assuming that, we finish the proof. Let $B_n(x,r)$ be an arbitrary ball in S_n, whose projection intersects the tripods $G_{i_1}, \dots, G_{i_{N_0}}, G_{i_{N_0+1}}, i_1 < \dots < i_{N_0+1} \le n$. Also, assume that these are the smaller possible such indices, namely there exists no $i \notin \{i_1, \dots, i_{N_0+1}\}$ with $i < i_{N_0+1}$ such that the projection of $B_n(x,r)$ intersects G_i. Then we have

$$\begin{aligned}
\sum_{k=1}^{n} \mu_n\left(\bigcup_{E \sim G_k} B_n(x,r) \cap E\right) &= \sum_{j=1}^{N_0} \mu_n\left(\bigcup_{E \sim G_{i_j}} B_n(x,r) \cap E\right) \\
&\quad + \mu_n\left(\bigcup_{E \sim G_{i_{N_0+1}}} B_n(x,r) \cap E\right) \\
&\quad + \sum_{i_{N_0+1} < k \le n} \mu_n\left(\bigcup_{E \sim G_k} B_n(x,r) \cap E\right) \\
&\le N_0 \cdot 24\pi r^2 + r^2 + \sum_{i_{N_0+1} < k \le n} \mu_n\left(\bigcup_{E \sim G_k} E\right),
\end{aligned}$$

where we used the bound (3.4) for the first term, and condition (3.5) for the second term. By condition (3.6), the last term is bounded by

$$\mu_n\left(\bigcup_{E\sim G_{i_{N_0+1}}} E\right)\cdot\sum_{k=1}^{\infty}\frac{1}{2^k}\leq r^2.$$

This concludes the proof, by (3.3), with constant $c=\pi+24N_0\pi+2$.

Now, we focus on our claim, which we will prove by induction on $n\in\mathbb{N}$. For $n=1,\dots,N_0$ we have nothing to show, since the projection of a ball $B_n(x,r)$ to $\mathbb{R}^2$ will always intersect at most N_0 tripods. We can also adjust the heights h_i, $i=1,\dots,N_0$, to be so small, depending on $G_1,\dots,G_{N_0}$, that (3.6) holds. We assume that the statements hold for some $n\in\mathbb{N}$, $n\geq N_0$, and consider the tripod G_{n+1} and the flap-plane (S_{n+1},d_{n+1}). We also choose h_{n+1} to be so small that (3.6) holds; note that by $(G8)$ the rectangles $E\sim G_i$, $i<n+1$, with metric d_{n+1} are isometric to Euclidean rectangles, so we only have to choose h_{n+1} to be small small enough so that (3.6) holds for the (last) index $i=n$. Later, we will make h_{n+1} even smaller in order to achieve (3.5).

If $B_{n+1}(x,r)$ has a projection to the plane that intersects $G_{i_1},\dots,G_{i_{N_0}}$, $G_{i_{N_0+1}}$, with $i_1<\dots<i_{N_0+1}\leq n+1$ then we split in two cases:

Case 1 $i_{N_0+1}\neq n+1$. If $P\colon S_{n+1}\to S_n$ denotes the natural projection, then in this case the projection of $B_n(P(x),r)$ to $\mathbb{R}^2$ also intersects $G_{i_1},\dots,G_{i_{N_0}},G_{i_{N_0+1}}$. This is essentially because the projection P from S_{n+1} to S_n is 1-Lipschitz by $(G1)$, and also the projections are compatible by $(G4)$. Now, using the induction assumption we obtain

$$\mu_n\left(\bigcup_{E\sim G_{i_{N_0+1}}} E\right)\leq r^2.$$

The measure μ_n, restricted to E, is identical to the Lebesgue measure, and also to μ_{n+1} by $(G8)$. This completes the proof of (3.5) in this case.

Case 2 $i_{N_0+1}=n+1$. Then there exist points $a_{i_j}\in G_{i_j}\subset\mathbb{R}^2$ for $j=1,\dots,N_0$ and a point $a_{n+1}\in G_{n+1}$ such that

$$|a_{i_j}-a_{n+1}|<2r$$

for all $j=1,\dots,N_0$. This is because the projection of $B_{n+1}(x,r)$ to $\mathbb{R}^2$ intersects the corresponding tripods, by assumption, and also it is 1-Lipschitz by $(G1)$.

We now introduce auxiliary vertices on G_{n+1} as follows. We partition each edge of G_{n+1} in finitely many edges such that the interior of each (new) edge of G_{n+1} does not contain any vertex of G_i, $i\leq n$, and also each (new) edge of G_{n+1} has one "free" vertex that does not lie on any G_i, $i\leq n$; this is possible because the tripods $\{G_i\}_{i\in\mathbb{N}}$ possess property (G) and in particular the intersection of G_{n+1} with the union of the tripods G_i, $i \leq n$, contains finitely many points. For $i \leq n$ we set $\delta_i > 0$ to be to be the minimum (Euclidean) distance of the (new) edges of G_{n+1} from G_i, excluding the edges of G_{n+1} that intersect G_i. We then set $\delta = \min_{1\leq i\leq n} \delta_i > 0$. The partitioning of the edges of G_{n+1} is only used to define δ in this proof, and is not supposed to alter the tripod G_{n+1} for any other consideration.

If $|a_{i_j} - a_{n+1}| \geq \delta$ for some $j \in \{1, \dots, N_0\}$, then we have $\delta \leq 2r$. Hence, if we set h_{n+1} to be so small (depending on δ) that

$$\mu_{n+1}\left(\bigcup_{E\sim G_{n+1}} E\right) \leq \delta^2/4 \leq r^2,$$

then we obtain the desired conclusion. Note that δ, and thus h_{n+1}, is chosen depending only on $G_1, \dots, G_{n+1}$.

If $|a_{i_j} - a_{n+1}| < \delta$ for all $j \in \{1, \dots, N_0\}$, then by the definition of δ this means that a_{n+1} necessarily lies on an edge e of G_{n+1} that intersects G_{i_j}, for all $j \in \{1, \dots, N_0\}$. However, the edge e by construction has a "free" vertex, so only one of the two vertices of the edge e can intersect tripods G_i, $i \leq n$. Hence, there are $N_0 + 1$ tripods (including G_{n+1}) meeting at a vertex of the edge e. This implies that there are $N_0 + 1$ edges of the graph T_{n+1}, meeting at a vertex of the edge e. We now have a contradiction to the assumption that the degree of the planar graph T_{n+1} is at most N_0. □

3.3 Linear local connectivity

A metric space (X, d) is *linearly locally connected* (LLC) if there exists a constant $c \geq 1$ such that for each ball $B_d(x, r)$ and for any two points $z, w \in X$ we have:

(i) If $z, w \in B_d(x, r)$, then there exists a continuum $F \subset B_d(x, cr)$ connecting z and w.
(ii) If $z, w \notin B_d(x, r)$, then there exists a continuum $F \subset X \setminus B_d(x, r/c)$, connecting z and w.

In this case, we say that X is c-LLC.

As we remarked earlier, the LLC property and the upper mass bound of Ahlfors regularity in the 2-dimensional setting can yield the lower mass bound:

Proposition 3.12 *Let (X, d) be a metric space homeomorphic to $\mathbb{R}^2$. If (X, d) has locally finite Hausdorff 2-measure and is c-LLC, then there exists a constant $c' > 0$ depending only on c such that*

$$\mathcal{H}^2_d(B_d(x, r)) \geq c' r^2$$

for all $x \in X$ *and* $0 < r < diam(X)$.

This is a folklore statement, and it is discussed in [33, Section 16]; see also [23, Corollary 1.4]. We give a quick proof for the sake of completeness.

Proof Let $x \in X$ and $r < \operatorname{diam}(X)$. Consider the function $\phi(z) = d(x, z)$, which is 1-Lipschitz. Let $y \notin B(x, r)$. Then for each $t \in (0, r)$ there exists a component K_t of $\phi^{-1}(t)$ that separates x from y [29, IV Theorem 26]. We claim that $\mathcal{H}^1_d(K_t) \geq C_1 r$ for all $t \in [r/4, r/2]$ and for a constant $C_1 > 0$ depending only on c. If this is the case, then by the co-area formula [1, Proposition 3.1.5], there exists a uniform constant $C_2 > 0$ such that we have

$$\mathcal{H}^2_d(B_d(x, r)) \geq C_2 \int_{r/4}^{r/2} \mathcal{H}^1_d(\phi^{-1}(t))\, dt \geq \frac{C_2 C_1}{4} r^2,$$

as desired.

To prove our claim, note that for any $w \in K_t$, $t \in [r/4, r/2]$, we have $x, y \notin B_d(w, r/4)$. By condition (ii) in the definition of linear local connectivity, it follows that there exists a continuum $F \subset X \setminus B_d(w, r/(4c))$ that connects x and y. However, any such continuum has to intersect K_t. It follows that K_t cannot be contained in $B_d(w, r/(4c))$ for any $w \in K_t$, hence $\mathcal{H}^1_d(K_t) \geq \operatorname{diam}(K_t) \geq r/(4c)$. Our claim is proved with $C_1 = (4c)^{-1}$. □

Note that each space (S_n, d_n) satisfies condition (i) with constant 1, since the space is endowed with its internal metric the distance between two points $x, y \in S_n$ is equal to the length of the shortest path between the two points.

Proposition 3.13 *Assume that the degree of* T_n *is bounded by* N_0*, for all* $n \in \mathbb{N}$*. If the heights* $\{h_n\}_{n \in \mathbb{N}}$ *are chosen to be sufficiently small, then the spaces* (S_n, d_n) *are c-LLC, with the constant c depending only on* N_0*.*

We remark, once again, that the choice of the heights is to be interpreted as follows: the height h_n of the rectangles attached to G_n has to be chosen to be sufficiently small, depending on $G_1, \dots, G_n$; see also the comments after Theorem 3.7.

We first prove a version of that proposition for the flap-plane corresponding single tripod G, with constants independent of the geometry of G.

Lemma 3.14 *Let* $G \subset \mathbb{R}^2$ *be a tripod and consider a flap-plane* $S = S(G)$*. If the height h of the rectangles* $E \sim G$ *is less than the width of E, then S is c-LLC for a universal constant* $c > 1$*.*

In particular, the constant c is independent of the lengths of the edges of G and of their angles.

Proof Note that we only have to prove condition (ii), i.e., for any ball $B_d(x,r)$ in S and any two points $z, w \in S\setminus B_d(x,r)$ there exists a continuum $F \subset B_d(x, r/c)$ connecting z and w, where $c > 1$ is a universal constant to be determined.

In fact, it suffices to show that there exists a constant $c > 1$ such that for each ball $B_d(x,r)$ and $z \notin B_d(x,r)$ there exists a polygonal path $\gamma_z \subset S\setminus B_d(x, r/c)$ that connects z to a point z', whose projection to $\mathbb{R}^2$ lies outside $B(\tilde{x}, r/c)$. Here $\tilde{x}$ is the projection of x to $\mathbb{R}^2$. Indeed, if this is true, then the same statement holds for a point $w \notin B_d(x,r)$, and there exists a polygonal path γ_w and a point w' with the corresponding properties. One can then connect the projections $\tilde{z}'$ and $\tilde{w}'$ with a polygonal path $\tilde{\gamma} \subset \mathbb{R}^2\setminus B(\tilde{x}, r/c)$. By properties $(G3)$ and $(G1)$, the path $\tilde{\gamma}$ lifts to a polygonal path γ that connects z' and w' and lies outside $B(x, r/c)$. Then the concatenation of γ_z, γ, and γ_w yields the desired path in the LLC (ii) condition.

Assume that $z \notin B_d(x,r)$. We denote by $P\colon S \to \mathbb{R}^2$ the projection of S to the plane. Also, for a point $y \in S$ we use the notation $\tilde{y} = P(y)$. We now split the argument in two cases:

Case 1 $r \geq 12h$. Then the projected point $\tilde{z}$ does not lie in $B(\tilde{x}, r/2)$. Indeed, if $|\tilde{x} - \tilde{z}| < r/2$, then by the property $(G2)$ we have

$$d(x,z) \leq |\tilde{x} - \tilde{z}| + 6h < r/2 + r/2 = r,$$

a contradiction. Hence, we can take $z' = z$, and $c \geq 2$.

Case 2 $r < 12h$. We set $r_1 = r/48 < h/4$.

We connect z to x with a geodesic $\gamma \subset S$. If γ (or rather its projection $\tilde{\gamma}$) does not intersect G, then by $(G5)$ it projects isometrically to a geodesic from $\tilde{z}$ to $\tilde{x}$, which has to be a straight line segment. Then

$$|\tilde{x} - \tilde{z}| = d(x,z) > r$$

so $\tilde{z} \notin B(\tilde{x}, r)$, and we can set $z' = z$ and $c \geq 1$.

If $\tilde{\gamma}$ does intersect G, we consider y to be the first entry point of γ into a rectangle $E \sim G$ as it travels from z to x; we could have $y = z$ in case z lies in a rectangle attached to G. In particular, the segment $[z, y] \subset \gamma_z$ does not intersect the rectangles attached to G, except possibly at the point y, and projects isometrically to a line segment $[\tilde{z}, \tilde{y}]$ in $\mathbb{R}^2$. If $y \in B_d(x, 5r_1)$, then using the 1-Lipschitz property $(G1)$ of the projection and also $(G5)$ we have

$$\begin{aligned} |\tilde{z} - \tilde{x}| &\geq |\tilde{z} - \tilde{y}| - |\tilde{y} - \tilde{x}| \geq d(z,y) - d(y,x) \\ &\geq d(z,x) - 2d(y,x) > r - 10r_1 > r_1, \end{aligned}$$

since $r = 48r_1 > 11r_1$. It follows that $\tilde{z}$ lies outside $B(\tilde{x}, r_1)$. Hence, we may take $z' = z$ and $c \geq 48$.

Finally, we have to treat the case that $y \in E$ but $y \notin B_d(x, 5r_1)$. We claim that y can be connected to a point $y' \in E$ with a polygonal path $\gamma_y \subset E$ outside $B_d(x, r_1)$ such that y' projects to a point $\tilde{y}'$ lying outside $B(\tilde{x}, r_1)$. In this case, note that we also have $[z, y] \cap B_d(x, r_1) = \emptyset$, since $y \notin B_d(x, r_1)$ and $[z, y] \subset \gamma_z$, where γ_z is a geodesic from z to x. Then one can concatenate the path $[z, y]$ with γ_y to obtain the desired polygonal path. Here, we have $z' = y'$ and $c \geq 48$.

We now prove our last claim. If $B_d(x, r_1) \cap E = \emptyset$, then we connect y with a polygonal path in E to a point $y' \in E$, whose projection lies outside $B(\tilde{x}, r_1)$. This can be done because E projects onto a line segment of length equal to the width of E, and thus greater than the height h, by assumption. On the other hand, the ball $B(\tilde{x}, r_1)$ has diameter $2r_1 < h/2$. Next, assume that $B_d(x, r_1) \cap E \neq \emptyset$ and that the intersection contains a point $a \in E$. We have

$$y \notin B_d(x, 5r_1) \supset B_d(a, 4r_1) \supset B_d(a, 2r_1) \supset B_d(x, r_1).$$

The metric d is isometric to the Euclidean metric when restricted to E by $(G8)$, hence $B_d(a, 2r_1) \cap E$ is the intersection of a round ball with the rectangle E. Since $2r_1 < h/2$, the ball $B_d(a, 2r_1)$ cannot intersect both the top and bottom "long" sides of E. This implies that the set $E \backslash B_d(a, 2r_1)$ has at most two connected components, one of which contains a "long" side $\mathcal{E}$ of E, with length equal to the width of E, and thus greater than the height h.

If $E \backslash B_d(a, 2r_1)$ has only one component then it is path connected. In this case, the point $y \in E \backslash B_d(a, 2r_1)$ can be connected with a polygonal path $\gamma_y \subset E \backslash B_d(a, 2r_1)$ to a point $y' \in E$, whose projection to the plane lies outside $B(\tilde{x}, r_1)$. Again, this is because the set $E \backslash B_d(a, 2r_1)$ projects onto an interval in $\mathbb{R}^2$ whose length is larger than $h > 4r_1 > \operatorname{diam}(B(\tilde{x}, r_1))$.

The scenario in which $E \backslash B_d(a, 2r_1)$ has two components can only occur if $B_d(a, 2r_1)$ intersects two adjacent sides of the rectangle E. The point y then has to lie in the "large" component that also contains the side $\mathcal{E}$ of E. Indeed, the other component is contained in $B(a, 4r_1)$, hence it cannot contain y. Now, as before, we can connect y with a polygonal path outside $B_d(a, 2r_1) \supset B_d(x, r_1)$ to a point y' with the desired property.

Summarizing, we proved that S is c-LLC with $c = 48$, provided that h is smaller than the length of the edges of G. □

Remark 3.15 The proof of Lemma 3.14 shows that one may obtain the following stronger conclusion which implies that the space is c-LLC:
There exists a constant $c > 1$ such that for each ball $B_d(x, r)$ and for each $z \notin B_d(x, r)$ there exists a path γ_z outside $B_d(x, r/c)$ that connects z to a point z', whose projection $\tilde{z}'$ to the plane lies outside $B(\tilde{x}, r/c)$. The path γ_z can be taken to be polygonal.

Next, we have a version of the previous lemma for $N \leq N_0$ tripods.

Lemma 3.16 *Let* $\{G_1, \dots, G_N\}$ *be a family of tripods possessing property* (G) *and suppose that* $N \leq N_0$. *Consider a flap-plane* $S = S(G_1, \dots, G_N)$. *If the height* h_i *of each rectangle* $E \sim G_i$ *is less than the width of* E *for all* $i \in \{1, \dots, N\}$, *then the flap-plane* S *is* c_0*-LLC, with constant* c_0 *depending only on* N_0.

In fact, Remark 3.15 also applies here. It is important in this lemma that the height h_i can be chosen to depend only on G_i and not on G_j for $i \neq j$. In particular, we can set h_i to be less than smallest among the lengths of the edges of G_i. Moreover, the dependence of c_0 on the number of tripods N_0 cannot be relaxed.

