ON THE DISTORTION OF BOUNDARY SETS UNDER CONFORMAL MAPPINGS

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Introduction

Let Ω be a Jordan domain in the complex plane $\mathbb{C}$. If $w_0 \in \Omega$, we let $\omega \equiv \omega(\cdot\,; \Omega, w_0)$ denote the *harmonic measure* on $\partial\Omega$ evaluated at w_0. It can be defined as the image of normalized Lebesgue measure under the boundary correspondence induced by a conformal mapping f of the unit disc $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ onto Ω with $f(0) = w_0$.

The purpose of this paper is to study metric properties of the harmonic measure or, more precisely, to study possible relations between ω and the Hausdorff measures.

Let φ be a continuous, increasing function on $[0, t_0)$ such that $\varphi(0) = 0$. Let E be a bounded plane set. For $\delta > 0$ consider all coverings of E with a countable number of discs Δ_j with radii $r_j < \delta$ and define

$$\Lambda_\varphi^\delta(E) = \inf\{\sum_j \varphi(r_j)\},$$

the infimum being taken over all such coverings. The limit

$$\Lambda_\varphi(E) = \lim_{\delta \to 0} \Lambda_\varphi^\delta(E)$$

is called the *Hausdorff measure* of E with respect to the measure function φ. If $\varphi(t) = t^\alpha$, for some $\alpha > 0$, then Λ_φ is denoted by Λ_α. See [7] for more information about Hausdorff measures.

The harmonic measure is said to be *absolutely continuous* with respect to Λ_φ if

$$\Lambda_\varphi(E) = 0 \;\Rightarrow\; \omega(E) = 0$$

for all Borel subsets $E \subset \partial\Omega$. We write $\omega \ll \Lambda_\varphi$ in this case. The harmonic measure ω is by definition *singular* with respect to Λ_φ if there exists a Borel subset $E \subset \partial\Omega$ such that $\Lambda_\varphi(E) = 0$ and $\omega(\partial\Omega \setminus E) = 0$. (Notation: $\omega \perp \Lambda_\varphi$.) Clearly, these notions do not depend on the choice of w_0.

We now recall some well-known theorems. If Ω is a Jordan domain with rectifiable boundary, a theorem by F. and M. Riesz (see [**8**, Chapter 10, § 1, Theorem 2]) states that ω is equivalent to arc length ($\equiv \Lambda_1$). However, for non-rectifiable boundaries this is no longer true. Lavrentiev [**10**] was the first to give an example of a Jordan domain such that $\omega \not\ll \Lambda_1$. A simpler example with the stronger property $\omega \perp \Lambda_1$ is due to McMillan and Piranian [**13**]. Subsequently, Carleson [**6**] constructed for each $\alpha < \frac{1}{2}$ a domain such that $\omega \not\ll \Lambda_\varphi$ where

$$\varphi(t) = t\exp\{\log^\alpha t^{-1}\}.$$

Finally, Kaufman and Wu [**9**] proved that there exists a Jordan domain such that for each $A > 0$, $\omega \perp \Lambda_\varphi$ where

$$\varphi(t) = t\exp\{A\sqrt{(\log t^{-1})}\}.$$

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In the other direction, it was shown that the *compression* of boundary sets cannot be too strong. It is a consequence of Beurling's projection theorem that ω is always absolutely continuous with respect to $\Lambda_{\frac{1}{2}}$ (see [12]). In his remarkable paper [6] Carleson proved that there exists a number $\beta > \frac{1}{2}$ such that $\omega \ll \Lambda_\beta$ for all Jordan domains. To the best of my knowledge the question of *whether ω is always absolutely continuous with respect to Λ_α for all $\alpha < 1$* remained open.

As to the *expansion* of boundary sets, a theorem due to Øksendal is of importance (see [14, 15]): ω is always singular with respect to area measure ($\equiv \Lambda_2$). It was asked [15, p. 183] *whether ω is always singular with respect to Λ_α for all $\alpha > 1$.*

The principal results of the paper are presented in the following three theorems. Theorems 1 and 3 answer in *affirmative* the two questions stated above. Theorem 2 refines the result of Kaufman and Wu. Our results are very nearly sharp.

THEOREM 1. *There exists a universal constant $C > 0$ such that for any Jordan domain Ω the harmonic measure on $\partial\Omega$ is absolutely continuous with respect to the Hausdorff measure Λ_φ where*

$$\varphi(t) = t\exp\{C\sqrt{(\log t^{-1}\log\log\log t^{-1})}\}.$$

THEOREM 2. *There exist a positive number $c > 0$ and a Jordan domain Ω such that the harmonic measure on $\partial\Omega$ is singular with respect to the Hausdorff measure Λ_φ where*

$$\varphi(t) = t\exp\{c\sqrt{(\log t^{-1}\log\log\log t^{-1})}\}.$$

THEOREM 3. *For any simply connected domain Ω and for any measure function φ such that*

$$\lim_{t\to\infty} (\varphi(t)/t) = 0,$$

the harmonic measure on $\partial\Omega$ is singular with respect to Λ_φ.

The first two theorems show that the distortion on the boundary obeys, in some sense, *the law of the iterated logarithm* (LIL). We intend now to explain what this law has to do with conformal mappings.

The distortion of boundary sets depends on the boundary behaviour of the derivative of the conformal mapping. The well-known correspondence exists between the derivatives of univalent functions and the class $\mathscr{B}$ of *Bloch functions* (i.e. holomorphic functions b in $\mathbb{D}$ satisfying

$$\| b \|_{\mathscr{B}} \equiv |b(0)| + \sup_{\mathbb{D}}(1-|z|^2)|b'(z)| < \infty,$$

see [3]). The lacunary series

$$b(z) = \sum_{k\geq 0} z^{2^k}$$

provides an important example of the Bloch function. Let $S_n(z) = \sum_{k=0}^{n} z^{2^k}$. The LIL for lacunary series [20] asserts that for almost all $\zeta \in \partial\mathbb{D}$,

$$\overline{\lim_{n\to\infty}} \frac{|S_n(\zeta)|}{\sqrt{(n\log\log n)}} = 1$$

or, which is equivalent,

$$\overline{\lim_{r\to 1-}} \frac{|b(r\zeta)|}{\sqrt{(\log(1-r)^{-1}\log\log\log(1-r)^{-1})}} = 1.$$

It seems natural that among Bloch functions, lacunary series have the 'greatest possible' boundary growth (in a sense imposed by the latter assertion). The following theorem is a clue for the proof of Theorem 1.

