A HISTORICAL SURVEY OF QUASICONFORMAL MAPPINGS

Olli Lehto

Introduction

By the classical definition, a quasiconformal mapping is a sense-preserving diffeomorphism of a plane domain onto another plane domain which maps infinitesimal circles onto infinitesimal ellipses with a uniformly bounded ratio of axes. Later it was found preferable to relax a priori differentiability conditions and define a quasiconformal mapping in the plane as a sense-preserving homeomorphism which leaves some conformal invariant quasi-invariant. The most general conformal invariant suitable for this purpose is the module of a path family. The precise requirement for quasiconfrormality is the existence of a fixed constant K such that the module of every path family lying in the domain in which the homeomorphism is considered increases at most K times.

A closer analysis of the relations between the classical and the more general definition led to the characterization of quasiconformality in terms of the Beltrami differential equation. It then turned out that quasiconformal mappings had in fact been studied for a very long time, within the theory of partial differential equations. However, this was not realized until quite a while after quasiconformal mappings were introduced into complex analysis and were given a name. In retrospect, it is amazing that it took so long before the connection between these two approaches to the theory of quasiconformal mappings finally became clear, in the late fifties.

Around 1960 the general theory of plane quasiconformal mappings had reached a satisfactory level. Interest was then focused on quasiconformal mappings in higher dimensional euclidean spaces R^n. At the outset, the theory of such mappings has no longer anything in common with complex analysis. But somewhat surprisingly, there appear to be striking analogues between (not necessarily injective) quasiconformal mappings and analytic functions of one complex variable.

1. Beltrami Differential Equation

Every sense-preserving diffeomorphism is locally quasiconformal. It follows that there is no undisputed criterion to determine the first appearance of quasiconformal mappings in analysis. I have not found any direct connection with Euler, but we have good reason to open the survey of quasiconformal mappings with Gauss.

The problem of Gauss in which quasiconformal mappings are involved is to map a surface locally conformally into the plane. Let S be a smooth orientable surface in the euclidean space R^3. Given an arbitrary point $p \in S$, let $f = (f_1, f_2, f_3)$ be the inverse of a local parameter near p, i.e., f is a diffeomophism of a domain in the plane R^2 onto a neighborhood of p on S. Gauss wanted to find a *mapping* f which is conformal in the sense that it preserves angles.

By means of f, the line element ds of S can be expressed in the form

$$ds^2 = \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial x} dx + \frac{\partial f_i}{\partial y} dy \right)^2$$

$$= E\,dx^2 + 2F\,dx\,dy + G\,dy^2 , \tag{1.1}$$

with

$$E = \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial x}\right)^2 , \quad F = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x}\frac{\partial f_i}{\partial y} , \quad G = \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial y}\right)^2 .$$

The expression (1.1) is invariant in that it does not depend on the local representation of S.

Here it is advisable to use complex notation $dz = dx + i\,dy$, $d\bar{z} = dx - i\,dy$ (even though Gauss did not). We then obtain from (1.1)

$$ds = \lambda \left| dz + \mu\, d\bar{z} \right| , \tag{1.2}$$

where

$$\lambda^2 = \frac{1}{4}(E + G + 2\sqrt{EG - F^2}), \quad \mu = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}.$$

It is important to note that

$$|\mu|^2 = \frac{E + G - 2\sqrt{EG - F^2}}{E + G + 2\sqrt{EG - F^2}} < 1.$$

Gauss proved that the mapping f is conformal if and only if $E = G$, $F = 0$; the proof is standard calculus. This condition is equivalent to μ being identically zero. In this case $ds^2 = E(dx^2 + dy^2)$ or

$$ds = \lambda|dz|.$$

Local coordinates $z = x + iy$ of S with this property are called isothermal.

Gauss was thus led to the problem of finding isothermal coordinates for the given surface S. The idea is to transform suitably the coordinates z given by f^{-1}, i.e., to consider a diffeomorphism $z \to w$ of the domain of f onto another plane domain. Then $|dw| = |\partial w dz + \bar{\partial} w d\bar{z}|$, where $\partial = (\partial/\partial x - i\partial/\partial y)/2$ and $\bar{\partial} = (\partial/\partial x + i\partial/\partial y)/2$ denote complex derivatives. Suppose that

$$\bar{\partial} w = \mu \partial w. \tag{1.3}$$

Then $|dw| = |\partial w||dz + \mu d\bar{z}|$, and comparison with (1.2) shows that $ds = (\lambda/|\partial w|)|dw|$. We see that if we can find an injective solution of the differential equation (1.3), then the w-coordinates are isothermal. (For a diffeomorphic solution, $|\partial w| \neq 0$, because the condition $|\partial w| \neq 0$ is equivalent to the Jacobian being non-zero.)