Proof We give the argument in case $N = 2$ and then sketch the almost identical induction argument required to prove the statement for arbitrary $N \leq N_0$. Assume that we have two tripods G_1 and G_2, and let $h_1, h_2 > 0$ be smaller than the length of each edge of G_1, G_2, respectively. Without loss of generality, we assume that $h_1 \leq h_2$.

Consider a flap-plane $S = S(G_1, G_2)$ with metric d, and let $\Sigma = S(G_2)$ with metric σ be the flap-plane that arises by collapsing (or projecting) in S the rectangles $E \sim G_1$ to the plane. Also, consider the natural projection $P^*\colon S \to \Sigma$. For a point $x \in S$ we denote $x^* = P^*(x)$.

As remarked in the beginning of the proof of Lemma 3.14 and also in Remark 3.15, it suffices to show that for each ball $B_d(x, r)$ and $z \notin B_d(x, r)$ there exists a path $\gamma_z \subset S \setminus B_d(x, r/c_0)$ that connects z to a point z', whose projection to $\mathbb{R}^2$ lies outside $B(\tilde{x}, r/c_0)$. Here, $c_0 > 1$ is a constant that depends only on N_0 and, as usual, $\tilde{x}$ denotes the projection of x to the plane.

As in the proof of Lemma 3.14 we split in two cases. If $r \geq 12h_1$, then we have $z^* \in \Sigma \setminus B_\sigma(x^*, r/2)$. Indeed, otherwise, by $(G2)$ we would have

$$d(x, z) \leq \sigma(x^*, z^*) + 6h_1 < r/2 + r/2 = r,$$

a contradiction. By Lemma 3.14 and Remark 3.15 there exists a universal constant $c > 1$ and there exists a polygonal path γ^* outside $B_\sigma(x^*, r/2c)$ that connects z^* to a point $(z^*)'$, whose projection to the plane lies outside $B(\tilde{x}, r/2c)$. Using $(G1)$ and $(G3)$, we can lift γ^* to a polygonal path $\gamma \subset S \setminus B_d(x, r/2c)$ that connects z to a point z', whose projection to the plane agrees with the projection of $(z^*)'$ (by compatibility $(G4)$), and therefore lies outside $B(\tilde{x}, r/2c)$. Hence, we may choose $c_0 \geq 2c$.

If $r < 12h_1$, then we also have $r < 12h_2$. Then the same argument as in Case 2 of the proof of Lemma 3.14 can be used to obtain the conclusion and here we only need to choose $c_0 \geq 48$. Summarizing, one has to choose $c_0 = \max\{2c, 48\}$.

Assume that the statement holds for any family of $N - 1 \leq N_0$ tripods $G_1, \dots, G_{N-1}$ satisfying the assumptions. Namely, there exists a constant

c depending only on N_0 such that any flap-plane $S = S(G_1, \dots, G_{N-1})$ with the correct choice of heights is c-LLC and satisfies the condition of Remark 3.15. We now consider $N \le N_0$ tripods $G_1, \dots, G_N$ and a flap-plane $S = S(G_1, \dots, G_N)$ with metric d as in the statement. By reordering the tripods, we may assume that $h_1 \le \dots \le h_N$. Let Σ with metric σ be the flap-plane arising by collapsing all rectangles $E \sim G_1$ to the plane. Then $\Sigma = S(G_2, \dots, G_N)$ and it satisfies the condition of Remark 3.15, by the induction assumption.

We consider $z \in S \backslash B_d(x, r)$ and split in two cases. If $r \ge 12h_1$, then $z^* \in \Sigma \backslash B_\sigma(x^*, r/2)$, where $z^* = P^*(z)$, and $P^* \colon S \to \Sigma$ is the projection. By the induction assumption it follows that z^* can be connected with a path $\gamma^* \subset \Sigma \backslash B_\sigma(x^*, r/2c)$ to a point $(z^*)'$, whose projection to the plane lies outside $B(\tilde{x}, r/2c)$. Lifting the path γ^* to S yields the desired path. Hence, it suffices to choose the LLC constant to be $c_0 \ge 2c$.

If $r < 12h_1$ then in fact $r < 12h_i$ for all $i \in \{1, \dots, N\}$. Thus, the argument in Case 2 of the proof of Lemma 3.14 can be used, and the LLC constant has to be $c_0 \ge 48$. Summarizing, one has to choose $c_0 = \max\{2c, 48\}$, which depends only on N_0, since by assumption c depends only on N_0. In fact, one can choose $c_0 = 2^{N_0 - 1}48$. □

Proof of Proposition 3.13 The metric of $S_n = S(G_1, \dots, G_n)$ is denoted by d_n and the ball around x of radius r will be denoted by $B_n(x, r)$.

We argue by induction on n. We claim that for each $n \in \mathbb{N}$ there exists a constant $C_n > 1$, increasing and uniformly bounded in n such that if the height h_n is chosen to be small enough depending on $G_1, \dots, G_n$ and also smaller than the width of the rectangles $E \sim G_n$, then the following holds:
whenever $z \notin B_n(x, r)$ for some $x \in S_n$ and $r > 0$, we can connect z to a point z' with a polygonal path $\gamma \subset S_n \backslash B_n(x, r/C_n)$ such that the projection $\tilde{z}'$ of z' to $\mathbb{R}^2$ lies outside $B(\tilde{x}, r/C_n)$.
This suffices by Remark 3.15, and shows that S_n is C_n-LLC. Since C_n is bounded above by a constant C, it follows that S_n is C-LLC for all $n \in \mathbb{N}$, which is the desired conclusion.

Now, we focus on proving our claim. For $n = 1$ the statement holds with the constant C_1 given by Lemma 3.16, provided that h_1 is sufficiently small depending on G_1. We assume that the claim holds for $S_1, \dots, S_n$, so, in particular, the height h_i of each rectangle $E \sim G_i$ has been chosen to be less than the width of E for $i \in \{1, \dots, n\}$. Our goal is to choose the height h_{n+1} of the rectangles $E \sim G_{n+1}$ so that our claim holds. To begin with, we choose h_{n+1} to be smaller than the width of all rectangles $E \sim G_{n+1}$, and later we will choose it to be even smaller. Consider a ball $B_{n+1}(x, r) \subset S_{n+1}$ and $z \notin B_{n+1}(x, r)$. We split into two main cases:

Case 1 The projection of $B_{n+1}(x, r)$ to the plane does not intersect both G_{n+1} and $\bigcup_{i<n+1} G_i$.

Assume first that the projection of $B_{n+1}(x,r)$ to the plane intersects only G_{n+1}. We denote by $\Sigma = S(G_{n+1})$ the flap-plane that arises by collapsing all rectangles $E \sim G_i$, $i < n+1$, to the plane. Also, we denote by $P^*\colon S(G_1,\dots,G_{n+1}) \to \Sigma$ the natural projection, and by σ the metric of Σ. An application of $(G6)$ yields $(P^*)^{-1}(B_\sigma(x^*,r)) = B_{n+1}(x,r)$, where $x^* = P^*(x)$. We now have that $z^* = P^*(z) \notin B_\sigma(x^*,r)$, hence, by Lemma 3.16 and Remark 3.15, there exists a polygonal path $\gamma^* \subset \Sigma \setminus B_\sigma(x^*, r/C_1)$ that connects z^* to a point $(z^*)'$, whose projection to $\mathbb{R}^2$ lies outside $B(\tilde{x}, r/C_1)$. Using $(G3)$ and the 1-Lipschitz property $(G1)$, we lift the path γ^* under P^* to a polygonal path $\gamma \subset S_{n+1}\setminus B_{n+1}(x, r/C_1)$ that connects z to a point z', whose projection to $\mathbb{R}^2$ is the same as the projection of $(z^*)'$ to $\mathbb{R}^2$ (by compatibility $(G4)$), so it lies outside $B(\tilde{x}, r/C_1)$. The constant C_{n+1} will be chosen later so that $C_{n+1} \geq C_n \geq C_1$. Hence, γ lies outside $B_{n+1}(x, r/C_{n+1})$ and the projection of z' to the plane lies outside $B(\tilde{x}, r/C_{n+1})$, as desired.

Next, assume that the projection of $B_{n+1}(x,r)$ to the plane intersects only $\bigcup_{i<n+1} G_i$. We denote here by P^* the projection of S_{n+1} to S_n. Then by $(G6)$ we have $(P^*)^{-1}(B_n(x^*,r)) = B_{n+1}(x,r)$. Since $z^* \notin B_n(x^*,r)$, by the induction assumption there exists a polygonal path $\gamma^* \subset S_n \setminus B_n(x^*, r/C_n)$ that connects z^* to a point $(z^*)'$, whose projection to $\mathbb{R}^2$ lies outside $B(\tilde{x}, r/C_n)$. Lifting this path and noting that $C_{n+1} \geq C_n$, as before, yields again the desired path γ and point z'.

Case 2 The projection of $B_{n+1}(x,r)$ to the plane intersects both G_{n+1} and $\bigcup_{i<n+1} G_i$. Let $J \subset \{1,\dots,n\}$ be the set of indices j such that the projection of $B_{n+1}(x,r)$ to the plane intersects G_j.

Assume first that $\#J \leq N_0 - 1$ (here N_0 is by assumption the bound on the degree of the graph T_{n+1}), and let $\Sigma = S(\{G_{n+1}\} \cup \{G_j : j \in J\})$ with metric σ be the flap-plane arising by collapsing the rectangles $E \sim G_i$, $i \notin J$, $i \neq n+1$, to the plane. We note that we can apply Lemma 3.16 to Σ, since the heights of the rectangles attached to the corresponding tripods are smaller than the widths, by the induction assumption and the choice we have made for h_{n+1}. Since $B_{n+1}(x,r)$ (or rather its projection to $\mathbb{R}^2$) intersects only tripods that are also "present" in Σ, we can use as before property $(G6)$ and path-lifting to reduce the statement to Σ. By Lemma 3.16 and Remark 3.15, the conclusion holds in Σ with the constant C_1 given by Lemma 3.16.

Assume now that $\#J \geq N_0$. Then there exist points $a_j \in G_j$, $j \in J$, and a point $a_{n+1} \in G_{n+1}$ such that

$$|a_j - a_{n+1}| < 2r$$

for all $j \in J$. This follows from the 1-Lipschitz property $(G1)$ of the projections.

As in the proof of Lemma 3.11, we introduce auxiliary vertices on G_{n+1} as follows. We partition each edge of G_{n+1} in finitely many edges such that the interior of each (new) edge of G_{n+1} does not contain any vertex of G_i, $i \leq n$, and also each (new) edge of G_{n+1} has one "free" vertex that does not lie on any G_i, $i \leq n$. For $i \leq n$ we set $\delta_i > 0$ to be to be the minimum distance of the (new) edges of G_{n+1} from G_i, excluding the edges of G_{n+1} that intersect G_i. We then set $\delta = \min_{1 \leq i \leq n} \delta_i$. The partitioning of the edges of G_{n+1} is only used to define δ in this proof, and is not considered to alter the tripod G_{n+1} for any other consideration. Note that δ depends only on $G_1, \dots, G_{n+1}$.

As in the proof of Lemma 3.11, there has to exist some $j \in J$ such that $|a_j - a_{n+1}| > \delta$, since the degree of T_{n+1} is at most N_0. Hence, $\delta < 2r$. Now, for any number $\alpha \in (0, 1)$, we can choose the height h_{n+1} to be so small, depending only on N_0, δ, and α, that

$$B_{n+1}(x, r) \supset (P^*)^{-1}(B_n(x^*, \alpha r)), \tag{3.7}$$

where P^* denotes the projection from S_{n+1} onto S_n. Indeed, for any $y^* \in B_n(x^*, \alpha r)$ and any preimage $y \in (P^*)^{-1}(y^*)$ we have by $(G2)$

$$d_{n+1}(x, y) \leq d_n(x^*, y^*) + 6h_{n+1} < \alpha r + (1 - \alpha)\delta/2 < r,$$

provided that $h_{n+1} < (1 - \alpha)\delta/12$. (3.7) implies that $z^* \notin B_n(x^*, \alpha r)$, so by the induction assumption there exists a path $\gamma^* \subset S_n \backslash B_n(x^*, \alpha r/C_n)$ that connects z^* to a point $(z^*)' \notin B_n(x^*, \alpha r/C_n)$, whose projection to $\mathbb{R}^2$ lies outside $B(\tilde{x}, \alpha r/C_n)$. The path γ^* lifts to a path $\gamma \subset B_{n+1}(x, \alpha r/C_n)$, so our claim holds with $C_{n+1} = C_n/\alpha$. Now, we choose $\alpha = 1 - 1/(n+1)^2$, so

$$C_{n+1} = C_n \left(1 - \frac{1}{(n+1)^2}\right)^{-1} > C_n \geq C_1.$$

With this choice we have

$$C_{n+1} \leq \frac{C_1}{\prod_{i=1}^{\infty}(1 - 1/(i+1)^2)} =: C < \infty$$

for all $n \in \mathbb{N}$. Note that C depends only on C_1, and thus only on N_0. □

3.4 Proof of Theorem 3.7

We will use Theorem 3.9. We note first that the assumptions of the theorem are satisfied by the spaces (S_n, d_n) with uniform constants. Indeed, each of the flap-planes (S_n, d_n) is unbounded since the projection onto the plane is 1-Lipschitz by $(G1)$. Also, (S_n, d_n) is complete since it is obtained by attaching finitely many rectangles to the plane; cf. proof of Proposition 3.2. Furthermore, if the heights $h_1, \dots, h_n$ are chosen (inductively) to be sufficiently small, then by Propositions 3.10 and 3.13 we conclude that (S_n, d_n) is Ahlfors 2-regular and LLC with constants independent of n. We also choose the heights to be even smaller, if necessary, so that the conclusions of Propositions 3.2 and 3.5 hold.

Theorem 3.9 now yields for each $n \in \mathbb{N}$ a quasisymmetry f_n from (S_n, d_n) onto $\mathbb{R}^2$. Since the statement of the theorem is quantitative, we may assume that the distortion function η of f_n is independent of n. We would like to pass to a limiting quasisymmetry $f : S_\infty \to \mathbb{R}^2$. This will be obtained by applying Lemma 2.12, after normalizing the functions f_n.

Consider the limiting space (S_∞, d_∞), given by Proposition 3.2. By Proposition 3.5, for a fixed point $p \in S_\infty$ we may choose points $p_n \in S_n$ such that the sequence (S_n, d_n, p_n) converges to the space (S_∞, d_∞, p) in the pointed Gromov–Hausdorff sense of Definition 2.10.

Since all of the spaces S_n are Ahlfors 2-regular with uniform constants, it follows that they are uniformly doubling; see comments after Definition 2.10. For each $n \in \mathbb{N}$ we consider a point $x_n \in S_n$ such that $d_n(p_n, x_n) = 1$; recall that the space S_n is a length space. By postcomposing f_n with a Möbius transformation of $\mathbb{R}^2$, we may obtain a sequence $g_n : S_n \to \mathbb{R}^2$ such that $g_n(p_n) = 0$ and $g_n(x_n) = 1$ for all $n \in \mathbb{N}$. The functions g_n will still be η-quasisymmetric, since the distortion function is not affected under compositions with scalings and translations. Lemma 2.12 (with $Y_n \equiv \mathbb{R}^2$) now yields a subsequence of g_n that converges to an η-quasisymmetry $g : S_\infty \to \mathbb{R}^2$. By Lemma 2.11 it also follows that S_∞ is Ahlfors 2-regular. □

4 The continuous case

In this section we prove first the non-removability of the gasket for continuous $W^{1,2}$ functions (Theorem 1.6) and then the non-removability of homeomorphic copies of the gasket (Theorem 1.7). Also, we include proofs of the general statements in Theorem 1.3 and Proposition 1.4, regarding the (non)-removability of sets of positive measure.

4.1 Terminology and geometry of the gasket

We first recall the definition of the Sierpiński gasket, introduce some terminology, and discuss its combinatorial properties.

The Sierpiński gasket is constructed as follows. We consider an equilateral triangle of sidelength 1 and subdivide it into four equilateral triangles of sidelength $1/2$. After removing the middle triangle, we proceed inductively with subdividing each of the remaining three triangles into four equilateral triangles of sidelength $1/2^2$, and so on. The remaining compact set K is the Sierpiński gasket; see Fig. 1. From the definition it is immediate that K has area zero. Indeed, at the n-th step of the construction K is contained in the union of 3^n equilateral triangles of sidelength $1/2^n$, hence

$$m_2(K) \leq 3^n \cdot \frac{\sqrt{3}}{4}\frac{1}{4^n},$$

which converges to 0 as $n \to \infty$. We will assume in what follows that $K \subset B(0,2) \subset \mathbb{R}^2$.

We call *w-triangles* the complementary triangles of K that are removed in each step. Making abuse of terminology we also call the unbounded component of $\mathbb{R}^2 \setminus K$ a w-triangle of sidelength 1. In the construction of K, at each step we remove a *central* w-triangle W_0 from an equilateral triangle V_0 having double the sidelength of W_0, after subdividing V_0 into four equilateral triangles. We call *v-triangles* the triangles arising as V_0. w-triangles and v-triangles are by definition open sets. Hence, using the previous notation $V_0 \setminus \overline{W_0}$ is the union of three v-triangles. We say that the *level* of a w-triangle W_0 is equal to n if the sidelength of W_0 is equal to 2^{-n}. In particular, the unbounded w-triangle has level 0, and the central w-triangle of the first step of the construction has level 1. For $n \geq 1$ there exist 3^{n-1} w-triangles of level n. Similarly, we say that the level of a v-triangle V_0 is equal to n if its sidelength is 2^{-n}. Note that there exists one v-triangle of level 0 and 3^n v-triangles of level n, for each $n \in \mathbb{N}$.

We denote by $\mathcal{W}$ be the collection of w-triangles, and

$$W_\infty := \bigcup_{W \in \mathcal{W}} \overline{W}.$$

Also, we use the notation K° for the points of K that do not lie on the boundary of any w-triangle, so in particular we have

$$K^\circ = K \setminus W_\infty.$$

In the proofs, if z is a point of the gasket, we will often have to distinguish between three cases, depending on whether z is a vertex of a w-triangle, or a point on an edge of a w-triangle but not a vertex, or none of the above, i.e., $z \in K^\circ$. In the first case that a point $z \in K$ is a vertex of a w-triangle, we say that z is *of vertex type*. In the second case that z lies on the boundary of a w-triangle but it is not a vertex, we say that it is *of edge type*.