THEOREM A. *There exists a universal constant $C > 0$ such that if $b \in \mathscr{B}$ then for almost all $\zeta \in \partial\mathbb{D}$,*

$$\varlimsup_{r\to 1-} \frac{|b(r\zeta)|}{\sqrt{(\log(1-r)^{-1}\log\log\log(1-r)^{-1})}} \leqslant C\,\|b\|_{\mathscr{B}}.$$

Theorem A will be proved in § 1. In § 2 we shall derive Theorem 1 by combining Theorem A with some geometrical considerations due essentially to Carleson [**6**]. In § 3, which does not depend on the results of §§ 1 and 2, we shall prove Theorems 2 and 3.

We shall use the following notation:

$|\cdot|$ is the normalized ($|\partial\mathbb{D}| = 1$) Lebesgue measure on the unit circle $\partial\mathbb{D}$;
L^p, with $1 \leqslant p \leqslant \infty$, denotes the Lebesgue space on $(\partial\mathbb{D}, |\cdot|)$;
$\|\cdot\|_p$ is the L^p-norm;
$\langle\ ,\ \rangle$ denotes the scalar product in L^2;
$\hat{f}(n)$ denotes the nth Fourier coefficient $\int_{\partial\mathbb{D}} \bar{\zeta}^n f(\zeta)\,|d\zeta|$ of the function $f \in L^1$;
$B(z, r)$ is the open disc of radius r and centre z.

The letters C and c will be used to denote various constants which may differ from one formula to the next, even within a single string of estimates.

1. *LIL for Bloch functions*

In this section we shall prove Theorem A. The proof is preceded by a sequence of lemmas. The main auxiliary result is Lemma 1.3 which yields an estimate for the growth of L^p-means of Bloch functions.

The basic tool in our investigation will be the description of the Bloch class in terms of convolutions with the *Valleé–Poussin type kernels* $\{W_n\}_{n\geqslant 0}$. By definition, $W_0(z) = 1+z$. If $n > 0$, then W_n is an analytic polynomial satisfying the following conditions:

$\hat{W}_n(2^n) = 1$;
$\hat{W}_n \equiv 0$ outside $(2^{n-1}, 2^{n+1})$;
$\hat{W}_n$ is linear on $[2^{n-1}, 2^n]$ and on $[2^n, 2^{n+1}]$.

Obviously, $f = \sum_{n\geqslant 0} f * W_n$ for any polynomial f ($*$ is the sign of convolution).

The Bloch class $\mathscr{B}$ can be characterized as follows. (See [**4, 16**].) *Let b be a function holomorphic in $\mathbb{D}$. Then b is a Bloch function if and only if*

$$\sup_{n\geqslant 0} \|b * W_n\|_\infty < \infty.$$

To state the first lemma, we need some more notation. By $\mathfrak{M}_n$ is meant the set of all multiindices $\alpha = (\alpha_0, \ldots, \alpha_n)$ with $\alpha_j \geqslant 0$ integers. As usual,

$$|\alpha| = \alpha_0 + \ldots + \alpha_n,$$

$$\alpha! = \alpha_0! \ldots \alpha_n!.$$

Let $P_{(m)}$ denote the projection in L^2 onto the subspace $\operatorname{span}\{z^j: j \geqslant 2^m\}$. Given a

Bloch function b, we define

$$b_n = b * W_n,$$

$$b^\alpha = b_0^{\alpha_0} \dots b_n^{\alpha_n}.$$

Finally, we define the following quantities associated with a fixed Bloch function b and integers $n \geqslant 0$ and $p, \nu \geqslant 2$:

$$X(n,p) = \sup_{|x_\alpha| \leqslant 1} \left\| \sum_{\alpha \in \mathfrak{M}_n, |\alpha| = p} x_\alpha \frac{b^\alpha}{\alpha!} \right\|_2,$$

$$\tilde{X}(n,p,\nu) = \sup_{|x_\alpha| \leqslant 1} \left\| P_{(n+\nu)} \sum_{\alpha \in \mathfrak{M}_n, |\alpha| = p} x_\alpha \frac{b^\alpha}{\alpha!} \right\|_2,$$

the supremums being taken over all collections of complex numbers $\{x_\alpha\}$ such that $|x_\alpha| \leqslant 1$.

1.1. Lemma. *There exists a universal constant $C > 0$ such that the inequalities*

$$[X(n,p)]^2 \leqslant C^p \frac{1}{p!}(n+p)^p, \tag{1}$$

$$[\tilde{X}(n,p,\nu)]^2 \leqslant C^p 4^{1-\nu} \frac{1}{p!}(n+p)^p, \tag{2}$$

hold for any b in $\mathscr{B}$ with $\| b \|_{\mathscr{B}} \leqslant 1$.

Proof. We can assume that $\| b_j \|_\infty \leqslant 1$ for any j. If $p = 1$ then

$$\begin{aligned} [X(n,1)]^2 &= \sup_{|x_j| \leqslant 1} \left\| \sum_{j=0}^n x_j b_j \right\|_2^2 \\ &\leqslant \sum_{0 \leqslant i,j \leqslant n} |\langle b_i, b_j \rangle| \\ &\leqslant \sum_{|i-j| \leqslant 1} 1 \\ &\leqslant 3(n+1). \end{aligned}$$

Besides, $\tilde{X}(n,1,\nu) = 0$ for all $\nu \geqslant 1$.

For $p > 1$ we proceed by induction. Assume that (1) and (2) are valid for the values $1, 2, \dots, p-1$ instead of p. First, we shall establish the inequality (1). Note that

$$\sum_{\alpha \in \mathfrak{M}_n, |\alpha| = p} x_\alpha \frac{b^\alpha}{\alpha!} = \sum_{j=0}^n b_j \sum_{\beta \in \mathfrak{M}_j, |\beta| = p-1} y_{j,\beta} \frac{b^\beta}{\beta!}$$

with $|y_{j,\beta}| \leqslant 1$. Therefore,

$$\begin{aligned} [X(n,p)]^2 &\leqslant 2 \sup \sum_{0 \leqslant i \leqslant j \leqslant n} \left| \left\langle b_j \sum_{\substack{\beta \in \mathfrak{M}_j \\ |\beta| = p-1}} y_{j,\beta} \frac{b^\beta}{\beta!}, P_{(j-1)} b_i \sum_{\substack{\gamma \in \mathfrak{M}_i \\ |\gamma| = p-1}} y_{i,\gamma} \frac{b^\gamma}{\gamma!} \right\rangle \right| \\ &\leqslant 2 \sum_{j=0}^n X(j,p-1) \left[3X(j,p-1) + \sum_{\nu=1}^{j-2} \tilde{X}(j-2-\nu, p-1, \nu) \right]. \end{aligned}$$

By assumption this does not exceed

$$10C^{p-1}\sum_{j=0}^{n}\frac{(j+p-1)^{p-1}}{(p-1)!}\leqslant\frac{10C^{p-1}}{(p-1)!}\int_0^{n+p}t^{p-1}\,dt$$
$$\leqslant C^p\frac{1}{p!}(n+p)^p,$$

provided that $C\geqslant 10$.