Gauss [5] solved the equation (1.3), which was later named after Beltrami, using real notation and assuming μ to be sufficiently smooth. Thus he solved the problem of finding a locally conformal mapping of a smooth orientable surface into the plane. This result, highly interesting as such, allows important interpretations if we use modern terminology, and it leads to far-reaching generalizations.

First of all, the transition $z \to w$ from the original coordinates to the new ones is a *locally quasiconformal mapping*, and $\mu = \bar{\partial}w/\partial w$ is its complex dilatation. Complex dilatation is a function which describes the local geometric properties of the quasiconformal mapping. In solving (1.3), Gauss solved a basic problem in the theory of quasiconformal mappings by showing that the complex dilatation can be prescribed.

Another interpretation brings Gauss's result into contact with complex analysis. Suppose that z and w are both isothermal coordinates corresponding to the same portion of S. Then the induced mapping $z \to w$ is conformal, so that isothermal coordinates define a conformal structure of S. In other words, Gauss proved that a smooth orientable surface in R^3 can always be made into a Riemann surface. The method used by Gauss can be applied to a much more general situation, and it follows that an abstract surface with a Riemannian metric can be given a complex-analytic structure. Solving an equation (1.3), i.e., finding a quasiconformal mapping with a prescribed complex dilatation, is again the crucial step.

In 1825, the terminology could of course not be used by Gauss, who considered the problem of finding conformal mappings of a surface from the point of view of differential geometry. It took some more decades before Riemann fully recognized the fundamental connection between conformal mappings and complex analysis.

After Gauss, it took well over a century before Beltrami equations (1.3) became an inseparable part of complex function theory. There were hints of an intimate connection: An easy computation shows that if w_1 and w_2 are diffeomorphic solutions of the same equation (1.3), then $w_2 \circ w_1^{-1}$ is a conformal mapping. In other words, the solutions of (1.3) are unique up to conformal transformations. This uniqueness theorem, combined with the general uniformization theorem, yields an important result about the existence of global solutions of (1.3). Suppose that (1.3) can be solved locally injectively in a simply connected domain A of the complex plane. By the uniqueness theorem, these local solutions define a conformal structure for A which thus becomes a Riemann surface. By the general uniformization theorem, this Riemann surface is conformally equivalent to the Riemann surface A with its natural conformal structure defined by the identity mapping. The mapping function is a globally injective solution of (1.3) in A.

From the point of view of quasiconformal mappings it is of interest to note the result, due independently to Lichtenstein and Korn from around the year 1915, that if μ is Hölder continuous, then the solutions of (1.3) are diffeomorphic. In the other direction, the complex dilatation of a diffeomorphism is continuous but not necessarily Hölder continuous. However, attempts to extend the result of Lichtenstein and Korn to a continuous μ failed, and since the 1950's we know that a continuous μ does not necessarily produce a continuously differentiable solution of (1.3).

It meant decisive progress for the theory of Beltrami equations when Morrey [10] in 1938 realized that one should not look for solutions of (1.3) in the classical sense but, as we would say today, in the sense of distributions. More precisely, let μ be a measurable function in a plane domain with $\|\mu\|_\infty < 1$. A function w is an L^2-solution of (1.3) if w is continuous, belongs locally to the Sobolev space W_2^1 (i.e., has generalized first order partial derivatives which are locally in L^2) and satisfies (1.3) almost everywhere. Using suitable approximation, Morrey proved that (1.3) always has a homeomorphic L^2-solution. He also showed that the solution is unique up to conformal mappings and that it is absolutely continuous with respect to the two-dimensional Lebesgue measure. These generalized solutions of (1.3) are nothing else but quasiconformal mappings in the modern sense. However, it took almost twenty years before the importance of Morrey's paper for complex analysis was understood.

2. Quasiconformal Mappings in the Plane

Let f be a complex-valued diffeomorphism of a domain A of the complex plane and $\partial_\alpha f(z)$ the derivative of f at $z \in A$ in the direction α. The function

$$z \to D(z) = \frac{\max_\alpha |\partial_\alpha f(z)|}{\min_\alpha |\partial_\alpha f(z)|}$$

is called the dilatation quotient of f. Clearly f is a conformal mapping of A if and only if $D(z) = 1$ at every point $z \in A$.