Two w-triangles W_1, W_2 are *adjacent* if a vertex of W_1 lies on ∂W_2, or vice versa. Note that if W_1 has a vertex on ∂W_2 then the level of W_2 is strictly smaller than that of W_1, i.e., W_2 is a strictly larger triangle than W_1.

We now study some important properties of the combinatorics of the gasket. For each point $z \in K$ there exists a sequence $\{V_n\}_{n\in\mathbb{N}}$ of nested v-triangles with

$$\{z\} = \bigcap_{n=1}^{\infty} \overline{V}_n.$$

In fact, this sequence is unique if the following hold:

(i) z is not a vertex of a w-triangle, or it is a vertex of the unbounded w-triangle of level 0,
(ii) V_1 has level 0 (so it is the very first triangle in the construction of K), and V_n has level $n-1$ for $n \in \mathbb{N}$,
(iii) $V_{n+1} \subset V_n$ for $n \in \mathbb{N}$.

If z is a vertex of a w-triangle of level at least 1 then there are precisely two distinct sequences shrinking to z and satisfying (ii) and (iii).

The following two lemmas describe how the sequence $\overline{V}_n$ shrinks to the point z. In fact, the first lemma refers to v-triangles and the second lemma to w-triangles. We could have incorporated both lemmas in one, but this would complicate the statements, so we state them separately.

Lemma 4.1 *Let $\{V_n\}_{n\geq 1}$ be a nested sequence of v-triangles satisfying* (ii) *and* (iii)*, and converging to a point $z \in K$, in the sense that*

$$\{z\} = \bigcap_{n=1}^{\infty} \overline{V}_n.$$

In case z is a vertex of a w-triangle of level at least 1 *we also consider the other sequence $\{V_n'\}_{n\in\mathbb{N}}$ that is distinct from $\{V_n\}_{n\in\mathbb{N}}$ and converges to z. If z is a vertex of the unbounded w-triangle of level* 0 *we set $V_n' = V_n$ for $n \in \mathbb{N}$.*

(I) *If z is of vertex type, then there exist two (possibly non-distinct) w-triangles A and B with $z \in \partial A \cap \partial B$ such that for each $n \in \mathbb{N}$ the set $\overline{A} \cup \overline{B} \cup \overline{V}_n \cup \overline{V}_n'$ contains all sufficiently small open neighborhoods of z.*
(II) *If z is of edge type, then $z \in \partial V_n$ for all $n \in \mathbb{N}$ and moreover, there exists a w-triangle B with $z \in \partial B$ such that for each $n \in \mathbb{N}$ the set $\overline{B} \cup \overline{V}_n$ contains all sufficiently small open neighborhoods of z.*
(III) *If $z \in K^\circ$ then $z \notin \partial V_n$ for all $n \in \mathbb{N}$, so for each $n \in \mathbb{N}$ the set V_n contains all sufficiently small open neighborhoods of z.*

The following lemma describes essentially Fig. 5.

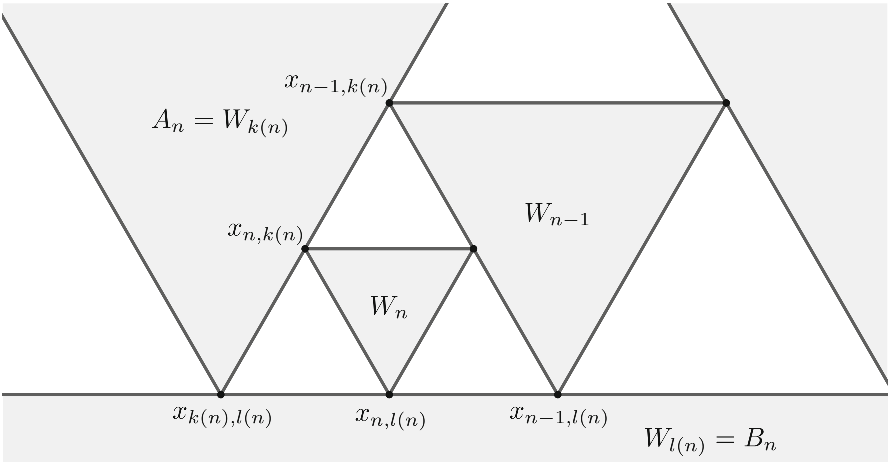

Fig. 5 A typical situation as described in Lemma 4.2(III)

Lemma 4.2 *Let* $\{V_n\}_{n\geq 1}$ *be a nested sequence of* v*-triangles satisfying* (ii) *and* (iii)*, and converging to a point* $z \in K$*, in the sense that*

$$\{z\} = \bigcap_{n=1}^{\infty} \overline{V}_n.$$

Also, for each $n \in \mathbb{N}$ *consider the central* w*-triangle* $W_n \subset V_n$ *of level* n*. Then, for* $n \geq 2$*,* W_n *has one vertex on* ∂W_{n-1}*, and two vertices on the boundaries of some* w*-triangles* A_n*,* B_n *(we could have* $A_n = B_n$ *if they are the unbounded* w*-triangle of level* 0*). Assume that the level of* B_n *is at most the level of* A_n *(so* B_n *is a larger triangle than* A_n*). Furthermore:*

- (I) *If* z *is a vertex of a* w*-triangle* A *(i.e.,* z *is of vertex type), then there exists another* w*-triangle* B *with* $z \in \partial A \cap \partial B$ *such that for all sufficiently large* $n \in \mathbb{N}$ *we have* $A_n = A$ *and* $B_n = B$*. In this case,* ∂V_n *is contained in* $\partial W_{n-1} \cup \partial A \cup \partial B$ *and contains* z*,* $\partial W_{n-1} \cap \partial A$*,* $\partial W_{n-1} \cap \partial B$*, and also the vertices of* $W_n \subset V_n$*.*
- (II) *If* z *is of edge type, then there exists a* w*-triangle* B *such that* $z \in \partial B$ *and* $B_n = B$ *for all sufficiently large* $n \in \mathbb{N}$*, but no other* w*-triangle* A *has the property that* $A_n = A$ *infinitely often. Moreover, for all sufficiently large* $n \in \mathbb{N}$*,* W_{n-1} *has a vertex on* ∂B *and a vertex on* ∂A_n*, and* A_n *has a vertex on* ∂B*. In fact, there exists a sequence* $\{k(n)\}_{n\in\mathbb{N}}$ *with* $k(n) \to \infty$ *as* $n \to \infty$ *such that* $A_n = W_{k(n)}$ *for all sufficiently large* $n \in \mathbb{N}$*. In this case,* ∂V_n *is contained in* $\partial W_{n-1} \cup \partial W_{k(n)} \cup \partial B$ *and contains the vertices* $\partial W_{n-1} \cap \partial B$*,* $\partial W_{k(n)} \cap \partial B$*,* $\partial W_{n-1} \cap \partial W_{k(n)}$*, and also the vertices of* W_n*. Finally, the vertices* $x_{n-1,k(n)} = \partial W_{n-1} \cap \partial W_{k(n)}$*,* $x_{n,k(n)} = \partial W_n \cap \partial W_{k(n)}$*,*

and $x_{k(n),B} = \partial W_{k(n)} \cap \partial B$ are contained in a half-edge of $\partial W_{k(n)}$, and $x_{n,k(n)}$ lies between the two other vertices.

(III) *If $z \in K^\circ$, then no w-triangle W has the property that $A_n = W$ or $B_n = W$ infinitely often. Moreover, for all sufficiently large $n \in \mathbb{N}$, W_{n-1} has a vertex on ∂A_n and a vertex on ∂B_n, and A_n has a vertex on ∂B_n. In fact, there exist sequences $\{k(n)\}_{n\in\mathbb{N}}$, $\{l(n)\}_{n\in\mathbb{N}}$ that diverge to ∞ such that $A_n = W_{k(n)}$ and $B_n = W_{l(n)}$ for all sufficiently large $n \in \mathbb{N}$. In the latter case, note that $l(n) < k(n) < n-1 < n$, and also $\partial V_n \subset \partial W_{n-1} \cup \partial W_{k(n)} \cup \partial W_{l(n)}$. Finally, the vertices $x_{n-1,k(n)} = \partial W_{n-1} \cap \partial W_{k(n)}$, $x_{n,k(n)} = \partial W_n \cap \partial W_{k(n)}$, and $x_{k(n),l(n)} = \partial W_{k(n)} \cap \partial W_{l(n)}$ are contained in a half-edge of $\partial W_{k(n)}$, and $x_{n,k(n)}$ lies between the two other vertices. The same statement holds with the roles of $k(n)$ and $l(n)$ reversed; see Fig. 5.*

The proofs of both lemmas are elementary and can be done by induction, so we leave them to the reader. Especially the second lemma will be crucially used in the proof of continuity of f in the next theorem, which is a restatement of Theorem 1.6.

Theorem 4.3 (Theorem 1.6). *There exists a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f \in W^{1,2}(\mathbb{R}^2 \setminus K)$, but $f \notin W^{1,2}(\mathbb{R}^2)$. In particular, K is non-removable for $W^{1,2}$.*

The function f will be almost a constant on each w-triangle, and will rapidly change near the vertices. The (almost constant) value of f on each w-triangle will be the average of the values on neighboring triangles of the previous level, with the exception of the central w-triangle of level 1; see Fig. 6.

The construction will be done in several steps. We will give an inductive construction of the function f, and ensure that it has *finite energy*, i.e., $\nabla f \in L^2(\mathbb{R}^2)$. In fact, we will show that $\|\nabla f\|_{L^2(\mathbb{R}^2)}$ can be made arbitrarily small, and this will prevent f from lying in $W^{1,2}(\mathbb{R}^2)$. Finally, we will focus on proving that our function f with the inductive definition is continuous on $\mathbb{R}^2$. The proof of the latter property is very delicate and occupies most of the section.

4.2 Building block

Here we describe the *building block functions* that will be used to define f on each w-triangle.

Let $W \subset \mathbb{R}^2$ be an open equilateral triangle with vertices x_1, x_2, x_3. Then for each $\varepsilon > 0$ and each choice of real numbers a, c_1, c_2, c_3 there exists a continuous function $g : \overline{W} \to \mathbb{R}$ with $g \in W^{1,2}(W)$ and balls $B(x_i, r_i)$, $i = 1, 2, 3$, such that

$(B1)$ $g \equiv a$ on $\overline{W} \setminus \bigcup_{i=1}^{3} B(x_i, r_i)$,
$(B2)$ $g(x_i) = c_i$ for $i = 1, 2, 3$,
$(B3)$ g is monotone increasing or decreasing (not necessarily strictly) on each half-edge of ∂W, from its midpoint to a vertex,
$(B4)$ $\int_W |\nabla g|^2 < \varepsilon$.

Furthermore, the balls $B(x_i, r_i)$ can be chosen to be arbitrarily small. Hence, by $(B1)$ we may have that

$(B5)$ g has the value a at the midpoints of the edges of W.

The value a is called the *height* of g. See Fig. 6 for an illustration of the graph of four such functions. Finally, we require a monotonicity property:

$(B6)$ $\mathrm{osc}_{\overline{W}}(g) = \mathrm{osc}_{\partial W}(g)$.

To construct such a function near the vertex x_i of W, we may assume that $a = 0$, $c_i = 1$, and that $x_i = 0 \in \mathbb{R}^2$. The conceptual fact behind this construction is that the 2-capacity of a point is equal to 0; see [11, Section 3]. For $0 < r < R$ consider the function

$$g(x) = \begin{cases} (\log(R/r)^{-1} \log(R/|x|), & r \le |x| \le R \\ 1, & |x| < r \\ 0, & |x| > R. \end{cases}$$

Then there exists a constant $C > 0$ such that $\int |\nabla g|^2 \le C \log(R/r)^{-1}$, which converges to 0 as $r \to 0$. In fact, making R smaller one sees that g can be supported in an arbitrarily small neighborhood of 0. Hence, the ball $B(x_i, r_i)$ with $r_i = R$ can be made arbitrarily small. One can now glue together three such functions, one near each vertex of W, to obtain the desired building block function. However, in order to prove the continuity of the function f in Theorem 4.3, we will not use this particular function g, but we will need to make a more careful construction, towards the end of the proof, so that a certain modulus of continuity is satisfied.

Remark 4.4 For the continuity of the function f of Theorem 4.3 we will need the properties $(B2)$, $(B3)$, $(B5)$, and $(B6)$ of the building block functions. Properties $(B1)$ and $(B4)$ are only used to show that f does not lie in $W^{1,2}(\mathbb{R}^2)$ in the next section.

4.3 Avoidance of $W^{1,2}(\mathbb{R}^2)$

The function f will be defined inductively in the next section so that $f \equiv 0$ in the unbounded component of $\mathbb{R}^2 \setminus K$ and in particular outside a fixed ball $B(x_0, R_0)$, $f \equiv 1$ in a fixed ball $B(x_0, r_0)$ contained in the central w-triangle of

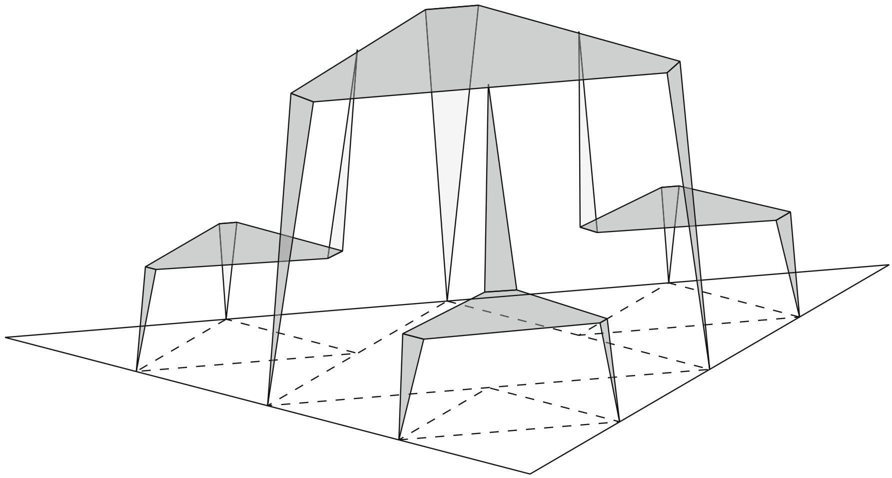

Fig. 6 Illustration of the graph of f on w-triangles of level at most 2

sidelength $1/2$, and $0 \leq f \leq 1$. Inside each w-triangle W the function f will be equal to a suitable building block function g_W, so that global continuity is ensured; see Fig. 6. We fix $\varepsilon > 0$. By choosing $\|\nabla g_W\|_{L^2(W)}$ to be sufficiently small for each W, we may have

$$\|\nabla f\|_{L^2(\mathbb{R}^2)} = \|\nabla f\|_{L^2(\mathbb{R}^2 \setminus K)} = \sum_{W \in \mathcal{W}} \|\nabla g_W\|_{L^2(W)} < \varepsilon.$$

We remark that the ball $B(x_0, r_0)$ on which $f \equiv 1$ and the ball $B(x_0, R_0)$ outside of which $f \equiv 0$ are independent of ε.

Now, we wish to prevent f from lying in $W^{1,2}(\mathbb{R}^2)$. Suppose that $f \in W^{1,2}(\mathbb{R}^2)$, and in particular that f is absolutely continuous on almost every line; see e.g. [36, Section 26]. Let I_t, $0 \leq t \leq r_0$, be the family of horizontal segments $[0, 1] \times \{t\}$, translated and scaled, so that I_t starts inside $B(x_0, r_0)$ and ends outside $B(x_0, R_0)$ for each t. Since $f \equiv 0$ outside $B(x_0, R_0)$ and $f \equiv 1$ in $B(x_0, r_0)$, by the absolute continuity on almost every line we obtain

$$1 \leq \int_{I_t} |\nabla f| \, ds$$

for a.e. $t \in [0, r_0]$. Integrating over $t \in [0, r_0]$ and applying Fubini's theorem and the Cauchy-Schwarz inequality we have

$$r_0 \leq \int_{B(x_0, R_0)} |\nabla f| \leq \|\nabla f\|_{L^2(\mathbb{R}^2)} \pi^{1/2} R_0 < \varepsilon \pi^{1/2} R_0.$$

If ε is sufficiently small, we obtain a contradiction. Hence, by choosing a small $\varepsilon > 0$ we may have that $f \notin W^{1,2}(\mathbb{R}^2)$.

4.4 Inductive choice of parameters

Here, we give the inductive construction of f.

We let $f = 0$ on the closure of the w-triangle of level 0 (i.e., the closure of the unbounded component of $\mathbb{R}^2 \setminus K$), and we define f in the closure of the central w-triangle of level 1 to be a building block function with parameters $a = 1$ and $c_i = 0$ for $i = 1, 2, 3$. In particular, $f \equiv 1$ in a fixed ball $B(x_0, r_0)$, by the property $(B1)$ of the building block function.

Once f has been defined on the closure of w-triangles of level $m - 1$, we define f on each triangle $W \in \mathcal{W}$ of level m as follows. Note that the vertices x_i, $i = 1, 2, 3$, of the triangle W lie on the boundaries of triangles of level at most $m - 1$. Hence, the function f has already been defined on the vertices of W. We now set

$$c_i = f(x_i),\ i = 1, 2, 3, \quad \text{and}$$
$$a = \frac{1}{3}(c_1 + c_2 + c_3).$$

Define f on $\overline{W}$ to be equal to a building block function with these parameters; see Fig. 6. We also set

$$\mathcal{O}(W) = \max_{i=1,2,3} |a - c_i|,$$

which controls the oscillation of f on W. In particular,

$$\operatorname{osc}_{\overline{W}}(f) = \operatorname{osc}_{\partial W}(f) \leq 2\mathcal{O}(W), \tag{4.1}$$

by properties $(B3)$ and $(B6)$. Proceeding inductively, f is defined on $W_\infty = \bigcup_{W \in \mathcal{W}} \overline{W}$.