It remains to establish the inequality (2). For $\nu=1$ it follows from (1). Hence, we suppose that $\nu\geqslant 2$. Since

$$\sum_{\alpha\in\mathfrak{M}_n,|\alpha|=p}x_\alpha\frac{b^\alpha}{\alpha!}=\sum_{j=0}^{n}\sum_{k=1}^{p}\frac{b_j^k}{k!}\sum_{\beta\in\mathfrak{M}_{j-1},|\beta|=p-k}y_{j,k,\beta}\frac{b^\beta}{\beta!}$$

with $|y_{j,k,\beta}|\leqslant 1$, and since $\widehat{b_j^k}(m)=0$ for $m\geqslant 2^{\nu+j-1}$ provided $k\leqslant 2^{\nu-2}$, we have

$$\tilde{X}(n,p,\nu)\leqslant\sum_{j=1}^{n}\sum_{k=1}^{2^{\nu-2}}\frac{1}{k!}\tilde{X}(j-1,p-k,n-j+\nu)\tag{3}$$
$$+\sum_{j=1}^{n}\sum_{k=2^{\nu-2}+1}^{p}\frac{1}{k!}X(j-1,p-k).$$

(Note that $\tilde{X}(n,p,\nu)=0$ for $p\leqslant 2^{\nu-1}$.)

By induction, the first sum in (3) does not exceed

$$\frac{(\sqrt{C})^{p-1}}{2^{\nu-1}}\sum_{j=1}^{n}\sum_{k=1}^{2^{\nu-2}}\frac{1}{k!}2^{j-n}\frac{(\sqrt{(j-1+p-k)})^{p-k}}{\sqrt{(p-k)!}}$$
$$\leqslant\frac{2(\sqrt{C})^{p-1}}{2^{\nu-1}}\sum_{k=1}^{2^{\nu-2}}\frac{1}{k!}\frac{(\sqrt{(n+p-k-1)})^{p-k}}{\sqrt{(p-k)!}}$$
$$\leqslant\frac{2(\sqrt{C})^{p-1}}{2^{\nu-1}}\sqrt{\left(\sum_{k\geqslant 0}\frac{1}{k!}\right)}\sqrt{\left(\sum_{k=0}^{p}\frac{(n+p-1)^{p-k}}{k!(p-k)!}\right)}$$
$$=\frac{2\sqrt{e}(\sqrt{C})^{p-1}}{2^{\nu-1}}\frac{1}{\sqrt{p!}}\sqrt{(n+p)^p}.$$

The second sum in (3) does not exceed

$$\sum_{j=1}^{n}\sum_{k=2^{\nu-2}+1}^{p}\frac{1}{k!}(\sqrt{C})^{p-k}\frac{(\sqrt{(j-1+p-k)})^{p-k}}{\sqrt{(p-k)!}}$$
$$\leqslant\frac{(\sqrt{C})^{p-1}}{2^{\nu-1}}\sum_{k=2}^{p}\frac{(\sqrt{(n+p-1)})^{p-k+2}}{(k-2)!\sqrt{(p-k+2)!}}$$
$$\leqslant\frac{(\sqrt{C})^{p-1}}{2^{\nu-1}}\sum_{l=0}^{p}\frac{1}{\sqrt{l!}}\frac{(\sqrt{(n+p-1)})^{p-l}}{\sqrt{(l!(p-l)!)}}$$
$$\leqslant\frac{\sqrt{e}(\sqrt{C})^{p-1}}{2^{\nu-1}}\frac{1}{\sqrt{p!}}\sqrt{(n+p)^p}.$$

Hence,

$$[\tilde{X}(n,p,\nu)]^2\leqslant 9eC^{p-1}4^{1-\nu}(p!)^{-1}(n+p)^p,$$

which implies (2) provided that $C\geqslant 9e$. This completes the proof.

1.2. COROLLARY. *If b is a Bloch function, with $\| b \|_{\mathscr{B}} \leqslant 1$, and if $p > 0$ is an integer, then*

$$\| b^p * W_n \|_2^2 \leqslant C^p p!\,(n+p)^{p-1},$$

C being a universal constant.

Proof. We have

$$\| b^p * W_n \|_2^2 = \left\| \left(\sum_{j=0}^{n+1} b_j \right)^p * W_n \right\|_2^2$$
$$= (p!)^2 \left\| \sum_{\alpha \in \mathfrak{M}_{n+1}, |\alpha| = p} \frac{b^\alpha}{\alpha!} * W_n \right\|_2^2.$$

Since $\widehat{b^\alpha}$ is zero on $[p2^{j+1}, \infty)$, provided that $\alpha \in \mathfrak{M}_j$ and $| \alpha | = p$, we have

$$\| b^p * W_n \|_2^2 \leqslant (p!)^2 \left\| \sum_{n-2-(\log p)/(\log 2) < j \leqslant n+1} b_j \sum_{\substack{\beta \in \mathfrak{M}_j \\ |\beta| = p-1}} \frac{1}{(\beta_j + 1)} \frac{b^\beta}{\beta!} \right\|_2^2$$
$$\leqslant (p!)^2 \Big[\sum_{n-2-(\log p)/(\log 2) < j \leqslant n+1} X(j, p-1) \Big]^2$$
$$\leqslant (p!)^2 \left(3 + \frac{\log p}{\log 2} \right)^2 C^{p-1} \frac{1}{(p-1)!} (n+p)^{p-1}$$
$$\leqslant p!\, C_1{}^p (n+p)^{p-1}.$$

1.3. LEMMA. *Let b be a Bloch function, with $\| b \|_{\mathscr{B}} \leqslant 1$, and let $p > 0$ be an integer. Then for $r \in (1 - e^{-p}, 1)$,*

$$\int_{\partial \mathbb{D}} | b(r\zeta) |^{2p} | d\zeta | \leqslant C^p p! \log^p \frac{1}{1-r},$$

C being a universal constant.

Proof. Without loss of generality, we can assume that $b(0) = 0$. Then

$$\int_{\partial \mathbb{D}} | b(r\zeta) |^{2p} | d\zeta | = \sum_{k \geqslant 1} r^{2k} | \widehat{b^p}(k) |^2$$
$$\leqslant \sum_{n \geqslant 0} r^{2^{n+1}} \sum_{k=2^n}^{2^{n+1}-1} | \widehat{b^p}(k) |^2$$
$$\leqslant \sum_{n \geqslant 0} r^{2^{n+1}} \| b^p * W_n + b^p * W_{n+1} \|_2^2$$
$$\leqslant 4C^p p! \sum_{n \geqslant 1} r^{2^n} (n+p)^{p-1}.$$

It is easy to check that if $p < \log(1-r)^{-1}$, then

$$\sum_{n \geqslant 1} r^{2^n} (n+p)^{p-1} \leqslant C^p \log^p (1-r)^{-1}$$

for some $C > 0$.