One important observation is that the function f has a monotonicity property outside the central w-triangle of level 1; here monotonicity is to be understood in the sense that the maximum and minimum on open sets is attained at the boundary of these sets. Of course, validity of such a monotonicity property depends partly on the building block functions. We now formulate more precisely and prove the form of monotonicity that we will need.

Lemma 4.5 *For each v-triangle V of level $m \geq 1$ we have*

$$\sup_{x \in \overline{V} \cap W_\infty} f(x) = \max_{x \in \partial V} f(x), \quad \textit{and}$$
$$\inf_{x \in \overline{V} \cap W_\infty} f(x) = \min_{x \in \partial V} f(x).$$

In particular,

$$osc_{\overline{V} \cap W_\infty}(f) = \sup_{x \in \overline{V} \cap W_\infty} f(x) - \inf_{x \in \overline{V} \cap W_\infty} f(x) = osc_{\partial V}(f).$$

Note that ∂V is contained in the union of the boundaries of the w-triangles, so f is already defined there and all the expressions that appear in the lemma make sense.

Proof Assume that $W_1 \subset V$ is the w-triangle whose vertices $x_i, i = 1, 2, 3$, lie on ∂V. Then, by the averaging definition of f and the monotonicity properties $(B3)$ and $(B6)$ of the building block functions, it follows that the maximum and minimum of f on $\overline{W}_1$ are attained at the vertices of W_1, i.e.,

$$\min_{i \in \{1,2,3\}} f(x_i) \leq f(z) \leq \max_{i \in \{1,2,3\}} f(x_i)$$

for all $z \in \overline{W}_1$. Hence,

$$\max_{x \in \partial V} f(x) \leq f(z) \leq \max_{x \in \partial V} f(x)$$

for all $z \in \overline{W}_1$.

Let V_2 be one of the three v-triangles of $V \setminus \overline{W}_1$, and let $W_2 \subset V_2$ be the central w-triangle whose vertices lie on ∂V_2. If the vertices of W_2 are y_i, $i = 1, 2, 3$, then as before we have

$$\min_{i \in \{1,2,3\}} f(y_i) \leq f(z) \leq \max_{i \in \{1,2,3\}} f(y_i)$$

for all $z \in \overline{W}_2$. The vertices y_i lie on $\partial V_2 \subset \partial W_1 \cup \partial V$, hence

$$\max_{i \in \{1,2,3\}} f(y_i) \leq \max_{x \in \partial W_1 \cup \partial V} f(x) = \max_{x \in \partial V} f(x),$$

by our conclusion for W_1. The analog of this statement also holds for the minimum, hence, we obtain in this case

$$\max_{x \in \partial V} f(x) \leq f(z) \leq \max_{x \in \partial V} f(x)$$

for all $z \in \overline{W}_2$. The proof of the general statement follows with the same argument by induction. □

4.5 Proof of Continuity

Proposition 4.6 *The function $f : W_\infty \to \mathbb{R}$ is uniformly continuous and thus, it has a continuous extension to $\mathbb{R}^2$, which is the closure of $W_\infty = \bigcup_{W \in \mathcal{W}} \overline{W}$.*

The way to interpret this statement is that *there exists* a choice of building block functions that makes f continuous. As we remarked in Sect. 4.2, we cannot use a "generic" building block function, but we have to make a careful construction.

The proof of continuity relies on the next crucial lemma. Recall the definition of $\mathcal{O}(W)$ from Sect. 4.4.

Lemma 4.7 *For each $\varepsilon > 0$ there exist at most finitely many w-triangles W with*

$$\mathcal{O}(W) > \varepsilon.$$

In particular, for each $\varepsilon > 0$ there exist at most finitely many w-triangles W with

$$osc_{\overline{W}}(f) > \varepsilon.$$

Assuming the lemma, we prove Proposition 4.6.

Proof of Proposition 4.6 Using Lemma 4.7, we fist prove:

Claim 1 For each $\varepsilon > 0$ there exist at most finitely many v-triangles V with

$$\operatorname{osc}_{\overline{V} \cap W_\infty}(f) > \varepsilon. \tag{4.2}$$

We argue by contradiction, assuming that there exists $\varepsilon > 0$ such that the above holds for infinitely many v-triangles. Let V be one of them. Then by the monotonicity of f from Lemma 4.5 we have $\operatorname{osc}_{\partial V}(f) > \varepsilon$. Each edge of ∂V is contained in an edge of a w-triangle, so there exist three (possibly non-distinct) w-triangles W_1, W_2, and W_3 such that

$$\varepsilon < \operatorname{osc}_{\partial V}(f) \le \operatorname{osc}_{\partial V \cap \partial W_1}(f) + \operatorname{osc}_{\partial V \cap \partial W_2}(f) + \operatorname{osc}_{\partial V \cap \partial W_3}(f).$$

In particular, for one of them, say for W, we have $\operatorname{osc}_{\partial V \cap \partial W}(f) > \varepsilon$. We thus see that each v-triangle of the set $\{V : \operatorname{osc}_{\overline{V} \cap W_\infty}(f) > \varepsilon\}$ corresponds to a w-triangle W such that $\operatorname{osc}_{\partial V \cap \partial W}(f) > \varepsilon/3$. Moreover, this correspondence is finite-to-one. Indeed, if there were infinitely many v-triangles V_n, $n \in \mathbb{N}$, corresponding to a single w-triangle W, then the diameters of V_n would shrink to 0. However, the uniform continuity of the restriction of f to ∂W would imply that $\mathrm{osc}_{\partial V_n \cap \partial W}(f) \to 0$, a contradiction. It follows that there exist infinitely many w-triangles W with the property that there exists a v-triangle V such that

$$\varepsilon/3 < \mathrm{osc}_{\partial V \cap \partial W}(f) \leq \mathrm{osc}_{\partial W}(f) \leq 2\mathcal{O}(W),$$

where we used (4.1). This contradicts Lemma 4.7.

Now, we prove that f is uniformly continuous on W_∞. We argue by contradiction, assuming that there exists $\varepsilon > 0$ and sequences $x_n, y_n \in W_\infty$ with $|x_n - y_n| \to 0$ such that $|f(x_n) - f(y_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. The sequences x_n, y_n cannot escape to ∞ since f is identically equal to 0 in a neighborhood of ∞. Consider an accumulation point z of x_n and y_n, and by passing to a subsequence, assume that $x_n, y_n \to z$. Note that z cannot lie in the interior of any w-triangle, since the function f is already continuous there. Hence, $z \in K$ and we split into three cases.

Suppose first that z is of vertex type. By Lemma 4.1(I) there exist two (possibly non-distinct) w-triangles A and B containing z on their boundary and two (possibly non-distinct) sequences of v-triangles $\overline{V}_k$ and $\overline{V}_k'$ shrinking to z such that for each $k \in \mathbb{N}$ the set $\overline{A} \cup \overline{B} \cup \overline{V}_k \cup \overline{V}_k'$ contains all sufficiently small neighborhoods of z. We fix k and a small $r > 0$ such that $B(z, r) \subset \overline{A} \cup \overline{B} \cup \overline{V}_k \cup \overline{V}_k'$. Since $z \in \partial A \cap \partial B \cap \overline{V}_k \cap \overline{V}_k'$, for each $p \in B(z, r)$ we have

$$\begin{aligned} |f(p) - f(z)| \leq \mathrm{osc}_{\overline{A} \cap B(z,r)}(f) + \mathrm{osc}_{\overline{B} \cap B(z,r)}(f) + \mathrm{osc}_{\overline{V}_k \cap W_\infty}(f) \\ + \mathrm{osc}_{\overline{V}_k' \cap W_\infty}(f). \end{aligned}$$

By choosing a sufficiently large k, we may have that $\mathrm{osc}_{\overline{V}_k \cap W_\infty}(f) < \varepsilon/8$ and $\mathrm{osc}_{\overline{V}_k' \cap W_\infty}(f) < \varepsilon/8$, by Claim 1. By choosing $r > 0$ to be sufficiently small, using the uniform continuity of the restriction of f on $\overline{A}$ and $\overline{B}$ we may also have $\mathrm{osc}_{\overline{A} \cap B(z,r)}(f) < \varepsilon/8$ and $\mathrm{osc}_{\overline{B} \cap B(z,r)}(f) < \varepsilon/8$. It follows that $|f(p) - f(z)| < \varepsilon/2$ for all $p \in B(z, r)$. Now, if n is sufficiently large, then $x_n, y_n \in B(z, r)$, hence

$$|f(x_n) - f(y_n)| \leq |f(x_n) - f(z)| + |f(y_n) - f(z)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which is a contradiction.

If z is of edge type, then applying Lemma 4.1(II) we obtain a w-triangle B with $z \in \partial B$ and a sequence of v-triangles $\overline{V}_k$ shrinking to z such that for each $k \in \mathbb{N}$ the set $\overline{B} \cup \overline{V}_k$ contains all sufficiently small neighborhoods of z. One now argues exactly as in the previous case, using Claim 1 or using the uniform continuity of the restriction of f on $\overline{B}$. It follows that for each $k \in \mathbb{N}$ there exists a small $r > 0$ such that for all $p \in B(z, r)$ we have

$$|f(p) - f(z)| \leq \operatorname{osc}_{\overline{B} \cap B(z,r)}(f) + \operatorname{osc}_{\overline{V}_k \cap W_\infty}(f) < \varepsilon/8 + \varepsilon/8 = \varepsilon/4.$$

Since $x_n, y_n \in B(z, r)$ for all sufficiently large n, we obtain again a contradiction to the assumption that $|f(x_n) - f(y_n)| \geq \varepsilon$ for $n \in \mathbb{N}$.

Finally, suppose that $z \in K^\circ$. By Lemma 4.1(III) there exists a sequence of v-triangles V_k shrinking to z such that $z \in V_k$ for all $k \in \mathbb{N}$. It follows that for each k there exists a large n such that $x_n, y_n \in V_k$. In particular, we have

$$|f(x_n) - f(y_n)| \leq \operatorname{osc}_{\overline{V}_k \cap W_\infty}(f).$$

If we choose a sufficiently large $k \in \mathbb{N}$ then the latter is less than ε by Claim 1 and we obtain a contradiction. □

Finally, we prove the basic Lemma 4.7.

Proof of Lemma 4.7 We argue by contradiction, assuming that for some $\varepsilon_0 > 0$ we have

$$\operatorname{osc}_{\overline{W}}(f) > \varepsilon_0 \tag{4.3}$$

for infinitely many w-triangles W. We split in three cases.

Case 1 There exist infinitely many w-triangles satisfying (4.3) and converging to a point $z \in K$ of vertex type.

By Lemma 4.2(I), the vertex z is (contained in) the intersection of the closures of two fixed w-triangles A and B (these might be non-distinct if they are the unbounded w-triangle). Since the restriction of f is uniformly continuous on $\overline{A \cup B}$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \overline{A \cup B}$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

Using the notation from Lemma 4.2(I), we consider the sequences of triangles W_n and V_n. For sufficiently large n, the triangle V_n has z on its boundary, and is contained in the $\delta/2$-neighborhood of z. Note that each triangle W_n has its vertices on ∂W_{n-1}, ∂A, and ∂B for all sufficiently large n. Assume that all of the above hold for $n \geq N$.

We denote by c_n the height of f on W_n (recall the definition of the height of a building block function in Sect. 4.2), and note that for $n > N$ we have

$$c_n = \frac{1}{3}(c_{n-1} + c_{n,A} + c_{n,B}),$$

where $c_{n,A}$ and $c_{n,B}$ are the values of f on the vertex of W_n lying on ∂A and ∂B, respectively. Note that the vertex of W_n lying on ∂W_{n-1} is the midpoint of an edge of ∂W_{n-1}. Hence, by property $(B5)$, the value of f at this vertex is equal to the height of f on the triangle W_{n-1}, i.e., c_{n-1}.

Our goal is to find a sequence $\{\Delta_n\}_{n\in\mathbb{N}}$ such that

$$\mathcal{O}(W_n) = \max\{|c_n - c_{n-1}|, |c_n - c_{n,A}|, |c_n - c_{n,B}|\} \le \Delta_n \tag{4.4}$$

for $n \in \mathbb{N}$, and $\Delta_n \le 3\varepsilon$ for sufficiently large n. Then

$$\begin{aligned}\operatorname{osc}_{\partial V_n}(f) &\le \operatorname{osc}_{\partial V_n\cap\partial A}(f) + \operatorname{osc}_{\partial V_n\cap\partial B}(f) + \operatorname{osc}_{\partial V_n\cap\partial W_{n-1}}(f)\\ &\le 2\varepsilon + \operatorname{osc}_{\overline{W}_{n-1}}(f) \le 2\varepsilon + 2\Delta_{n-1} \le 8\varepsilon\end{aligned}$$

for all sufficiently large n, where we used (4.1).

Note that there are at most two nested sequences $\{V_n\}_{n\in\mathbb{N}}$ and $\{V_n'\}_{n\in\mathbb{N}}$ shrinking to z, for which the above bounds hold; see Lemma 4.1(I). If W is a small w-triangle near z satisfying (4.3), then it has to be contained in V_n or V_n' for some large n, by Lemma 4.1(I). Using the monotonicity of f we see that

$$\operatorname{osc}_{\overline{W}}(f) \le \max\{\operatorname{osc}_{\partial V_n}(f), \operatorname{osc}_{\partial V_n'}(f)\} \le 8\varepsilon.$$

This contradicts (4.3) if we choose $\varepsilon < \varepsilon_0/8$.

We proceed to the proof of (4.4). For $1 \le n \le N$ we use the trivial bound $\mathcal{O}(W_n) \le 1 =: \Delta_n$. Once Δ_{n-1} has been defined and satisfies (4.4), for $n > N$ we have

$$\begin{aligned}|c_n - c_{n-1}| &\le \frac{1}{3}(|c_{n,A} - c_{n-1}| + |c_{n,B} - c_{n-1}|)\\ &\le \frac{1}{3}(|c_{n,A} - c_{n-1,A}| + |c_{n-1,A} - c_{n-1}|\\ &\quad + |c_{n,B} - c_{n-1,B}| + |c_{n-1,B} - c_{n-1}|)\\ &\le \frac{1}{3}(\varepsilon + \Delta_{n-1} + \varepsilon + \Delta_{n-1})\\ &\le \frac{2\varepsilon}{3} + \frac{2}{3}\Delta_{n-1},\\ |c_n - c_{n,A}| &\le \frac{1}{3}(|c_{n-1} - c_{n,A}| + |c_{n,B} - c_{n,A}|)\\ &\le \frac{1}{3}(\varepsilon + \Delta_{n-1} + \varepsilon)\\ &\le \frac{2\varepsilon}{3} + \frac{1}{3}\Delta_{n-1}, \quad \text{and}\\ |c_n - c_{n,B}| &\le \frac{2\varepsilon}{3} + \frac{1}{3}\Delta_{n-1}.\end{aligned}$$

Here, we used the fact from Lemma 4.2(I) that the vertices $\partial W_n \cap \partial A$, $\partial W_{n-1} \cap \partial A$, and $\partial W_n \cap \partial B$, $\partial W_{n-1} \cap \partial B$ are all contained in ∂V_n, which lies in the $\delta/2$-neighborhood of z. Thus, we may choose $\Delta_n := \frac{2\varepsilon}{3} + \frac{2}{3}\Delta_{n-1}$ for $n > N$, which yields

$$\Delta_n = \frac{2\varepsilon}{3} + \cdots + \frac{2^{n-N}\varepsilon}{3^{n-N}} + \frac{2^{n-N}}{3^{n-N}} \leq 2\varepsilon + \frac{2^{n-N}}{3^{n-N}}.$$

Since $\Delta_n \leq 3\varepsilon$ for sufficiently large n, we have the desired conclusion.

Case 2 There exist infinitely many w-triangles satisfying (4.3) and converging to a point $z \in K \backslash K^\circ$ that is of edge type, i.e., it lies on an open edge of a w-triangle B.

We consider the unique sequences of triangles V_n and W_n converging to z, as in Lemma 4.2(II). We fix a small $\varepsilon > 0$ and consider $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in \partial B$. Assume that V_n is contained in the $\delta/2$-neighborhood of z for $n > N$. Then arguing as in Case 1, we wish to find a sequence Δ_n that bounds the oscillation of f on W_n, and Δ_n is sufficiently small, depending on ε.

This time we have

$$c_n = \frac{1}{3}(c_{n-1} + c_{n,A_n} + c_{n,B}).$$

for $n \geq 1$; also here, c_n is the height of f on W_n and we use again property $(B5)$. By Lemma 4.2(II), $A_n = W_{k(n)}$ for, say, $n > N$, where $k(n) \to \infty$. We set $c_{n,k(n)} := c_{n,A_n}$. In general, if a w-triangle W_n has a vertex on ∂W_m for some m, then the value of f at that vertex is denoted by $c_{n,m}$.

Our claim now is that for each $m \in \mathbb{N}$ there exists $N_m \in \mathbb{N}$ and Δ_m such that

$$\mathcal{O}(W_n) \leq \Delta_m$$

for $n > N_m$, and $\Delta_m \leq 5\varepsilon$ for sufficiently large m.

We assume this for the moment. If W is a small w-triangle sufficiently close to z, then $W \subset V_n$ for some large n, by Lemma 4.1(II). The boundary of V_n is contained in ∂W_{n-1}, $\partial W_{k(n)}$, and ∂B; see Lemma 4.2(II). Also V_n lies in the $\delta/2$-neighborhood of z for large n. Hence,

$$\begin{aligned}\operatorname{osc}_{\overline{W}}(f) \leq \operatorname{osc}_{\partial V_n}(f) &\leq \varepsilon + 2\mathcal{O}(W_{n-1}) + 2\mathcal{O}(W_{k(n)})\\ &\leq \varepsilon + 4\Delta_m\end{aligned}$$

provided that $n-1, k(n) > N_m$. If W is sufficiently close to z, then V_n can be chosen to be sufficiently close to z, so $n-1$, $k(n)$ and m can be large enough, in order to have $n-1, k(n) > N_m$, and $\Delta_m \leq 5\varepsilon$. Hence,

$$\operatorname{osc}_{\overline{W}}(f) \leq 21\varepsilon$$

and this contradicts (4.3), if we choose $\varepsilon < \varepsilon_0/21$.

Now, we prove our claim. For $m = 1$ we use the trivial bound $\mathcal{O}(W_n) \leq 1 =: \Delta_1$, which holds for all $n \in \mathbb{N}$. If N_{m-1} has been chosen, we choose $N_m > N$ to be so large that $n-1, k(n) > N_{m-1}$ for all $n > N_m$. This can be done since $k(n) \to \infty$ by Lemma 4.2(II). For $n > N_m$ we have

$$|c_n - c_{n-1}| \leq \frac{1}{3}(|c_{n,k(n)} - c_{n-1}| + |c_{n,B} - c_{n-1}|).$$

If $|c_{n,k(n)} - c_{n-1,k(n)}| \leq \Delta_{m-1}/2$ then we have

$$\begin{aligned} |c_n - c_{n-1}| &\leq \frac{1}{3}(|c_{n,k(n)} - c_{n-1,k(n)}| + |c_{n-1,k(n)} - c_{n-1}| \\ &\quad + |c_{n,B} - c_{n-1,B}| + |c_{n-1,B} - c_{n-1}|) \\ &\leq \frac{1}{3}(\Delta_{m-1}/2 + \Delta_{m-1} + \varepsilon + \Delta_{m-1}) \\ &\leq \frac{2\varepsilon}{3} + \frac{5}{6}\Delta_{m-1}. \end{aligned}$$

Here we used the fact from Lemma 4.2(II) that $\overline{V}_n$ contains the vertices $\partial W_{n-1} \cap \partial B$ and $\partial W_n \cap \partial B$, so they are δ-close to each other.

If $|c_{n,k(n)} - c_{n-1,k(n)}| > \Delta_{m-1}/2$, then we necessarily have $|c_{n,k(n)} - c_{k(n),B}| \leq \Delta_{m-1}/2$, where $c_{k(n),B}$ denotes the value of f at the vertex of $W_{k(n)}$ lying on ∂B. This is because the vertices $\partial W_{n-1} \cap \partial W_{k(n)}$, $\partial W_n \cap \partial W_{k(n)}$, $\partial W_{k(n)} \cap \partial B$ are ordered points, contained in a half-edge of $W_{k(n)}$ (by Lemma 4.2), where f is monotone increasing or decreasing by property $(B3)$ in Sect. 4.2. On the other hand, by the induction assumption, the oscillation of f on this half-edge is bounded by $\mathcal{O}(W_{k(n)}) \leq \Delta_{m-1}$, since $k(n) > N_{m-1}$. In this case, we have

$$\begin{aligned} |c_n - c_{n-1}| &\leq \frac{1}{3}(|c_{n,k(n)} - c_{k(n),B}| + |c_{k(n),B} - c_{n-1,B}| + |c_{n-1,B} - c_{n-1}| \\ &\quad + |c_{n,B} - c_{n-1,B}| + |c_{n-1,B} - c_{n-1}|) \\ &\leq \frac{1}{3}(\Delta_{m-1}/2 + \varepsilon + \Delta_{m-1} + \varepsilon + \Delta_{m-1}) \\ &= \frac{2\varepsilon}{3} + \frac{5}{6}\Delta_{m-1}. \end{aligned}$$

In the same way we also compute bounds for $|c_n - c_{n,k(n)}|$ and $|c_n - c_{n,B}|$, and we can show that they are all bounded by

$$\mathcal{O}(W_n) \leq \frac{2\varepsilon}{3} + \frac{5}{6}\Delta_{m-1} =: \Delta_m.$$

Observe that

$$\Delta_m \leq 4\varepsilon + \frac{5^{m-1}}{6^{m-1}} \leq 5\varepsilon$$

for sufficiently large m, as desired.

Case 3 There exist infinitely many w-triangles satisfying (4.3) and converging to a point $z \in K^{\circ}$.

In the previous two cases the proof was mostly combinatorial, based on Lemmas 4.1 and 4.2, on qualitative properties of the building block functions, and on the fact that the restriction of f on the union of finitely many w-triangles is uniformly continuous. However, in this case it will be crucial to make a suitable choice of the building block functions, so that they have a certain modulus of continuity on each w-triangle near the vertices.

Let W_0 be a w-triangle, and consider the corresponding oscillation $\mathcal{O}(W_0)$, which depends on the value of f at the vertices of W_0, by the inductive definition of f. We require the following:

4.5.1 Condition $(*)$

$(*)$ Assume that two w-triangles W_1 and W_2 are adjacent, and each has a vertex on a triangle ∂W_0 of strictly lower level. If $z_1 \in \partial W_1 \cap \partial W_0$ and $z_2 \in \partial W_2 \cap \partial W_0$ are these vertices, then

$$|f(z_1) - f(z_2)| \leq \mathcal{O}(W_0)/3.$$

Recall that W_1 is adjacent to W_2 if W_1 has a vertex on ∂W_2, or vice versa. Note at this point the vertices of W_0 can only lie on triangles of strictly lower level from that of W_0, by our observations in Sect. 4.1, so in particular they do not lie on W_1 or W_2. Of course, we still require the initial properties $(B1)$–$(B6)$ of the building block functions from Sect. 4.2.

To construct f on W_0 with the desired properties we work as follows. We fix an edge $I \subset \partial W_0$, and a vertex $z \in I$ of W_0. Assume that the edges of W_0 have length 1, and that $I = [0, 1]$, $z = 0$. We consider the points 2^{-k}, $k \geq 2$, on I, and define a radial function with the following procedure. For a fixed $N \in \mathbb{N}$ we define f on the annulus $A_1 := A(0; 2^{-3}, 2^{-2})$ to be a radial function that is equal to 0 in the outer circle and increases to $1/N$ in the inner circle, with slope $\simeq 2^3/N$. For $1 \leq k \leq N$, we define f in $A_k := A(0; 2^{-2-k}, 2^{-1-k})$ to be a radial function that is equal to $(k-1)/N$ in the outer circle and increases to k/N in the inner circle, with slope $\simeq 2^{2+k}/N$. In the ball $B(0, 2^{-2-N})$ we set $f \equiv 1$, outside the ball $B(0, 2^{-2})$ we set $f \equiv 0$, and then we restrict f to the triangle $\overline{W}_0$. Then $f \in W^{1,2}(W_0)$ and

$$\begin{aligned}\int_{W_0} |\nabla f|^2 &= \sum_{k=1}^{N} \int_{W_0 \cap A_k} |\nabla f|^2 \lesssim \sum_{k=1}^{N} \left(\frac{2^{2+k}}{N} \right)^2 m_2(W_0 \cap A_k) \\ &\simeq \sum_{k=1}^{N} \frac{1}{N^2} \simeq \frac{1}{N}.\end{aligned}$$

Hence, by choosing a large N we can achieve both that the f has small energy, and that

$$|f(2^{-k}) - f(2^{-k-1})| \leq \frac{1}{N} \leq \frac{1}{3}$$

for all $k \geq 1$. Note that the same bounds hold for the corresponding dyadic points lying on the other edge of W_0 that is connected to 0, and is a rotation of I by 60 degrees. Of course, the assumptions that $f = 1$ near $z = 0$ and $f = 0$ outside $B(0, 2^{-2})$ are not restrictive, since by rescaling f and choosing a sufficiently large N we can achieve the oscillation we wish with small energy.

We now consider dyadic points as above on each half-edge of W_0, converging to the corresponding vertex, and do a similar construction for all vertices. Near each vertex we may have that f oscillates radially from a given value to the desired height and also has the property that if x_1 and x_2 are adjacent dyadic points lying on an edge I of W_0 then

$$|f(x_1) - f(x_2)| \leq \mathcal{O}(W_0)/3. \tag{4.5}$$

Again, by dyadic points we mean points of the form 2^{-k} and $1 - 2^{-k}$ for $k \in \mathbb{N}$, once we scale W_0 so that $I = [0, 1]$ is an edge of W_0. The properties $(B1)$–$(B6)$ in Sect. 4.2 hold by construction. Especially, note that the property $(B3)$ holds, since f is radially increasing or decreasing near each vertex.

To check property $(*)$ we first use a scaling followed by a rotation of the Sierpiński gasket so that the points z_1, z_2 in question lie on the edge $I = [0, 1]$ of ∂W_0. One now has to observe that if $z_1, z_2 \in I \subset \partial W_0$ are vertices of adjacent triangles W_1, W_2, and they are not vertices of W_0, then they must both lie in one of the closed dyadic intervals of the form $[2^{-k-1}, 2^{-k}]$ or $[1 - 2^{-k}, 1 - 2^{-k-1}]$, $k \geq 1$. Hence, using the monotonicity of f on these intervals (property $(B3)$) and (4.5) we obtain

$$|f(z_1) - f(z_2)| \leq \mathcal{O}(W_0)/3.$$

To prove the observation mentioned, we first note that the two vertices of ∂W_0 that are endpoints I, actually lie on the edges of two triangles A and B. The triangles W_0, A, and B bound a v-triangle V, and each w-triangle that has a vertex on I must be contained in V. We consider the central w-triangle $W(1/2) \subset V$ that has a vertex at the midpoint of I. The points $2^{-2}, 1-2^{-2} \in I$ are vertices of triangles $W(2^{-2})$, $W(1-2^{-1})$, respectively, which also have a vertex on $\partial W(1/2)$. Inductively, the points $2^{-k}, 1-2^{-k} \in I$, $k \geq 2$, are the vertices of triangles $W(2^{-k})$, $W(1-2^{-k})$, which have vertices on $W(2^{-k+1})$, $W(1-2^{-k+1})$, respectively. Note that the v-triangles bounded by $\partial W_0, \partial W(2^{-k+1})$, and $\partial W(2^{-k})$, or by $\partial W_0, \partial W(1-2^{-k+1})$, and $\partial W(1-2^{-k})$ are disjoint for $k \geq 1$, and that the closures of these v-triangles cover I, except for its endpoints.

Now, if $W_1, W_2 \subset V$ are adjacent triangles as in the statement of $(*)$, and they are not equal to the "dyadic" triangles $W(2^{-k})$, $W(1-2^{-k})$, $k \in \mathbb{N}$, that have a vertex on a dyadic point of I, then their vertices $z_1, z_2 \in I$ cannot lie in distinct dyadic intervals. This is because in this case W_1 and W_2 would lie in disjoint v-triangles, and thus, they would not be adjacent. With a similar analysis one deals with the cases where one of W_1, W_2, or both, are equal to "dyadic" triangles. In any case the vertices $z_1 \in \partial W_1 \cap \partial W_0$ and $z_2 \in \partial W_2 \cap \partial W_0$ must both lie in one of the intervals of the form $[2^{-k-1}, 2^{-k}]$ or $[1-2^{-k}, 1-2^{-k-1}]$, $k \geq 1$, as desired. □

Now, we return to the main proof of Case 3, that (4.3) cannot occur for infinitely many w-triangles near a point $z \in K^\circ = K \setminus W_\infty$.

We consider the sequence of nested v-triangles $\{V_n\}_{n\geq 1}$ converging to z and the corresponding w-triangles $W_n \subset V_n$, given by Lemma 4.2(III). In this case, for sufficiently large $n \in \mathbb{N}$ the triangle W_n has a vertex on ∂W_{n-1} and two vertices on boundaries of some w-triangles $W_{k(n)}, W_{l(n)}$, with $k(n), l(n) \to \infty$; see Fig. 5.

Let $\Delta_1 = 1$. We claim that for each $m \geq 2$ there exists $N_m \in \mathbb{N}$ and $\Delta_m = 7\Delta_{m-1}/9$ such that for $n > N_m$ we have

$$\mathcal{O}(W_n) \leq \Delta_m.$$

If this is the case, then $\Delta_m = 7^{m-1}/9^{m-1}$ for $m \in \mathbb{N}$, and it follows that

$$\mathcal{O}(W_n) \leq 7^{m-1}/9^{m-1} \leq \varepsilon$$

for all sufficiently large m and $n > N_m$, where $\varepsilon > 0$ is to be chosen. The triangle V_n that contains W_n has boundary contained in ∂W_{n-1}, $\partial W_{k(n)}$, and $\partial W_{l(n)}$. Note that if W is a w-triangle that is sufficiently close to z, then $W \subset V_n$

for some large n, by Lemma 4.1(III). On the other hand, for any w-triangle $W \subset V_n$ by the monotonicity of f in Lemma 4.5 and (4.1) we have

$$\operatorname{osc}_{\overline{W}}(f) \le \operatorname{osc}_{\partial V_n}(f) \le 2\mathcal{O}(W_{n-1}) + 2\mathcal{O}(W_{k(n)}) + 2\mathcal{O}(W_{l(n)}) \le 6\varepsilon,$$

provided that n is sufficiently large, so that $n-1, k(n), l(n) > N_m$. Choosing $\varepsilon < \varepsilon_0/6$ yields a contradiction to (4.3).

Now, we proceed with the proof of our claim. For $m = 1$ we use the trivial bound $\mathcal{O}(W_n) \le 1 =: \Delta_1$, which holds for all $n \in \mathbb{N}$. If N_{m-1} has been chosen, we choose N_m to be so large that for $n > N_m$ the vertices of the triangle W_n lie on triangles ∂W_{n-1}, $\partial W_{k(n)}$, $\partial W_{l(n)}$, with $n-1, k(n), l(n) > N_{m-1}$. This can be done since $k(n), l(n) \to \infty$. Now we have

$$c_n = \frac{1}{3}(c_{n-1} + c_{n,k(n)} + c_{n,l(n)}),$$

using the notation of Case 2. We assume that the order of the triangles by level is $W_{l(n)}, W_{k(n)}, W_{n-1}, W_n$, i.e., $l(n) < k(n) < n-1 < n$, as in Lemma 4.2(III). In order to control $\mathcal{O}(W_n)$, we will find bounds for the differences $c_n - c_{n-1}$, $c_n - c_{n,k(n)}$, and $c_n - c_{n,l(n)}$.

We have

$$\begin{aligned} c_n - c_{n-1} &= \frac{1}{3}(c_{n,k(n)} - c_{n-1} + c_{n,l(n)} - c_{n-1}) \\ &= \frac{1}{3}[(c_{n,k(n)} - c_{n-1,k(n)}) + (c_{n-1,k(n)} - c_{n-1}) \\ &\quad + (c_{n,l(n)} - c_{n-1,l(n)}) + (c_{n-1,l(n)} - c_{n-1})] \end{aligned}$$

The first difference is bounded (in absolute value) by $\mathcal{O}(W_{k(n)})/3 \le \Delta_{m-1}/3$, by the induction assumption and the property $(*)$ of the function f; note that the triangles W_n and W_{n-1} are adjacent and they both have a vertex on $\partial W_{k(n)}$. With the same reasoning the third difference is also bounded by $\mathcal{O}(W_{l(n)})/3 \le \Delta_{m-1}/3$. Finally, by the definition of f on W_{n-1}, the sum of the second and fourth differences is equal to $c_{n-1} - c_{n-1,D}$, where $c_{n-1,D}$ is the value of f on the third vertex of W_{n-1} that does not lie on $\partial W_{k(n)}$ and $\partial W_{l(n)}$. Hence, for the absolute value of the sum of the second and fourth differences we obtain the upper bound $\mathcal{O}(W_{n-1}) \le \Delta_{m-1}$. Putting these altogether, we have

$$|c_n - c_{n-1}| \le \frac{1}{3}(\Delta_{m-1}/3 + \Delta_{m-1}/3 + \Delta_{m-1}) = \frac{5}{9}\Delta_{m-1}.$$

We now do the same analysis for the difference $c_n - c_{n,k(n)}$. We have

$$\begin{aligned} c_n - c_{n,k(n)} &= \frac{1}{3}(c_{n-1} - c_{n,k(n)} + c_{n,l(n)} - c_{n,k(n)}) \\ &= \frac{1}{3}\big[(c_{n-1} - c_{n-1,k(n)}) + (c_{n-1,k(n)} - c_{n,k(n)}) \\ &\quad + (c_{n,l(n)} - c_{k(n),l(n)}) + (c_{k(n),l(n)} - c_{n,k(n)})\big]. \end{aligned} \tag{4.6}$$

We bound the first difference by Δ_{m-1}, using the induction assumption. The second and fourth differences have opposite sign since the vertices $\partial W_{n-1} \cap \partial W_{k(n)}$, $\partial W_n \cap \partial W_{k(n)}$, $\partial W_{k(n)} \cap \partial W_{l(n)}$ are ordered points (see Fig. 5), contained in a half-edge of $W_{k(n)}$ (by Lemma 4.2), where f is monotone increasing or decreasing by property $(B3)$ in Sect. 4.2. Using the fundamental inequality

$$|x + y| \leq \max\{|x|, |y|\}$$

whenever $x, y \in \mathbb{R}$ and $xy < 0$ (or more generally $x, y \in \mathbb{C}$ and their angle is π), we conclude that

$$\begin{aligned} &|(c_{n-1,k(n)} - c_{n,k(n)}) + (c_{k(n),l(n)} - c_{n,k(n)})| \\ &\quad \leq \max\{|c_{n-1,k(n)} - c_{n,k(n)}|, |c_{k(n),l(n)} - c_{n,k(n)}|\} \leq \mathcal{O}(W_{k(n)}) \leq \Delta_{m-1}. \end{aligned}$$

For the third difference in (4.6), by Lemma 4.2(III) we have that $W_{k(n)}$ has a vertex on $\partial W_{l(n)}$. Hence, $W_{k(n)}$ and W_n are adjacent triangles having a vertex on the strictly larger triangle $\partial W_{l(n)}$. Property $(*)$ now implies that

$$|c_{n,l(n)} - c_{k(n),l(n)}| \leq \mathcal{O}(W_{l(n)})/3 \leq \Delta_{m-1}/3,$$

by the induction assumption. Summarizing,

$$|c_n - c_{n,k(n)}| \leq \frac{7}{9}\Delta_{m-1}.$$

Finally, we look at the difference $c_n - c_{n,l(n)}$. As before, we have

$$\begin{aligned} c_n - c_{n,l(n)} &= \frac{1}{3}(c_{n-1} - c_{n,l(n)} + c_{n,k(n)} - c_{n,l(n)}) \\ &= \frac{1}{3}\big[(c_{n-1} - c_{n-1,l(n)}) + (c_{n-1,l(n)} - c_{n,l(n)}) \\ &\quad + (c_{n,k(n)} - c_{k(n),l(n)}) + (c_{k(n),l(n)} - c_{n,l(n)})\big]. \end{aligned}$$

Exactly as in the previous computation, the first difference is bounded by Δ_{m-1}, the sum of the second and fourth differences is bounded by Δ_{m-1} using property $(B3)$, and the third difference is bounded by $\Delta_{m-1}/3$. Hence,

$$|c_n - c_{n,l(n)}| \leq \frac{7}{9}\Delta_{m-1}.$$

Summarizing, we have

$$\mathcal{O}(W_n) \leq \frac{7}{9}\Delta_{m-1} =: \Delta_m$$

for all $n > N_m$, and the proof is completed. □

4.6 Generalization to homeomorphic gaskets

Here we show that any image of the gasket under a homeomorphism of $\mathbb{R}^2$ is non-removable for $W^{1,2}$ as claimed in Theorem 1.7. We have to split in two cases, depending on whether the "homeomorphic gasket" has area zero or positive area. In the first case, our proof for the standard gasket applies with some modifications, while the second case can be treated with the general statement that sets of positive measure are non-removable for Sobolev spaces; see Theorem 4.9.