1.4. *Proof of Theorem A.* Assume that $\| b \|_{\mathscr{B}} \leqslant 1$. Fix an integer $p \geqslant 2$ and consider the maximal function

$$g_r(\zeta) = \max\{ | b(\rho\zeta) |^{2p} \colon 0 \leqslant \rho \leqslant r \},$$

where $\zeta \in \partial\mathbb{D}$ and $r \in (0,1)$. By Lemma 1.3 and the Hardy–Littlewood maximal theorem,

$$\int_{\partial\mathbb{D}} g_r(\zeta)\,|d\zeta| \leqslant C^p p^p \log^p(1-r)^{-1}$$

provided that $r \in (1-e^{-p}, 1)$. Define

$$u(r) = [(1-r)\log^{p+1}(1-r)^{-1}(\log\log(1-r)^{-1})^2]^{-1}.$$

Then

$$\int_{\partial\mathbb{D}} |d\zeta| \int_{1-e^{-p}}^{1} u(r) g_r(\zeta)\,dr \leqslant C^p p^p.$$

Hence, there exists a set E_p on $\partial\mathbb{D}$ enjoying the following properties:

(a) $|E_p| \geqslant 1-2^{-p}$;

(b) if $\zeta \in E_p$, then $\int_{1-e^{-p}}^{1} u(r) g_r(\zeta)\,dr \leqslant (2C)^p p^p$.

Now, fix $\zeta \in E_p$ and let $r \in (1-e^{-p}, 1)$. Since the function $r \mapsto g_r(\zeta)$ increases,

$$\int_r^1 u(t) g_t(\zeta)\,dt \geqslant g_r(\zeta) \int_r^1 u(t)\,dt$$
$$\geqslant g_r(\zeta)[\text{const. } p \log^p(1-r)^{-1}(\log\log(1-r)^{-1})^2]^{-1}.$$

Therefore, (b) implies that

$$g_r(\zeta) \leqslant C_1{}^p p^p \log^p(1-r)^{-1}[\log\log(1-r)^{-1}]^2,$$

and hence

$$|b(r\zeta)| \leqslant C(\sqrt{p})(\sqrt{\log(1-r)^{-1}})[\log\log(1-r)^{-1}]^{1/p}, \tag{4}$$

C being a universal constant.

For an arbitrary positive integer p_0, let

$$\zeta \in \bigcap_{p \geqslant p_0} E_p.$$

If $r \in (1-\exp\{-\exp\exp p_0\}, 1)$, we substitute $p = \log\log\log(1-r)^{-1}$ in (4) taking into account that $p \geqslant p_0$, $\zeta \in E_p$, and $r \in (1-e^{-p}, 1)$. Then, by (4),

$$|b(r\zeta)| \leqslant C\sqrt{\log(1-r)^{-1}}\sqrt{(\log\log\log(1-r)^{-1})}.$$

Hence, if ζ belongs to

$$E \equiv \bigcup_{p_0} \bigcap_{p \geqslant p_0} E_p,$$

we have

$$\overline{\lim_{r \to 1-}} \frac{|b(r\zeta)|}{\sqrt{(\log(1-r)^{-1}\log\log\log(1-r)^{-1})}} \leqslant C.$$

It remains only to note that (a) implies that $|E| = 1$.

2. *Absolute continuity of harmonic measure*

This section is devoted to the proof of Theorem 1. It will be accomplished in several stages. First, we shall establish two geometrical lemmas which, put together, show that the distortion on the boundary is, roughly speaking, the same as the distortion inside the domain. The latter, by the Koebe distortion theorem, depends entirely on the derivative of the conformal mapping, and this explains the relevance of our preceding results (see Introduction).

We shall make intensive use of extremal lengths. The notion is due to Ahlfors and Beurling [**2**]. For the convenience of the reader we shall recall the definition and some basic properties.

2.1. DEFINITION. Let Γ be a family of locally rectifiable curves in the complex plane. Consider all measurable (with respect to area measure), non-negative functions ρ defined on $\mathbb{C}$ such that

$$A(\rho) \equiv \iint_{\mathbb{C}} \rho^2 \neq 0, \infty.$$

For each ρ, we define

$$L(\rho) = \inf_{\gamma \in \Gamma} \int_{\gamma} \rho(z)\,|dz|.$$

(If ρ is not measurable with respect to $|dz|$, the integral is meant to be infinite.) The supremum

$$\lambda(\Gamma) = \sup_{\rho} \frac{[L(\rho)]^2}{A(\rho)}$$

is called *the extremal length* of the family Γ.

EXAMPLE. Let Γ be the family of all arcs in the annulus $\{z\colon r_1 < |z| < r_2\}$ which join the boundary circumferences. Then

$$\lambda(\Gamma) = \frac{1}{2\pi} \log \frac{r_2}{r_1}.$$

PROPERTIES. (a) The extremal length is a conformal invariant.

(b) If each curve $\gamma_2 \in \Gamma_2$ contains some curve $\gamma_1 \in \Gamma_1$ then $\lambda(\Gamma_1) \leqslant \lambda(\Gamma_2)$ (the comparison principle).

(c) If the families $\{\Gamma_j\}$ lie in disjoint, measurable sets and if $\Gamma = \bigcup \Gamma_j$ then

$$[\lambda(\Gamma)]^{-1} \geqslant \sum_j [\lambda(\Gamma_j)]^{-1}$$

(the Grötzsch principle).

For the proof see [**1**, Chapter 1D].

The following *theorem* (see [**17**]) provides a link between the extremal length and the harmonic measure:

Let Ω be a Jordan domain and let $K \subset \Omega$ be a continuum. For each Borel set E on $\partial\Omega$ denote by Γ_E the family of all arcs in Ω joining E with K. There exists a constant $A > 0$ independent of E, such that

$$\omega(E; \Omega, w_0) \leqslant A \exp\{-\pi\lambda(\Gamma_E)\}.$$

2.2. LEMMA. *Let ψ be a continuous, increasing function on $[0, \infty)$ such that $\psi(0) = 0$. Let $f\colon \mathbb{D} \to \Omega$ be a conformal mapping onto a Jordan domain Ω satisfying*

$$|f'(z)| \geqslant \psi(1 - |z|). \tag{1}$$

If l is a subarc of $\partial\mathbb{D}$ with endpoints z_1 and z_2, and if σ is a crosscut of Ω joining $f(z_1)$ and $f(z_2)$, then

$$\operatorname{diam}\sigma \geqslant c|l|\psi(|l|),$$

where $c > 0$ is a universal constant.