Theorem 4.8 *Let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism, and K be the Sierpiński gasket. If $m_2(h(K)) = 0$, then $h(K)$ is non-removable for $W^{1,2}$.*

Proof Our goal is to obtain a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f \in W^{1,2}(\mathbb{R}^2 \setminus h(K))$, with $0 \leq f \leq 1$, $f \equiv 0$ outside a ball $B(x_0, R_0)$, $f \equiv 1$ in a ball $B(x_0, r_0)$, such that $\|\nabla f\|_{L^2(\mathbb{R}^2)} = \|\nabla f\|_{L^2(\mathbb{R}^2 \setminus h(K))}$ is as small as we wish; here it is crucial that $m_2(h(K)) = 0$. Then, arguing as in Sect. 4.3, one can show that $f \notin W^{1,2}(\mathbb{R}^2)$, if $\|\nabla f\|_{L^2(\mathbb{R}^2)}$ is sufficiently small.

Our proof of Theorem 4.3 and more specifically the construction of f was combinatorial, except for the construction of the particular building block functions that satisfy $(B1)$–$(B6)$ and $(*)$. We remark that even in Cases 1 and 2 of the proof of Lemma 4.7 (which is the heart of the proof of continuity) we only used properties $(B1)$–$(B6)$ of the building block functions, together with the continuity of the restriction of f on each particular w-triangle, but we did not need any specific modulus of continuity. The property $(*)$ was only needed in Case 3 and requires some special care.

Since the combinatorics of the gasket are preserved under homeomorphisms of $\mathbb{R}^2$, it remains to show that building block functions satisfying $(B1)$–$(B6)$ and $(*)$ exist when the domain is an arbitrary Jordan region Ω, rather than a triangle. Assume that W is a w-triangle of K such that $h(W) = \Omega$. It suffices to do the construction near each vertex of Ω, i.e., near the images of the vertices of W. Let z be a vertex of W, and consider the two half-edges $I, J \subset \partial W$ that meet at z. Also, consider the dyadic points $x_k \in I$, $k \in \mathbb{N}$, and $y_k \in J$, $k \in \mathbb{N}$, converging to z, as in the proof of $(*)$ in Case 3; for example, if $I = [0, 1]$, then $x_k = 2^{-k}$ for $k \in \mathbb{N}$. Let $z' = h(z)$, $I' = h(I)$, $J' = h(J)$, $x_k' = h(x_k)$, and $y_k' = h(y_k)$.

Using a conformal map, we map Ω onto the upper half plane $\mathbb{H}$, and assume that $z' = 0$ and that $I' \subset [0, \infty)$, $J' \subset (-\infty, 0]$ are closed intervals meeting at 0. Furthermore, $x_k' \in I'$ is a strictly decreasing sequence converging to 0, and $y_k' \in J'$ is strictly increasing and converging to 0.

We wish to construct a continuous function $g : \overline{\mathbb{H}} \to \mathbb{R}$ with the following properties:

$(B1')$ g is supported in an arbitrarily small neighborhood of 0,
$(B2')$ $g(0) = 1$,
$(B3')$ g is monotone increasing or decreasing on each of I' and J'
$(B4')$ $\int_{\mathbb{H}} |\nabla g|^2$ is arbitrarily small,
$(B5')$ g has the value 0 at the endpoints of I' and J' that are distinct from 0,
$(B6')$ g is monotone, in the sense that $\operatorname{osc}_{\mathbb{H}}(g) = \operatorname{osc}_{\partial\mathbb{H}}(g)$, and
$(*')$ $|g(x_k') - g(x_{k+1}')| \le 1/3$ and $|g(y_k') - g(y_{k+1}')| \le 1/3$ for all $k \in \mathbb{N}$.

Since a conformal map from $\mathbb{H}$ onto Ω extends to a homeomorphism (using the spherical metric) from $\overline{\mathbb{H}} \cup \{\infty\}$ onto $\overline{\Omega}$, and also it does not change the Dirichlet energy $\int |\nabla g|^2$, all these properties can be transferred to Ω, and yield a function with the corresponding properties. The proof of the analog of $(*)$ then follows, as in the proof of Case 3 of the previous section; see Sect. 4.5.1.

The construction of g is very similar to the construction we did in Case 3 of the previous section. We fix a small $R_1 > 0$ and define $r_1 = R_1/2$. In the annulus $A_1 := A(0; r_1, R_1)$ we define g to be a radial function that is equal to 0 in the outer circle, and increases to $1/N$ in the inner circle, where $N \in \mathbb{N}$ is fixed. The slope of g is $\simeq \frac{1}{N(R_1 - r_1)}$ in A_1. Then we define $R_2 < r_1$ to be so small that the "transition" annulus $A(0; R_2, r_1)$ contains a point $x_k' \in I'$ and a point $y_l' \in J'$. Here we set g to be constant, equal to $1/N$. Then we define $r_2 = R_2/2$ and $A_2 := A(0; r_2, R_2)$, and we set g to be a radial function that increases from $1/N$ in the outer circle to $2/N$ in the inner circle. By construction, no interval $[x_{m+1}', x_m']$ or $[y_m', y_{m+1}']$ can intersect both annuli A_1 and A_2; this is because the sequences x_m' and y_m' are strictly monotone. The use of the transition annulus is crucial as we will see. We proceed in the same way, until the last annulus $A_N = A(0; r_N, R_N)$, where the function g increases from $(N-1)/N$ in the outer circle to 1 in the inner circle, with slope $\simeq \frac{1}{N(R_N - r_N)}$. Finally, we set $g \equiv 0$ outside $B(0, R_1)$, $g \equiv 1$ inside $B(0, r_N)$, and then restrict g to $\overline{\mathbb{H}}$.

By construction, each of the intervals $[x'_{k+1}, x'_k]$, $[y'_k, y'_{k+1}]$, $k \in \mathbb{N}$, intersects at most one annulus $A_m, m \in \{1, \dots, N\}$, where the function g increases by $1/N$. In the transition annuli of the form $A(0; R_m, r_{m-1})$ the function g is constant. Hence, we have

$$|g(x'_k) - g(x'_{k+1})| \leq \frac{1}{N} \quad \text{and} \quad |g(y'_k) - g(y'_{k+1})| \leq \frac{1}{N}$$

for all $k \in \mathbb{N}$. In particular, these are less than $1/3$ if N is sufficiently large.

Regarding the Dirichlet energy, we compute:

$$\begin{aligned}\int |\nabla g|^2 &= \sum_{m=1}^{N} \int_{\mathbb{H} \cap A_m} |\nabla g|^2 \lesssim \sum_{m=1}^{N} \frac{1}{N^2(R_m - r_m)^2} m_2(\mathbb{H} \cap A_m)\\ &\simeq \sum_{m=1}^{N} \frac{1}{N^2(R_m - r_m)^2}(R_m^2 - r_m^2) \simeq \sum_{m=1}^{N} \frac{1}{N^2 r_m^2} r_m^2 \simeq \frac{1}{N},\end{aligned}$$

where we used the fact that $R_m = 2r_m$. If N is sufficiently large, then $\int |\nabla g|^2$ can be as small as we wish, completing the proof. □

The case that $h(K)$ has positive Lebesgue measure has to be treated separately, and, in fact, the following more general statement holds in $\mathbb{R}^n$:

Theorem 4.9 (Theorem 1.3). *Let $K \subset \mathbb{R}^n$ be a compact set of positive Lebesgue measure and $1 \leq p < \infty$. Then K is non-removable for $W^{1,p}$.*

Proof We may assume that $\text{int}(K) = \emptyset$, otherwise K is trivially non-removable for $W^{1,p}$, $1 \leq p \leq \infty$, since one can simply consider a continuous function with no partial derivatives, supported on $\text{int}(K)$.

Define $\Omega := \mathbb{R}^n \setminus K$. Let $x_0 \in K$ be a Lebesgue point, i.e.,

$$\frac{m_n(B(x_0, r) \cap \Omega)}{m_n(B(x_0, r))} \to 0$$

as $r \to 0$. Hence, for each $i \in \mathbb{N}$ there exists an arbitrarily small $r_i > 0$ such that

$$m_n(B(x_0, r_i) \cap \Omega) \leq 2^{-ip} r_i^n. \tag{4.7}$$

Without loss of generality, we assume that $x_0 = 0$ and we set $B_i = B(x_0, r_i)$. We can also assume that the sequence $\{r_i\}_{i \in \mathbb{N}}$ satisfies

$$r_{i+1} < r_i/2 < 1/2 \tag{4.8}$$

for $i \in \mathbb{N}$.

Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be the 1-periodic extension of $|t|\chi_{[-1/2,1/2]}(t)$. Let c_i be a sequence of positive numbers and m_i be a sequence of positive integers, to be determined. We define

$$\phi_i(x) = c_i \phi\left(m_i\left(\frac{2|x|}{r_i} - 1\right)\right) \cdot \chi_{[r_i/2, r_i]}(|x|)$$

for $x \in \mathbb{R}^n$ and $i \in \mathbb{N}$. Roughly speaking, we changed the amplitude and frequency of $x \mapsto \phi(|x|)$, and also translated its support to the annulus $A(0; r_i/2, r_i)$. Observe that $\phi_i \in W^{1,p}(\mathbb{R}^n)$ for all $1 \le p < \infty$, ϕ_i is continuous in $\mathbb{R}^n$, and $|\phi_i| \le c_i$. Furthermore, for a.e. x in the annulus $A(0; r_i/2, r_i)$ we have

$$|\nabla \phi_i(x)| \simeq \frac{c_i m_i}{r_i},$$

with uniform constants. Hence,

$$\|\nabla \phi_i\|_{L^p(\mathbb{R}^n)} \simeq c_i m_i r_i^{n/p-1}$$

with constants depending only on the dimension n, and

$$\|\nabla \phi_i\|_{L^p(\Omega)} \lesssim \frac{c_i m_i}{r_i} 2^{-i} r_i^{n/p} \simeq 2^{-i} c_i m_i r_i^{n/p-1},$$

by (4.7).

We define

$$f = \sum_{i=1}^{\infty} \phi_i$$

and note that f is pointwise defined with $f(0) = 0$, since ϕ_i have disjoint supports by (4.8). Observe that if $c_i \to 0$, then the series $\sum_{i=1}^{\infty} \phi_i$ converges uniformly to a continuous function.

We have

$$\|\nabla f\|_{L^p(\Omega)} \le \sum_{i=1}^{\infty} \|\nabla \phi_i\|_{L^p(\Omega)} \lesssim \sum_{i=1}^{\infty} 2^{-i} c_i m_i r_i^{n/p-1},$$

so we wish to have that the latter series converges. If this is the case, then we will have indeed $f \in W^{1,p}(\Omega)$ by the completeness of the space.

If $f\in W^{1,p}(\mathbb{R}^n)$, then

$$\|\nabla f\|^p_{L^p(\mathbb{R}^n)}=\int\left|\sum_{i=1}^\infty \nabla\phi_i\right|^p=\sum_{i=1}^\infty \|\nabla\phi_i\|^p_{L^p(\mathbb{R}^n)}\simeq\sum_{i=1}^\infty c_i^p m_i^p r_i^{n-p}$$

because the functions ϕ_i have disjoint support. We wish the latter to be a divergent series, so that $f\notin W^{1,p}(\mathbb{R}^n)$.

Summarizing, we have to choose c_i, m_i such that $c_i\to 0$,

$$\sum_{i=1}^\infty 2^{-i}c_i m_i r_i^{n/p-1}<\infty, \quad \text{and} \quad \sum_{i=1}^\infty c_i^p m_i^p r_i^{n-p}=\infty.$$

If $p\geq n$, then we can choose $c_i=r_i^{1-n/p}\cdot i^{-1/p}$ and $m_i=1$ for all $i\in\mathbb{N}$. If $1\leq p<n$, then we choose $c_i=i^{-1/p}$ and m_i to be the smallest integer such that $m_i r_i^{n/p-1}\geq 1$. Then $(m_i-1)r_i^{n/p-1}<1$, so $m_i r_i^{n/p-1}\leq 2$. □

However, the conclusion fails for $W^{1,\infty}$:

Proposition 4.10 *There exists a compact set $K\subset\mathbb{R}^n$ of positive Lebesgue measure that is $W^{1,\infty}$-removable.*

Proof Let $C\subset\mathbb{R}$ be a Cantor set of positive Lebesgue measure, and define $K:=C^n$, so $m_n(K)>0$. We claim that K is $W^{1,\infty}$-removable. Let f be a continuous function on $\mathbb{R}^n$ that lies in $W^{1,\infty}(\mathbb{R}^n\setminus K)$. We wish to show that f is M'-Lipschitz, where $M'>0$ depends on $M=\|f\|_{W^{1,\infty}(\mathbb{R}^n\setminus K)}$. We fix a coordinate direction, say e_1, and a line L parallel to e_1. The function f is M-Lipschitz on each component of $L\setminus K$. On the other hand, by perturbing the line L we may obtain a line L' arbitrarily close and parallel to L such that $L'\cap K=\emptyset$. Hence, f is M-Lipschitz on L', and by continuity it is also M-Lipschitz on L. If $x,y\in\mathbb{R}^n$ are arbitrary points, then one can connect them with a polygonal path γ, each of whose segments is parallel to a coordinate direction such that the length of γ is comparable to $|x-y|$. The conclusion follows by using the Lipschitz bound on each of the segments of γ. □

Remark 4.11 In fact, the complement of the Cantor set K is a *quasiconvex set* in $\mathbb{R}^n$, i.e., there exists a constant $M>0$ such that for any two points $x,y\in\mathbb{R}^n\setminus K$ there exists a rectifiable path $\gamma\subset\mathbb{R}^n\setminus K$ that connects x and y, with

$$\text{length}(\gamma)\leq M|x-y|.$$

The argument in the proof of Proposition 4.10 can be modified to show that if the complement of compact set K with empty interior is quasiconvex, then K is $W^{1,\infty}$-removable.

5 Quasiconformal non-removability

We quickly sketch the strategy of constructing a homeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is quasiconformal on $\mathbb{R}^2 \setminus K$, but not globally quasiconformal; see also Sect. 1.2.

First we will define a continuous map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is the identity on the unbounded complementary component of the gasket K, but collapses each w-triangle to a tripod; see Fig. 7. This is done in Sect. 5.1.

Of course, this map is not a homeomorphism on the w-triangles, so we have to correct it. We do that by "folding" each w-triangle on top of each tripod; see Figs. 8 and 9. The folding map will be M-quasiconformal, in the sense of Definition 2.4, restricted on each w-triangle. The folding of a single equilateral triangle on top of a tripod is explained in Sect. 5.2. Moreover, the folding has to be compatible, in a sense, with f on the boundary of each w-triangle.

If the heights of the rectangles attached to each tripod are chosen to be sufficiently small, then we will obtain a homeomorphism Φ from $\mathbb{R}^2$ onto a limiting flap-plane S (Fig. 7), which is constructed out of infinitely many tripods, and thus falls into the setting of Proposition 3.2. The map Φ is the result of patching together the map f outside the w-triangles with the folding map of each w-triangle. The construction of the map Φ and of the flap-plane S is discussed in Sect. 5.3.

Finally, if one chooses the heights of the rectangles to be even smaller, then by Theorem 3.7 one obtains a quasisymmetric embedding Ψ of S onto $\mathbb{R}^2$. The composition $F = \Psi \circ \Phi$ will be a homeomorphism of $\mathbb{R}^2$ that is M'-quasiconformal on each w-triangle for some uniform $M' > 0$, but it cannot be globally quasiconformal, because it has to blow the gasket K to a set of positive area. Section 5.4 contains these details that finish the proof of non-removability.

5.1 Collapsing of w-triangles

Recall the definitions of w-triangles and v-triangles from Sect. 4.1. In this section we define a continuous map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is equal to the identity in the unbounded w-triangle, and collapses each bounded w-triangle to a tripod G. Further properties of f will be that it is injective on K°, and it maps the latter to a set of positive measure. Interestingly, the construction of such a map is rather a modification of the construction of the continuous function that we constructed in Sect. 4.