Proof. Assume that

$$C\operatorname{diam}\sigma < |l|\psi(|l|), \tag{2}$$

C to be chosen later. We define

$$Q = \{z \in \mathbb{D}\colon z|z|^{-1} \in l,\ |z| > 1-|l|\},$$
$$l' = \{z \in \partial Q\colon |z| = 1-|l|\}.$$

To simplify the notation, let $d = 1-|z|$ and $\delta = \operatorname{dist}(f(z), \partial\Omega)$. By (1) and the Koebe distortion theorem,

$$\delta \geqslant \tfrac{1}{4}d\psi(d).$$

If $z \in f^{-1}(\sigma)$ then, by (2), $\delta < |l|\psi(|l|)/C$, which implies that

$$Cd\psi(d) \leqslant 4|l|\psi(|l|).$$

Hence, $d < |l|$ provided that $C > 4$, and any arc in Q which joins l and l' must intersect $f^{-1}(\sigma)$.

Let Γ_z denote the family of all arcs in Q joining l' and $f^{-1}(\sigma)$. Comparing $\lambda(\Gamma_z)$ with the extremal length of the family of all arcs in Q which join l and l', we see that

$$\lambda(\Gamma_z) \leqslant \frac{1}{|l|}\log\frac{1}{1-|l|} \leqslant \frac{1}{1-|l|} \leqslant 2. \tag{3}$$

Set $\Gamma_w = f \circ \Gamma_z$. If $z \in l'$ then $d = |l|$ and $\delta \geqslant \frac{1}{4}|l|\psi(|l|) \geqslant \frac{1}{4}C\operatorname{diam}\sigma$. Each arc in Γ_w has an endpoint on $f(l')$, that is, outside $B(f(z_1), r_2)$ where $r_2 = \frac{1}{4}C\operatorname{diam}\sigma$, and an endpoint on σ, that is, inside $B(f(z_1), r_1)$ where $r_1 = \operatorname{diam}\sigma$. By the comparison principle,

$$\lambda(\Gamma_w) \geqslant \frac{1}{2\pi}\log\frac{r_2}{r_1} = \frac{1}{2\pi}\log\tfrac{1}{4}C,$$

which contradicts (3) provided C is big enough.

REMARK. As is known, the inequality (1) is valid with $\psi(t) = t$ for any function f univalent in $\mathbb{D}$ and such that $f'(0) = 1$. In this case Lemma 2.2 provides an estimate due to M. A. Lavrentiev, C. Gattegno, and A. Ostrowski (cf. [**18**, Chapter 11, §2, Corollary 5]).

2.3. LEMMA. *Let Ω be a Jordan domain, let $w_0 \in \Omega$, and let $\omega = \omega(\cdot\,; \Omega, w_0)$. If Δ is a disc of radius $\varepsilon \leqslant \varepsilon_0$ such that*

$$\omega(\partial\Omega \cap \Delta) \geqslant \varepsilon,$$

there exists a crosscut σ of Ω which lies in the disc Δ' of radius 2ε concentric with Δ and separates from w_0 a subarc β of $\partial\Omega$ satisfying

$$\omega(\beta) \geqslant \omega(\partial\Omega \cap \Delta)/4\pi\log\varepsilon^{-1}.$$

Proof. Fix a continuum K in Ω. We can assume that it does not intersect $\operatorname{clos}\Delta'$. Assume also that $w_0 \notin \operatorname{clos}\Delta'$. Let Ω_0 denote the component of $\Omega\backslash\Delta$ containing w_0. Let $\{U_j\}$ be the set of those components of $\Omega_0 \cap \Delta'$ whose boundary has an arc on $\partial\Delta$. Evidently, this set is not empty.

Fix j and consider the domain $U \equiv U_j$. It is easy to see that it is a Jordan domain. The intersection $\partial U \cap \{\Delta'\backslash\operatorname{clos}\Delta\}$ is the union of open subarcs of ∂U with endpoints on $\partial\Delta \cup \partial\Delta'$. Precisely two of these arcs join different circumferences. Their complement also consists of two subarcs of ∂U, and we denote by $d = d_j$ that one which intersects $\partial\Delta$. The set $(\partial U \cap \partial\Delta)\backslash\partial\Omega$ is relatively open with respect to $\partial\Delta$ and consists of open subarcs $\{l_\nu\}$ of $\partial\Delta$. Each arc l_ν forms a crosscut of Ω separating w_0 from some subarc α_ν of $\partial\Omega$. We denote the union $\bigcup_\nu \alpha_\nu$ by E_j.

Let Γ_j be the family of all arcs in Ω joining K with E_j. By the theorem stated in § 2.1,

$$\omega_j \equiv \omega(E_j) \leqslant A\exp\{-\pi\lambda(\Gamma_j)\}.$$

If Γ_j' denotes the family of all arcs in U_j which join $\partial\Delta$ and $\partial\Delta'$ then

$$\lambda(\Gamma_j') \leqslant \lambda(\Gamma_j).$$

On the other hand, by the Grötzsch principle,

$$\sum_j [\lambda(\Gamma_j')]^{-1} \leqslant 2\pi/\log 2.$$

Hence,

$$\operatorname{card}\{j\colon \omega_j \geqslant A\varepsilon^{\pi k}\} \leqslant (2\pi/\log 2)k\log\varepsilon^{-1}. \tag{4}$$

Consequently,

$$\begin{aligned}\sum_{\omega_j \leqslant A\varepsilon^\pi} \omega_j &\leqslant \sum_{k\geqslant 1} \Big[\sum_{A\varepsilon^{\pi(k+1)} \leqslant \omega_j \leqslant A\varepsilon^{\pi k}} \omega_j\Big]\\ &\leqslant (2\pi A/\log 2)\log\varepsilon^{-1} \sum_{k\geqslant 1}(k+1)\varepsilon^{\pi k}\\ &\leqslant \tfrac{1}{2}\varepsilon\end{aligned}$$

provided that ε is sufficiently small. Since $\partial\Omega \cap \Delta \subset \bigcup_j E_j$ and $\omega(\partial\Omega \cap \Delta) \geqslant \varepsilon$, this implies that

$$\sum_{\omega_j > A\varepsilon^\pi} \omega_j \geqslant \tfrac{1}{2}\omega(\Delta \cap \partial\Omega).$$

The latter sum, by (4), has at most $(2\pi/\log 2)\log\varepsilon^{-1}$ terms. Hence, for some j_0,

$$\omega_{j_0} \geqslant \frac{\log 2}{4\pi}\,\frac{\omega(\partial\Omega \cap \Delta)}{\log\varepsilon^{-1}}.$$

This proves the lemma with

$$\beta = E_{j_0} \cup (d_{j_0} \cap \partial\Omega)$$

and σ being an arbitrary crosscut of U_{j_0} whose endpoints coincide with that of β.