5.1.1 Building block

Recall from Sect. 3 that a tripod G is by definition the union of three line segments in the plane, which have a common endpoint, but otherwise they

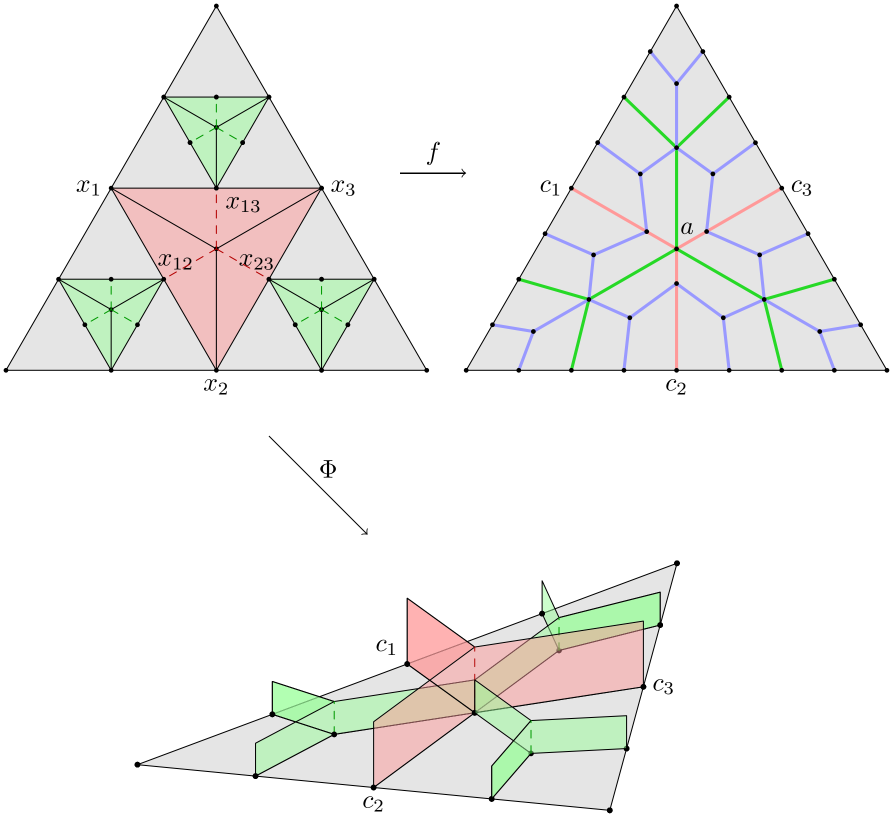

Fig. 7 Illustration of the collapsing map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and of the homeomorphism Φ from $\mathbb{R}^2$ onto the flap-plane S. For each $i = 1, 2, 3$ the collapsing map f sends the vertex x_i of the central pink triangle to the vertex c_i of the central pink tripod. For $i < j$ the midpoint x_{ij} of x_i is mapped to the barycenter a of the central tripod. The map Φ is a homeomorphism so in particular the green rectangles are *not* glued to the red rectangles, except at the three points $\Phi(x_{ij})$, $i < j$; these are the points that correspond to the barycenter a of the pink tripod (color figure online)

are disjoint; note that their length need not be the same. We call the common endpoint the central vertex of the tripod G. Every triple $c_1, c_2, c_3 \in \mathbb{R}^2$ of non-collinear points defines a *canonical* tripod, whose central vertex is the barycenter of c_1, c_2, c_3, i.e., it is

$$a = \frac{1}{3}(c_1 + c_2 + c_3).$$

In what follows, we will only be using canonical tripods, even if we do not mention it explicitly.

We consider an analog of the building block function discussed in Sect. 4.2. Let $W \subset \mathbb{R}^2$ an open equilateral triangle with vertices x_1, x_2, x_3. Then for each triple of non-collinear points $c_1, c_2, c_3 \in \mathbb{R}^2$ and for the canonical tripod G corresponding to these points there exists a continuous map $g: \overline{W} \to G$ such that

($\widetilde{B}2$) $g(x_i) = c_i$ for $i = 1, 2, 3$,
($\widetilde{B}3$) g is injective and *monotone* (see comments below), on each half-edge of ∂W, from its midpoint to a vertex,
($\widetilde{B}5$) g maps the midpoints of the edges of W to the central vertex a of G, and
($\widetilde{B}6$) g is monotone in the sense that $\operatorname{osc}_{\overline{W}}(g) = \operatorname{osc}_{\partial W}(g)$. Here, $\operatorname{osc}_{\overline{W}}(g) = \sup\{|g(x) - g(y)| : x, y \in \overline{W}\}$.

These properties should be compared to the properties of the building block function in Sect. 4.2. Note that ($\widetilde{B}6$) follows immediately from continuity and ($\widetilde{B}2$), since they imply that $g(\partial W) = G$. Also, ($\widetilde{B}2$), ($\widetilde{B}5$), and the injectivity from ($\widetilde{B}3$) imply that g maps each half-edge of W homeomorphically onto an edge of the tripod G. In particular, ($B3$) from Sect. 4.2 holds here, in the sense that if $I \subset \partial W$ is a half-edge of W and $J_1 \subset J_2 \subset I$ are segments, then $\operatorname{osc}_{J_1}(g) \leq \operatorname{osc}_{J_2}(g)$. This explains the use of the word *monotone* in the statement of ($\widetilde{B}3$).

From now on, a *building block map* will be a map g as above, and we will say that its *parameters* are c_1, c_2, c_3. At this moment we are not interested in the definition of the map g in the interior of the triangle W (which could be anything as long as $g(\overline{W}) = G$ and g is continuous), but we only focus on its boundary. The construction of such a continuous map g is elementary. For example, one can first collapse $\overline{W}$ to the canonical tripod defined by its vertices y_1, y_2, y_3, so that the midpoints of the edges are mapped to the barycenter of the triangle W, and so that the map is injective on each edge of ∂W. Then one can use an affine map to map this tripod to the canonical tripod G defined by c_1, c_2, c_3 such that the vertices y_1, y_2, y_3 are mapped to c_1, c_2, c_3, respectively. Note that affine maps preserve barycenters.

5.1.2 Inductive definition

A fundamental lemma that we will use is the following. A *convex quadrilateral* is the open region in the plane that is bounded by a polygon with four sides and (interior) angles strictly less than π (we wish to exclude degenerate cases).

Lemma 5.1 *Let $U \subset \mathbb{R}^2$ be convex quadrilateral and consider points $c_1, c_2, c_3 \in \partial U$ such that c_1 is a vertex of U, and c_2, c_3 lie on the interior of distinct sides of U that are not congruent to the vertex c_1. Also, consider the canonical tripod G corresponding to c_1, c_3, c_3, which are necessarily non-collinear points. Then $G \subset \overline{U}$ and each component Z of $U \setminus G$ is a convex quadrilateral. Furthermore, two of the sides of such a component Z are two* *of the edges of the tripod G and are congruent to its central vertex, while the other two sides of Z are contained in distinct sides of U.*

The proof is elementary and is omitted. Now, we define the desired map f inductively, in a very similar way, as the map we defined in Sect. 4.4. We define f to be the identity in the closure of the w-triangle of level 0 (i.e., the closure of the unbounded component of $\mathbb{R}^2 \setminus K$).

Note that the map f is already defined on the vertices of the central w-triangle W_1 of level 1. We define f on $\overline{W}_1$ to be a building block map that collapses this triangle to a tripod G_1. This tripod is contained in the v-triangle V_1 of level 0, which is convex. Each component U_2 of $V_1 \setminus G_1$ is a convex quadrilateral. We will be calling U_2 a u-quadrilateral. Note that each v-triangle V_2 of level 1 (i.e., V_2 is a component of $V_1 \setminus \overline{W}_1$) corresponds to a u-quadrilateral U_2, and in fact f maps ∂V_2 to ∂U_2 homeomorphically, in an orientation-preserving way, by property ($\widetilde{B}3$). Moreover, the midpoint x_1 of an edge of V_2 is mapped to a vertex c_1 of U_2 and the other two edges of V_2 are mapped to the other edges of U_2 that are not congruent to c_1.

We claim that we can define f on all w-triangles, so that each v-triangle corresponds to a u-quadrilateral as above. We now formulate and prove the inductive step.

Let V be a v-triangle and suppose that $f\big|_{\partial V}$ has been defined and maps ∂V homeomorphically onto the boundary of a convex quadrilateral U. Moreover, suppose that f maps the midpoint x_1 of an edge of V to a vertex c_1 of U and that each of the other two edges of V is mapped to one of the other two sides of U that are not congruent to c_1.

Consider the central w-triangle $W \subset V$ that has its vertices x_1, x_2, x_3 on ∂V. By the assumptions on the map $f\big|_{\partial V}$, the points $c_2 := f(x_2)$ and $c_3 := f(x_3)$ lie on distinct sides of U that are not congruent to $c_1 = f(x_1)$. We define f on $\overline{W}$ to be a building block map that collapses the triangle $\overline{W}$ to a tripod G with vertices c_1, c_2, c_3.

By Lemma 5.1 we see that each component U' of $U \setminus G$ is a convex quadrilateral. Two of the sides of U' are edges of the tripod G, each of which is the homeomorphic image of a half-edge of W under f, by property ($\widetilde{B}3$) of the building block map on $\overline{W}$. Suppose, for instance, that the edge of W is the segment $[x_1, x_2]$. The other two sides of U' are contained in two distinct sides of U. These two sides of U' have to correspond under the homeomorphism $f\big|_{\partial V}$ to an arc of ∂V that connects x_1 and x_2. Among the two such arcs, there is only one possibility, since it follows by the assumptions that $f^{-1}\big|_{\partial U}$ maps each side of U into one edge of V.

We thus see that there exists a v-triangle $V' \subset V \setminus \overline{W}$ such that f maps $\partial V'$ homeomorphically onto $\partial U'$. The midpoint $x_1' = \frac{x_1 + x_2}{2}$ of the edge $\partial V' \cap \overline{W}$ of V' is mapped to a point c_1' that is a vertex of U' (in fact it is the central vertex of G as follows from Lemma 5.1) and the other two edges of V' are mapped to the other two sides of U' by the mapping properties of f.

This completes the proof of the inductive step and shows that f can be defined on all of $W_\infty = \bigcup_{W\in\mathcal{W}} \overline{W}$.

5.1.3 Properties of f

(a) Each w-triangle $\overline{W}$ is mapped to a tripod G with vertices c_1, c_3, c_3 and central vertex $a = (c_1 + c_2 + c_3)/3$. Following the notation of Sect. 4, we define

$$\mathcal{O}(W) = \max_{i=1,2,3} |a - c_i|.$$

Note that $\mathcal{O}(W)$ is the length of the largest edge of the tripod G. The map f is *monotone* in the following sense. For each v-triangle V of level $m \geq 1$ we have

$$\operatorname{osc}_{\overline{V}\cap W_\infty}(f) = \sup_{x,y\in\overline{V}\cap W_\infty} |f(x) - f(y)| = \operatorname{osc}_{\partial V}(f).$$

This is the analog of Lemma 4.5. The reason it holds in our case is that, by its inductive definition, f maps ∂V homeomorphically onto a quadrilateral ∂U, and all w-triangles contained in V are mapped to tripods contained in $\overline{U}$.

We define the *level* of a u-quadrilateral U, which corresponds as above to a v-triangle V, to be an integer equal to the level of V; see also Sect. 4.1. More precisely, if the sidelength of V is 2^{-n}, $n \in \mathbb{N}$, then the level of V and U is n.

(b) The map f has a continuous extension to $\mathbb{R}^2$. The proof is the same as the proof of Proposition 4.6, with minor modifications. One observes that we have given essentially the same definition for f and it satisfies the properties $(\widetilde{B}2)$, $(\widetilde{B}3)$, $(\widetilde{B}5)$, and $(\widetilde{B}6)$, as in Sect. 4.2; see also Remark 4.4. Namely, what we called “height” of f on a w-triangle W in Sect. 4.2 is now the barycenter $a = (c_1 + c_2 + c_3)/3$ of the tripod G. Hence, the “height” of f here is the average of the values of f on the vertices of W, precisely as it was the case in the definition of f in Sect. 4.4. The only difference is that f is now complex-valued, instead of real-valued, but this does not affect the proofs. The proof of continuity goes through, if we ensure that f has on each w-triangle a certain modulus of continuity, as described in Case 3 of the proof of the basic Lemma 4.7 (see Sect. 4.5.1):

$(*)$ Assume that the triangles W_1, W_2 are adjacent and each has a vertex z_1, z_2, respectively, lying on a triangle ∂W_0 of a strictly lower level. Then

$$|f(z_1) - f(z_2)| \leq \mathcal{O}(W_0)/3.$$

As already discussed in Sect. 4.5.1, this boils down to looking at the dyadic points $\{x_k\}_{k\in\mathbb{N}}$ contained in each half-edge I of ∂W_0 and accumulating in the corresponding vertex of W_0, and requiring that

$$|f(x_k) - f(x_{k+1})| \leq \mathcal{O}(W_0)/3, \qquad \text{(Dyadic} *)$$

for all $k \in \mathbb{N}$, where x_k and x_{k+1} are "consecutive" dyadic points (for example, if $I = [0, 1]$ and 0 is a vertex of W_0, then $x_k = 2^{-k}$ for $k \in \mathbb{N}$). This will be ensured in the next section, where we construct more carefully the map f on each w-triangle, so that it still has properties ($\widetilde{B}2$), ($\widetilde{B}3$), ($\widetilde{B}5$), and ($\widetilde{B}6$), and also has this particular modulus of continuity.

(c) By continuity, the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is surjective. The u-quadrilaterals induce subdivisions in the image side, exactly as the v-triangles do in the domain. Let U_1 be the equilateral triangle of sidelength 1 that is the image of the v-triangle V_1 of level 0 (in fact, $U_1 = V_1$ since f is the identity on ∂V_1). Then U_1 contains three disjoint u-quadrilaterals of level 1, which lie in the complement of a "removed" tripod. Each of these quadrilaterals is the image of a v-triangle of level 1. This follows by the continuity and the monotonicity of f. In general, the closure of each u-quadrilateral of level m is the union of the closures of three disjoint u-quadrilaterals of level $m+1$. These u-quadrilaterals are the images of v-triangles of level $m+1$; see Fig. 7. By continuity, the diameters of u-quadrilaterals of level m converge to 0 as $m \to \infty$.

For each point $z \in \overline{U}_1$ we can find a sequence U_n, $n \in \mathbb{N}$, of nested u-quadrilaterals such that

$$\{z\} = \bigcap_{n=1}^{\infty} \overline{U}_n.$$

We set L° to be the set of points of $\overline{U}_1$ that do not lie on any "removed" tripod. The correspondence between U_n and V_n and the uniqueness of a sequence V_n shrinking to a point $x \in K^\circ$ (see the comments before Lemma 4.1) imply that f maps K° onto L°, and in fact f is injective on K°.

(d) The tripods contained in $\overline{U}_1$, which are the images of the w-triangles, have σ-finite length. Thus, the area of the tripods is equal 0. This implies that the image of K° has full measure inside $\overline{U}_1$.

5.2 Folding equilateral triangles to tripods

So far, we have a continuous surjective map $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is injective outside the union W_∞ of the closures of the w-triangles; of course continuity is still subject to choosing suitably the building block maps so that they have a certain modulus of continuity. We wish to change the definition of the map f

only inside each w-triangle so that f becomes injective everywhere. However, this is not possible if the target is $\mathbb{R}^2$. Thus, we change the target to a flap-plane S by attaching rectangles to each of the tripods; see Sect. 3 for the definition of a flap-plane. We then change the definition of f inside the w-triangles and allow them to be mapped onto the rectangles attached to the corresponding tripod. We wish to do this in such a way that the resulting map $\Phi\colon \mathbb{R}^2 \to S$ is a homeomorphism, quasiconformal inside the w-triangles. This will be discussed in detail in the next section.

In this section we show how one can "fold" an equilateral triangle $W \subset \mathbb{R}^2$ onto a space X obtained by attaching rectangles E to a single tripod G. The space X is constructed very similarly to the flap-planes discussed in Sect. 3.1. One first cuts the plane along the edges of the tripod G, and then attaches two rectangles on each slit arising from an edge. The width of each rectangle is equal to the length of the corresponding edge of G, and the height is a prescribed constant $h > 0$, which is the same for all 6 rectangles. The barycenter a of G must "lift" to three line segments, along which neighboring rectangles are glued; see Fig. 2 for the gluing pattern. We remark that the lengths of the edges of G, and thus the widths of the rectangles, need not be equal to each other. The space X is a topological disk, and we endow it with its natural length metric d, so that each rectangle $E \sim G$ (i.e., E is glued to an edge of G) is isometric to a rectangle with the Euclidean metric.

We wish to construct a homeomorphism $\phi\colon \overline{W} \to X$ that has the following properties:

(A) the composition of ϕ with the natural projection $X \to G$ satisfies ($\widetilde{B}2$), ($\widetilde{B}3$), ($\widetilde{B}5$), and ($\widetilde{B}6$),
(B) the same composition has the modulus of continuity in (Dyadic $*$), and
(C) ϕ is M-quasiconformal (in the sense of Definition 2.4) in the interior of the preimage of each rectangle $E \sim G$, where M is independent of the tripod G and of the height h of the rectangles $E \sim G$,

provided that h is sufficiently small.

To do this, we first draw the heights of W, which split it into three quadrilaterals, each of which is a rotation of the other, and contains in its boundary a vertex of W together with two congruent half-edges of W; see Fig. 8. Let Z be one of these quadrilaterals. We will "fold" Z with a piecewise linear map to the two rectangles attached to an edge of e of G. Assume that the length of e is ℓ. We assume that the height h of the rectangles is less than $\ell/6$. Then there exists a unique $N \in \mathbb{N}$ such that

$$\ell = Nh + q,$$

where $h \leq q < 2h$ is a remainder term.

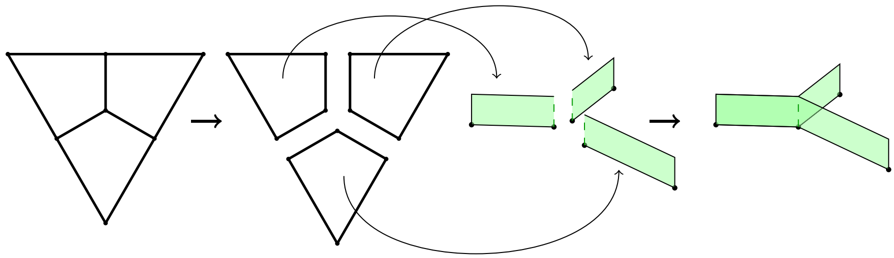

Fig. 8 An equlateral triangle W is split into three quadrilaterals Z_i, $i = 1, 2, 3$. Then each Z_i is folded over an edge of the tripod G. Finally, the resulting rectangles are glued to obtain the space X

We now divide each of the two rectangles E attached to e into N squares of dimensions $h \times h$, and a rectangle of dimensions $q \times h$, as in Fig. 9. We also subdivide Z by considering $N + 1$ dyadic points on each half-edge of W that is contained in ∂Z, and drawing trapezoids as in Fig. 9. Each of these $2N$ trapezoids is similar to the trapezoid $ABCD$, in the sense that it can be obtained by applying a Euclidean similarity to $ABCD$.