2.4. COROLLARY. *Let ψ be a continuous, increasing function on $[0, \infty)$ such that $\psi(0) = 0$, and let χ be the inverse of the mapping $t \mapsto t\psi(t)$. If the conformal mapping $f\colon \mathbb{D} \to \Omega$ onto a Jordan domain Ω satisfies*

$$|f'(z)| \geqslant \psi(1-|z|)$$

then

$$\omega \ll \Lambda_{\chi(t)\log(1/t)}. \tag{5}$$

Proof. We can assume that $w_0 = f(0)$. All we need is to check that if Δ is a disc of radius $\varepsilon \leqslant \varepsilon_0$ then

$$\omega(\partial\Omega \cap \Delta) \leqslant C\chi(\varepsilon)\log \varepsilon^{-1}.$$

This inequality is obvious when $\omega(\partial\Omega \cap \Delta) \leqslant \varepsilon$. Otherwise, we can make use of Lemma 2.3, and construct an arc β and a crosscut σ with the corresponding properties. Let l denote $f^{-1}(\beta)$. Since $\operatorname{diam}\sigma \leqslant 2\varepsilon$, by Lemma 2.2 we have

$$2\varepsilon \geqslant c|l|\psi(|l|).$$

Hence,

$$|l| \leqslant \chi(C\varepsilon) \leqslant C\chi(\varepsilon).$$

(Clearly, for the proof of (5) it can be assumed that $\chi(2t) \leqslant C\chi(t)$.) By Lemma 2.3,

$$\begin{aligned}\omega(\partial\Omega \cap \Delta) &\leqslant C\omega(\beta)\log\varepsilon^{-1}\\ &= C|l|\log\varepsilon^{-1}\\ &\leqslant C\psi(\varepsilon)\log\varepsilon^{-1}.\end{aligned}$$

Our next lemma is a consequence of Theorem A. We obtain an estimate for the derivative of the conformal mapping. For $\zeta \in \mathbb{D}$ we define

$$\mathscr{D}(\zeta) = \{z \in \mathbb{D} : |\arg(z-\zeta)| < (1-|z|)\sqrt{\log(1-|z|)^{-1}}\}.$$

To simplify the notation, we write $\log_{(3)}$ instead of $\log\log\log$.

2.5. Lemma. *There exists a universal constant $C > 0$ such that if f is a univalent function in $\mathbb{D}$ then for almost every $\zeta \in \partial\mathbb{D}$,*

$$\text{(6)}\qquad \varlimsup_{\substack{|z|\to 1-\\ z\in\mathscr{D}(\zeta)}} \frac{|f'(z)|^{-1}}{\exp\{C\sqrt{(\log(1-|z|)^{-1}\log_{(3)}(1-|z|)^{-1})}\}} = 0.$$

Proof. Remark first that if b is in $\mathscr{B}$ and $\|b\|_{\mathscr{B}} \leqslant 1$, and if

$$\text{(7)}\qquad \varlimsup_{r\to 1-} \frac{|b(r\zeta)|}{\sqrt{(\log(1-r)^{-1}\log_{(3)}(1-r)^{-1})}} \leqslant C$$

for some $\zeta \in \partial\mathbb{D}$ (according to Theorem A it is valid almost everywhere on $\partial\mathbb{D}$), then

$$\text{(8)}\qquad \varlimsup_{\substack{|z|\to 1-\\ z\in\mathscr{D}(\zeta)}} \frac{|b(z)|}{\sqrt{(\log(1-|z|)^{-1}\log_{(3)}(1-|z|)^{-1})}} \leqslant C.$$

Indeed, for $z \in \mathscr{D}(\zeta)$ let l be the smaller subarc of the circle of radius $|z|$ with endpoints in z and $\zeta|z|$. The length of l does not exceed $(1-|z|)\sqrt{\log(1-|z|)^{-1}}$, and, since $\|b\|_{\mathscr{B}} \leqslant 1$, we have that $|b'| \leqslant (1-|z|^2)^{-1}$ on l. Hence,

$$|b(z)-b(\zeta|z|)| \leqslant \sqrt{\log(1-|z|)^{-1}},$$

which together with (7) implies (8).

To prove (6) we can assume that $f'(0) = 1$. Define $b = \frac{1}{6}\log f'$. By the distortion estimates (see [8, Chapter 2, §4]),

$$\left|\frac{f''(z)}{f'(z)}\right| \leqslant \frac{6}{1-|z|^2}.$$

Therefore, $b \in \mathscr{B}$ and $\|b\|_{\mathscr{B}} \leqslant 1$. Let (8) be valid for some ζ on $\partial\mathbb{D}$. For $z \in \mathscr{D}(\zeta)$ with

$1-|z|$ sufficiently small, we have

(9) $$|\log f'(z)| \leqslant C\sqrt{(\log(1-|z|)^{-1}\log_{(3)}(1-|z|)^{-1})},$$

which implies (6).

REMARK. From (9) it also follows that, almost everywhere on $\partial\mathbb{D}$,

$$|f'(z)| = o(\exp\{C\sqrt{\log(1-|z|)^{-1}\log_{(3)}(1-|z|)^{-1}}\})$$

when $z \to \zeta$ and $z \in \mathscr{D}(\zeta)$. This considerably improves a result of Seidel and Walsh [**21**]:

$$|f'(z)| = o((1-|z|)^{-\frac{1}{2}}) \text{ a.e. on } \partial\mathbb{D}.$$

2.6. *Proof of Theorem* 1. Let f be a conformal mapping of $\mathbb{D}$ onto Ω and let

$$\varphi(t) = t\exp\{C\sqrt{(\log t^{-1}\log_{(3)}t^{-1})}\},$$

where C will be chosen later. Also let E_0 be a Borel set on $\partial\Omega$ such that $e_0 \equiv f^{-1}(E_0)$ has positive Lebesgue measure. We need to check that

(10) $$\Lambda_\varphi(E_0) > 0.$$

First, we make use of Lemma 2.5 to choose a closed subset e of $\partial\mathbb{D}$ enjoying the properties: $|e| > 0$, $e \subset e_0$, and

(11) $$|f'(z)|^{-1} \leqslant M\exp\{C\sqrt{(\log(1-|z|)^{-1}\log_{(3)}(1-|z|)^{-1})}\}$$

for all z in $\bigcup_{\zeta\in e}\mathscr{D}(\zeta)$ and some $M > 0$. It is easy to construct a Jordan domain $G \subset \mathbb{D}$ with smooth boundary such that

(a) $G \subset [\bigcup_{\zeta\in e}\mathscr{D}(\zeta)] \cup \{z: |z| < 0{\cdot}9\}$ (and thus (11) holds for all z in G),
(b) $e \subset \partial G$,
(c) if $n(z)$ denotes the unit exterior normal to ∂G at z then

(12) $$|n(z_1)-n(z_2)| \leqslant \frac{\text{const.}}{\sqrt{|\log|z_1-z_2||}}$$

for all $z_1, z_2 \in \partial G$.