Each of these trapezoids can be mapped with a piecewise linear map to the corresponding square of dimensions $h \times h$. In fact, this can be done by drawing one diagonal in each trapezoid and in each square, and and then gluing two linear maps. This piecewise linear map is M-quasiconformal for a uniform $M > 0$.

We have to treat specially the two triangles near the vertex of W that lies in ∂Z. The triangle HIK is a right triangle with angle $\widehat{HIK}$ equal to $\pi/6$. Hence, it can be mapped to the rectangle $H'I'J'K'$ (so that vertices are mapped to the corresponding vertices) with a piecewise linear map that is M-quasiconformal for a universal $M > 0$. Recall here that the ratio of the sides of the rectangle $H'I'J'K'$ is by construction bounded between 1 and 2. The construction of such a map is done by converting the triangle HIK to a quadrilateral, by introducing the midpoint J of the segment IK. Then one can draw the diagonals HJ and $H'J'$ and glue together two linear maps, one from the triangle HJK to the triangle $H'J'K'$, and one from the triangle HIJ to the triangle $H'I'J'$.

Hence, we obtain two piecewise linear maps, one defined on the triangle HIK and one defined on the triangle KIL. The first one maps linearly the segment IJ to the segment $I'J'$ and the second maps linearly the segment IJ to the segment $\tilde{I}'J'$. Upon folding, the segment $I'J'$ gets glued to the segment $\tilde{I}'J'$, so we obtain a homeomorphism ϕ from $\overline{Z}$ onto the two folded rectangles that are glued over an edge of the tripod G. We remark that the segment $A'D'$ is not glued to anything at this moment, and it will be glued to the corresponding segment that arises from the rectangles attached to another edge of G.

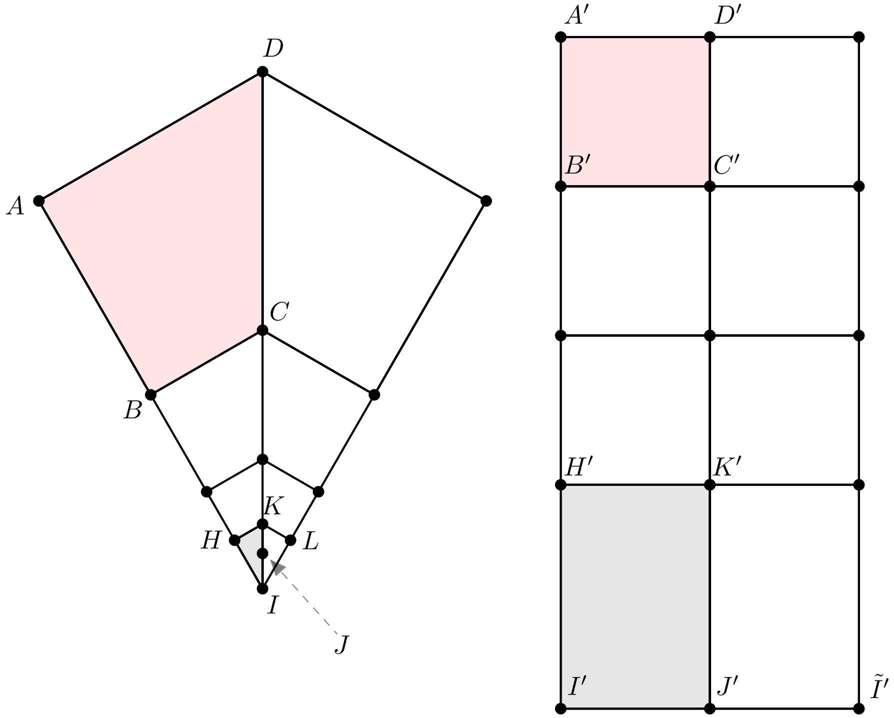

Fig. 9 Illustration of the folding map from a quadrilateral Z onto two rectangles

With the same procedure, we construct such a homeomorphism ϕ_i for each one of the three quadrilaterals Z_i, $i = 1, 2, 3$, in the subdivision of the original triangle W. The map ϕ_i maps $\overline{Z}_i$ to the two folded rectangles attached to the edge e_i of G, for $i = 1, 2, 3$. We remark that the heights of all these rectangles are equal to h, but their widths might vary if they are attached to distinct edges of G. The quadrilateral Z_1 is glued to Z_2 along edges of the form AD. The maps ϕ_1 and ϕ_2 are linear on these edges and they map AD to edges of "type" $A'D'$, whose length is h. Hence, the maps ϕ_1 and ϕ_2 can be "glued" together to obtain a homeomorphism from $\overline{Z}_1 \cup \overline{Z}_2$ onto the rectangles that are attached on two of the edges of the tripod G. Finally, one glues the third piece $\overline{Z}_3$ to obtain the desired homeomorphism $\phi\colon \overline{W} \to X$; see Fig. 8. The map ϕ is M-quasiconformal, in the sense of Definition 2.4, in the interior of the preimage of each rectangle $E \sim G$. Note that the set where ϕ is quasiconformal is the complement in W of finitely many line segments, along which the triangle W is "folded".

By construction, (A) and (C) hold, so it only remains to check that ϕ has the desired modulus of continuity (B). We reformulate the claim, using the metric d of the space X. The quantity $\mathcal{O}(W)$ is equal to the maximum length of the edges of the tripod G. We need to check that if x_k, x_{k+1} are consecutive "dyadic" points in a half-edge of W, then

$$d(\phi(x_k), \phi(x_{k+1})) \leq \mathcal{O}(W)/3.$$

Since the segment $[\phi(x_k), \phi(x_{k+1})] \subset X$ projects isometrically into an edge of $G \subset \mathbb{R}^2$, the desired claim will then follow. Note that $\phi(x_k)$ and $\phi(x_{k+1})$ are contained either in an edge of a square of dimensions $h \times h$, or in an edge of a rectangle of dimensions $q \times h$, where $h \leq q < 2h$; recall the subdivision of Z into "dyadic" trapezoids and the definition of ϕ in each trapezoid. Hence, $d(\phi(x_k), \phi(x_{k+1})) \leq 2h \leq \ell/3$, since we chose $h \leq \ell/6$, where ℓ is the length of the edge of G that contains x_k and x_{k+1}. On the other hand, $\ell \leq \mathcal{O}(W)$, so our claim is proved.

5.3 Homeomorphism onto a flap-plane

Here we show how to patch together the folding maps ϕ of each w-triangle W (from Sect. 5.2) with the continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ in order to obtain a homeomorphism Φ from $\mathbb{R}^2$ onto a flap-plane S. This will be done in three steps. (a) First, we explain how the folding maps ϕ of the w-triangles can induce the building block maps of $f : \mathbb{R}^2 \to \mathbb{R}^2$, so that f is continuous; recall the comments in Sect. 5.1.3(b). (b) Then we discuss how to construct a flap-plane S by gluing rectangles to the tripods provided by f. (c) Finally, we explain how one can "patch" together the map f with the folding maps ϕ to obtain a homeomorphism $\Phi : \mathbb{R}^2 \to S$ that is quasiconformal in $\mathbb{R}^2 \setminus K$.

(a) Let W be an equilateral triangle and G be a tripod. Consider a map ϕ that maps $\overline{W}$ to a metric space X as in Sect. 5.2, satisfying (A), (B), and (C). There is a natural projection $P : X \to G$, so that the composition $P \circ \phi$ satisfies $(\widetilde{B}2)$, $(\widetilde{B}3)$, $(\widetilde{B}5)$, $(\widetilde{B}6)$, and condition $(*)$. Note that ϕ depends on the height h of the rectangles that we attach to the tripod G, and condition $(*)$ is subject to choosing a sufficiently small height h.

Using the compositions $P \circ \phi$ as the building block maps of f we obtain a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$. More specifically, once f has been defined on w-triangles of level $m - 1$, then we know the tripods (as sets) that will correspond to the w-triangles of level m. Then one considers the folding maps ϕ with respect to the w-triangles of level m and the corresponding tripods. The compositions $P \circ \phi$ yield the building block maps that are used to define f on w-triangles of level m. The comments in Sect. 5.1.3(b) justify why f will extend continuously on all of $\mathbb{R}^2$.

A subtlety here is that if we change the height h of the rectangles attached to a specific tripod, then this changes the folding map ϕ, and alters the map f completely!

(b) Let $\mathcal{G}$ be the family of tripods arising from f, that is, the family $f(\overline{W})$, where W is a w-triangle of level at least 1. The family $\mathcal{G}$ is a family of tripods that has property (G), i.e., any two tripods have at most one point of intersection, in which case it is a non-central vertex of one of them; recall the definitions from Sect. 3.1.2. Also, the "graph" $T_\infty = \bigcup_{G\in\mathcal{G}} G$ has degree uniformly bounded by 6, as one can see inductively. Proposition 3.2 implies that if the heights of the rectangles attached to each tripod are sufficiently small, then one obtains a limiting flap-plane (S, d), which is a complete metric space. Recall Remark 3.6, which allows us to choose inductively the tripods G and the heights h of the corresponding rectangles, and still obtain the limiting flap-plane. The limiting space S can be regarded as the union of $\mathbb{R}^2 \setminus \bigcup_{G\in\mathcal{G}} G$ with the rectangles attached to each tripod G, after proper identifications; see also the comments in Sect. 3.1.3.

(c) The map f can be "patched" with the maps ϕ to yield naturally a map $\Phi\colon \mathbb{R}^2 \to S$. Namely, the maps f and Φ agree outside the closures of w-triangles, and inside a w-triangle $\overline{W}$ the map Φ is defined to be equal to the folding map ϕ that folds $\overline{W}$ on top of the corresponding tripod $f(\overline{W})$. There is possibly an ambiguity in the definition of Φ, whenever two w-triangles $\overline{W}_1, \overline{W}_2$ intersect at one point. In this case the corresponding tripods G_1, G_2 also intersect at one point (as described in property (G)). Then $\Phi(\overline{W}_1)$ and $\Phi(\overline{W}_2)$ also have to intersect at precisely one point in the space S, by our basic rules in the construction of a flap-plane; see Sect. 3.1 and Fig. 4. Hence, Φ can unambiguously be defined.

We claim that Φ is injective. Recall from Sect. 5.1.3(c) that f is injective on K° with $f(K^\circ) = L^\circ$ and it is the identity in the unbounded complementary component of the gasket K. Also, each of the maps ϕ is a homeomorphism from a w-triangle $\overline{W}$ onto the rectangles attached to the corresponding tripod. In order to show that Φ is injective, it remains to prove that if $x_1 \in \partial W_1$ and $x_2 \in \partial W_2$, where W_1, W_2 are distinct w-triangles with $\Phi(x_1) = \Phi(x_2)$, then $x_1 = x_2$. From the construction of S (looking at a finite stage of the construction) one sees that the equality $\Phi(x_1) = \Phi(x_2)$ is only possible if the triangles W_1 and W_2 are adjacent, so we necessarily have that $x_1 = x_2$ and it is the intersection point $\partial W_1 \cap \partial W_2$. This completes the proof of injectivity.

In fact, Φ is also surjective, since f maps K° surjectively onto L°; see the comments in Sect. 5.1.3(c). Note that Φ maps, in a sense, ∞ to ∞ and is continuous in a neighborhood of ∞, as it agrees with the identity there. Hence, if we show that $\Phi\colon \mathbb{R}^2 \to S$ is continuous, then it will be a a proper bijective map, and hence a homeomorphism, as desired.

The proof of continuity is very similar to the proof of Proposition 3.2, so we only provide a sketch. Assume for the sake of contradiction that x_k is a sequence in $\mathbb{R}^2$ converging to a point $x \in \mathbb{R}^2$, but the image points $y_k = \Phi(x_k)$, $y = \Phi(x)$ satisfy $d(y_k, y) \geq \delta$ for all $k \in \mathbb{N}$ and for some $\delta > 0$. The map Φ

is already continuous in the interior of all w-triangles, as it agrees there with the homeomorphisms ϕ, hence x cannot lie in the interior of a w-triangle. It follows that x must lie on the gasket K.

With the same reasoning, we cannot have that infinitely many terms x_k lie in the same w-triangle $\overline{W}$. Hence, we either have a subsequence of x_k all of whose terms lie in K°, or there exists a subsequence of x_k, still denoted by x_k, whose terms lie in distinct w-triangles $\overline{W}_k$.

In the first case, we assume that $x_k \in K^\circ$ for all $k \in \mathbb{N}$ and we consider two subcases. The first subcase is that $x \in K^\circ$, in which case $y \in L^\circ \subset \mathbb{R}^2 \setminus \bigcup_{G \in \mathcal{G}} G$; recall that $\Phi(K^\circ) = f(K^\circ) = L^\circ$ from Sect. 5.1.3(c). Then we necessarily have $y_k \to y$ with the Euclidean metric. This is because the map Φ "agrees" there with f, and f is continuous. We would like to argue that $y_k \to y$ with respect to the metric d of S. This follows because the metric d restricted to $\mathbb{R}^2 \setminus \bigcup_{G \in \mathcal{G}} G$ is topologically equivalent to the Euclidean metric; see Remark 3.3. Hence, we obtain a contradiction to the assumption that $d(y_k, y) \geq \delta$ for all $k \in \mathbb{N}$.

The other subcase is that x lies on an edge of a w-triangle W. Then one can find a nested sequence V_k of v-triangles and a subsequence of x_k, still denoted by x_k, such that $x_k \in V_k$ and $x \in \partial V_k$ for all $k \in \mathbb{N}$; see comments before Lemma 4.1. The triangles V_k correspond, under f, to a nested sequence of u-quadrilaterals U_k such that the projection $\tilde{y}$ of the point $\Phi(x) = y \in S$ to the plane lies in ∂U_k. In fact, y lies in the boundary of a rectangle E attached to the tripod G that corresponds to W, and also y is "accessible" from U_k; this statement can be made more precise if one looks at a finite stage (S_n, d_n) of the construction of (S, d), as discussed in Sect. 3. The line segments $(\tilde{y}, y_k] \subset \mathbb{R}^2$ are contained in the convex quadrilateral $\overline{U}_k$ and have the property that they do not intersect any tripod infinitely often. As in the proof of Proposition 3.2, this implies that for any given $\varepsilon > 0$ we have

$$d(y, y_k) \leq |\tilde{y} - y_k| + \varepsilon \leq \operatorname{diam}(U_k) + \varepsilon$$

for all sufficiently large k. Since $\operatorname{diam}(U_k) \to 0$ by the continuity of f, we obtain a contradiction.

The second case is that $x_k \in \overline{W}_k$, which are distinct w-triangles. We will reduce this to the previous case. One can find points $x_k' \in \partial W_k$ with $x_k' \to x$, since the diameters of W_k shrink to 0, such that the corresponding image points y_k' satisfy $d(y_k', y_k) \to 0$. This is because the heights of the rectangles attached to distinct tripods have to shrink to 0; see the statement of Proposition 3.2. It suffices to show that $d(y_k', y) \to 0$. Finally, arbitrarily close to x_k' one can find points $x_k'' \in K^\circ$ that converge to x such that the corresponding image points $y_k'' \in L^\circ$ satisfy $d(y_k', y_k'') \to 0$. This follows from the argument in the previous paragraph. However, we are now reduced to the previous case, so $d(y_k'', y) \to 0$, and therefore $d(y_k, y) \to 0$, a contradiction.

5.4 Finishing the proof of non-removability

We have constructed a homeomorphism $\Phi: \mathbb{R}^2 \to S$ that is M-quasiconformal in each w-triangle W, except at the finitely many line segments along which W is folded; recall property (C) of the folding maps in Sect. 5.2. If the heights of the rectangles attached to the tripods are chosen inductively to be sufficiently small, then by Theorem 3.7 there exists an η-quasisymmetry $\Psi: S \to \mathbb{R}^2$. Now, we consider the composition $F := \Psi \circ \Phi$ which is a homeomorphism of $\mathbb{R}^2$.

The map Ψ, restricted on a rectangle E attached to a tripod, is η-quasisymmetric, and thus M'-quasiconformal, where M' depends only on η; this follows from Lemma 2.6 and the fact that the metric d, restricted to E, is isometric to the Euclidean metric. Hence, by Lemma 2.7, in each w-triangle W the map F is $M \cdot M'$-quasiconformal in the complement of finitely many line segments; these are precisely the segments along which the triangle W is folded. Since these segments have finite length, the are removable for quasiconformal maps by Lemma 2.8, so F is $M \cdot M'$-quasiconformal on each w-triangle W.

We finally claim that F cannot be quasiconformal on $\mathbb{R}^2$. First, recall the continuous map $f: \mathbb{R}^2 \to \mathbb{R}^2$ that is in fact the composition of Φ with the natural projection P from S to $\mathbb{R}^2$; see Sect. 3.1 and Remark 3.4 for the definition of the projection. We note that $f(K^\circ) = L^\circ$ has positive Lebesgue measure, as remarked in Sect. 5.1.3(d). Since the projection P is 1-Lipschitz and L° projects to itself, if μ denotes the Hausdorff 2-measure of S, we have

$$\mu(L^\circ) \geq m_2(L^\circ) > 0.$$

Compare to the property $(G7)$ of the projections of flap-planes in Sect. 3.1.

Finally, by Lemma 2.9, we conclude that the pushforward measure $\Psi_*\mu$ and the Lebesgue measure on $\mathbb{R}^2$ are mutually absolutely continuous. This implies that $F(K^\circ) = \Psi(L^\circ) \subset \mathbb{R}^2$ has positive Lebesgue measure. Thus, the map F blows the set K° of measure zero to a set of positive measure. Using Lemma 2.3, we conclude that F cannot be globally quasiconformal. □

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