Let g: $\mathbb{D} \to G$ denote a conformal mapping onto G. Then (c) provides an estimate for the derivative g':

(13) $$\log|g'(z)|^{-1} \leqslant \text{const.}\sqrt{\log 2(1-|z|)^{-1}}.$$

The proof, which we outline, is based on the standard arguments (cf. [**8**, Chapter 10, § 1, Theorem 6]). Let $\tau = \arg g'$. By the Lindelöf theorem [**8**, Chapter 10, § 1, Theorem 4], τ is continuous up to the boundary and satisfies

(14) $$\tau(\zeta) = n(g(\zeta)) - \arg\zeta$$

on $\partial\mathbb{D}$. Since $g' \in L^2$, for ζ_1 and ζ_2 on $\partial\mathbb{D}$ we have $|g(\zeta_1)-g(\zeta_2)| \leqslant \text{const.}\sqrt{|\zeta_1-\zeta_2|}$. Hence, (12) and (14) imply that

$$|\tau(\zeta_1)-\tau(\zeta_2)| \leqslant \text{const.}/\sqrt{\log 2|\zeta_1-\zeta_2|^{-1}}.$$

For $z \in \mathbb{D}$ we have

$$\log g'(z) = i\int_{\partial\mathbb{D}}\frac{\zeta+z}{\zeta-z}[\tau(\zeta)-\tau(z|z|^{-1})]\,|d\zeta|$$
$$+i\tau(z|z|^{-1})+\log|g'(0)|$$

and

$$\log\frac{1}{g'(z)} \leqslant \text{const.} \int_0^1 \frac{dt}{|\exp(it)-|z||\sqrt{\log t^{-1}}},$$

which yields (13).

Further, we define the mapping $h = f \circ g$. Then $|h'(z)| = |g'(z)|\,|f'(z_1)|$ where z_1 stands for $g(z)$. By the distortion theorem,

$$1-|z_1| \geqslant \operatorname{dist}(z_1, \partial G) \geqslant \tfrac{1}{4}(1-|z|)|g'(z)|.$$

Hence, (11) and (13) imply that

$$|h'(z)|^{-1} \leqslant M\exp\{C\sqrt{(\log(1-|z|)^{-1}\log_{(3)}(1-|z|)^{-1})}\},$$

C being a universal constant. Since $|e| > 0$, we have $|g^{-1}(e)| > 0$. By Corollary 2.4,

$$\Lambda_{\chi(t)\log(1/t)}(E_0) > 0,$$

where χ is the inverse of the mapping $t \mapsto \exp\{-C\sqrt{(\log t^{-1}\log_{(3)} t^{-1})}\}$. Clearly,

$$\chi(t) \leqslant t\exp\{C\sqrt{(\log t^{-1}\log_{(3)} t^{-1})}\},$$

and (10) follows.

3. *Singularity of harmonic measure*

Both Theorem 2 and Theorem 3 will be proved by a single method. The method is based on the notion of a dominating subset due to Rubel and Shields [**19**]. Let Ω denote a bounded domain in $\mathbb{C}$ and let $H^\infty(\Omega)$ denote, as usual, the Banach algebra of all bounded holomorphic functions on Ω in the supremum norm $\|\cdot\|_\infty$. A subset Λ of Ω is said to be a *dominating subset* of Ω if

$$\|f\|_\infty = \sup_{\lambda\in\Lambda}|f(\lambda)|$$

for any $f \in H^\infty(\Omega)$. In case Ω is $\mathbb{D}$, the dominating subsets are characterized as follows (see [**5**, Theorem 3]): a subset Λ of $\mathbb{D}$ is a dominating subset if and only if almost every boundary point $\zeta \in \partial\mathbb{D}$ is a non-tangential limit point of Λ. Clearly, the property of a set to be dominating is conformally invariant. In this section δ_λ denotes $\operatorname{dist}(\lambda, \partial\Omega)$ for $\lambda \in \Omega$.

3.1. Lemma. *Let Ω be a simply connected, bounded domain and let φ be a continuous, increasing function such that $\varphi(0) = 0$. If there exists a dominating subset Λ of Ω such that $\sum_{\lambda\in\Lambda}\varphi(\delta_\lambda) < \infty$, then $\omega \perp \Lambda_\varphi$.*

Proof. For each $\lambda \in \Lambda$ we define $E_\lambda = \partial\Omega \cap B(\lambda, 2\delta_\lambda)$. It is a consequence of the Milloux inequality [**8**, Chapter 8, §4, Theorem 6] that

$$\omega(E_\lambda; \Omega, \lambda) \geqslant c > 0.$$

We claim that

$$\omega\Big(\bigcup_{\lambda\in\Lambda} E_\lambda\Big) = 1.$$

Otherwise, there would exist a Borel subset $E_0 \subset \partial\Omega\setminus\bigcup_{\lambda\in\Lambda}E_\lambda$ of positive harmonic measure. We define $f = \exp\{h+i\tilde{h}\}$ where $h(\cdot) = \omega(E_0; \Omega, \cdot)$ and $\tilde{h}$ denotes the conjugate function. Then $\| f \|_\infty = \exp 1$ and

$$|f(\lambda)| = \exp\{\omega(E_0; \Omega, \lambda)\} \leqslant \exp\{1-c\},$$

provided $\lambda \in \Lambda$. This would contradict the definition of a dominating subset.

If $\tilde{\Lambda}$ is obtained from Λ by omitting a finite number of points, the union $\bigcup_{\lambda\in\tilde{\Lambda}} E_\lambda$ will still be of full harmonic measure. Since $\bigcup_{\lambda\in\tilde{\Lambda}} E_\lambda \subset \bigcup_{\lambda\in\tilde{\Lambda}} B(\lambda, 2\delta_\lambda)$ and since $\sum_{\lambda\in\tilde{\Lambda}} \varphi(\delta_\lambda)$ is arbitrarily small for an appropriate $\tilde{\Lambda}$, the assertion follows.

REMARK. I do not know whether the converse of this lemma is true. It can be shown that the converse is true for $\varphi(t) = t$.

The next lemma was proved in [11]. For convenience, we shall repeat that part of its proof which will be of use later. By $\mathrm{St}(\zeta)$ is meant the Stolz angle with vertex at $\zeta \in \partial\mathbb{D}$, that is, the interior of the convex hull of $\{z\colon |z| \leqslant \frac{1}{2}\} \cup \{\zeta\}$.

3.2. LEMMA. *Let Ω be a simply connected, bounded domain and let $f\colon \mathbb{D} \to \Omega$ be a conformal mapping onto Ω. Let φ be a continuous, increasing function on $[0, t_0)$ with $\varphi(0) = 0$ such that, for some $\alpha > 0$ and $\beta > 0$,*

(1) *the function $\varphi(t)/t^\alpha$ increases and the function $\varphi(t)/t^\beta$ decreases.*

Let $\psi(t) = \varphi^{-1}(t)/t$. The following are equivalent:

(a) *for almost every $\zeta \in \partial\Omega$,*

$$\underline{\lim}_{z\to\zeta,\, z\in \mathrm{St}(\zeta)} \frac{|f'(z)|}{\psi(1-|z|)} = 0; \tag{2}$$

(b) *there exists a dominating subset Λ of Ω such that*

$$\sum_{\lambda\in\Lambda} \varphi(\delta_\lambda) < \infty. \tag{3}$$

Proof of (a) $\Rightarrow$ (b). Fix $\varepsilon > 0$. For each ζ satisfying (2) there exists $z \equiv z_\zeta \in \mathrm{St}(\zeta)$ such that

$$1-|z| \leqslant \varepsilon, \quad |f'(z)| \leqslant \varepsilon\psi(1-|z|).$$

Let l_ζ denote a subarc of $\partial\mathbb{D}$ centred at ζ and of length $1-|z_\zeta|$. By the covering lemma we can select a sequence $\{\zeta_n\}$ such that the set $\bigcup_n l_n$ has full Lebesgue measure and $\sum_n l_n < \infty$ (l_n stands for l_{ζ_n}).

Let $\Lambda_{(\varepsilon)}$ denote the set $\{f(z_{\zeta_n})\}$. By the distortion theorem, for $z = z_\zeta$ and $l = l_\zeta$ we have

$$\delta_{f(z)} \leqslant 4|f'(z)|\,|l|.$$

Applying (1), we obtain

$$\begin{aligned}\varphi(\delta_{f(z)}) &\leqslant \varphi(4\varepsilon|l|\psi(|l|)) \\ &\leqslant (4\varepsilon)^\alpha \varphi(|l|\psi(|l|)) \\ &= (4\varepsilon)^\alpha |l|.\end{aligned}$$

Consequently,

$$\sum_{\lambda\in\Lambda_{(\varepsilon)}} \varphi(\delta_\lambda) \leqslant \text{const.}\,\varepsilon^\alpha.$$

The set $\Lambda = \bigcup_{k \geqslant 0} \Lambda_{(2^{-k})}$ is a dominating subset of Ω by the criterion stated at the beginning of the section. Obviously, (3) is valid.

3.3. *Proof of Theorem* 2. To construct a Jordan domain with the required properties, we shall make use of the Ahlfors–Becker univalence criterion (see [**18**, Chapter 6]): if f is holomorphic in $\mathbb{D}$, $f'(0) = 0$, and

$$(4) \qquad (1-|z|^2)|z(f''(z)/f'(z))| \leqslant c < 1,$$

then f is univalent and maps $\mathbb{D}$ onto a Jordan domain. We define f by

$$\log(1/f'(z)) = \beta \sum_{k \geqslant 0} z^{2^k}$$

for some $\beta \in (0, \frac{1}{4})$. It is easy to see that f satisfies (4) (cf. [**18**, Chapter 2]). Let Ω denote the Jordan domain $f(\mathbb{D})$.

A theorem of Salem and Zygmund [**20**]—the LIL for lacunary series—states that, for almost every $\zeta \in \partial\mathbb{D}$,

$$\overline{\lim_{r \to 1-}} \frac{\operatorname{Re}(\sum_{k \geqslant 0} (r\zeta)^{2^k})}{\sqrt{(\log(1-r)^{-1} \log_{(3)}(1-r)^{-1})}} = 1.$$

Consequently,

$$\underline{\lim_{r \to 1-}} \frac{|f'(r\zeta)|}{\exp\{-\frac{1}{2}\beta\sqrt{(\log(1-r)^{-1}\log_{(3)}(1-r)^{-1})}\}} = 0.$$

Let $\varphi(t) = t\exp\{\frac{1}{2}\beta\sqrt{(\log t^{-1} \log_{(3)} t^{-1})}\}$. Then

$$\psi(t) \equiv \varphi^{-1}(t)/t \sim \exp\{-\tfrac{1}{2}\beta\sqrt{(\log t^{-1}\log_{(3)} t^{-1})}\}$$

when t tends to zero. Hence, almost everywhere on $\partial\mathbb{D}$,

$$\underline{\lim_{r \to 1-}} \; |f'(r\zeta)|/\psi(1-r) = 0.$$

By Lemma 3.2 there exists a dominating subset Λ of Ω such that (3) holds. By Lemma 3.1, $\omega \perp \Lambda_\varphi$.

3.4. *Proof of Theorem* 3. Let $\lim_{t \to 0} t^{-1}\varphi(t) = 0$. We define a modified function φ_1 by

$$\varphi_1(t) = t\sup\{s^{-1}\varphi(s) \colon 0 \leqslant s \leqslant t\}.$$

Clearly, $\varphi_1(t) \geqslant \varphi(t)$, $\lim_{t \to 0} t^{-1}\varphi_1(t) = 0$, and the function $t^{-1}\varphi_1(t)$ increases. Thus, we can apply to φ_1 the implication (a) $\Rightarrow$ (b) of Lemma 3.2. (Only the first condition in (1) was used in the proof of this implication.) If $\psi_1(t)$ denotes ${\varphi_1}^{-1}(t)/t$ then $\lim_{t \to 0} \psi_1(t) = \infty$. The assertion (a) in Lemma 3.2 now follows by the Lusin–Privalov uniqueness theorem [**8**, Chapter 10, § 2, Theorem 1]—the derivative φ' has the infinite angular boundary values at every point ζ not satisfying (2). Hence, there exists a dominating subset Λ of Ω such that

$$\sum_{\lambda \in \Lambda} \varphi(\delta_\lambda) \leqslant \sum_{\lambda \in \Lambda} \varphi_1(\delta_\lambda) < \infty.$$

An application of Lemma 3.1 completes the proof.

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