In 1928, Grötzsch [3] introduced the class of sense-preserving diffeomorphisms with bounded dilatation quotients. He proved that a number of results holding for conformal mappings can be extended for these more general mappings, either as such or with obvious modifications. In particular, certain conformal invariants remain quasi-invariant. Let $Q(z_1,z_2,z_3,z_4)$ be a quadrilateral, i.e., a Jordan domain Q and a sequence of four points z_1,z_2,z_3,z_4 on the boundary ∂Q determining a positive orientation of ∂Q with respect to Q. Let $M(Q)$ denote the conformal module of $Q(z_1,z_2,z_3,z_4)$. (If Q is mapped conformally onto the rectangle $\{x + iy | 0 < x < M, 0 < y < 1\}$ such that z_1,z_2,z_3,z_4 correspond to the vertices, $z_1 \to 0$, then $M(Q) = M$. $M(Q)$ is also the module of the family of paths of Q joining the sides (z_1,z_2) and (z_3,z_4).) Grötzsch proved that for a diffeomorphism f of A, the inequality $D(z) \le K$ holds if and only if

$$M(f(Q)) \le KM(Q) \tag{2.1}$$

for all quadrilaterals of A.

It soon became apparent that the mappings of Grötzsch were not merely an interesting generalization of conformal mappings. It was realized that they were an important tool in analysis and even more, that they had an intrinsic role in complex function theory. Ahlfors was the first to call these mappings quasiconformal in 1935. (If $D(z) \le K$, the mapping is said to be K-quasiconformal.)

Around 1935, Lavrentjev introduced a class of homeomorphisms defined by certain geometric mapping properties. The Lavrentjev mappings were clearly more general than the Grötzsch mappings, but still possessed many properties similar to those of conformal transformations. In 1957, Bojarski proved that Lavrentjev mappings agree with those homeomorphic L^2-solutions of Beltrami equations which have a continuous μ.

In the late thirties, quasiconformal mappings rose to the forefront of complex analysis thanks to Teichmüller. Introducing novel ideas, Teichmüller showed the intimate interaction between quasiconformal mappings and Riemann surfaces (see especially Teichmüller [15].) After the second World War, Teichmüller's ideas were taken up and pursued further by Ahlfors. His paper [1] revived interest in Teichmüller's work, and it also meant an important event in the history of quasiconformal mappings.

the applications, Grötzsch mappings had exhibited certain drawbacks. They are not closed under uniform convergence, with the result that many natural extremal problems fail to have a solution within the class. Often one had to allow a quasiconformal mapping not to be continuously differentiable at isolated points or on certain arcs. A more general definition was therefore desirable. Such a definition was suggested by Pfluger [11] in 1951, and full use of it was made by Ahlfors in the afore-mentioned paper [1] in 1953.

The inequality (2.1) characterizes K-quasiconformal diffeomorphisms. According to Pfluger and Ahlfors, a mapping is K-quasiconformal if it is a sense-preserving homeomorphism (not a diffeomorphism as before) which satisfies (2.1). This is a standard definition of quasiconformality today. (It was later proved that this definition is equivalent to the quasi-invariance of the module of path families mentioned in the Introduction.)

With this definition, a number of new problems arose as it was asked to what extent these quasiconformal mappings generalize Grötzsch mappings. Questions of this type were answered in rapid succession in the fifties, most theorems belonging by their nature more to real analysis than to complex analysis. It was proved by Strebel and Mori that a quasiconformal mapping is absolutely continuous on lines and by Mori that it is differentiable a.e. For a K-quasiconformal mapping f, differentiability a.e. in conjunction with an old result of Grötzsch, gave the inequality

$$\max_{\alpha} |\partial_{\alpha} f(z)| \le K \min_{\alpha} |\partial_{\alpha} f(z)| \quad \text{a.e.} \tag{2.2}$$

This in turn showed that the partial derivatives of a quasiconformal mapping are locally in L^2.

In the opposite direction, it was possible to characterize quasiconformality by properties of this kind. After a gradual reduction of conditions (Yujobo, Bers [3], Pfluger), Gehring and Lehto proved in 1959 that a sense-preserving homeomorphism f is K-quasiconformal if f is absolutely continuous on lines and satisfies (2.2) a.e.

It was in this connection that the mere writing of (2.2) in a different form had an unbelievable effect. An elementary computation shows that

$$\max_\alpha |\partial_\alpha f| = |\partial f| + |\bar\partial f|, \quad \min_\alpha |\partial_\alpha f| = |\partial f| - |\bar\partial f|.$$

It follows that (2.2) is equivalent to the inequality

$$|\bar\partial f(z)| \le \frac{K-1}{K+1} |\partial f(z)| \qquad \text{a.e.}$$

If we write

$$\bar\partial f = \mu \partial f, \tag{2.3}$$

we see that a homeomorphic f is K-quasiconformal if and only if f is an L^2-solution of (2.3), where $|\mu(z)| \le (K-1)/(K+1) < 1$ a.e. In other words, quasiconformal mappings coincide with the solutions of Beltrami equations in the sense of Morrey. This fundamental observation was made by Bers [3] in 1957. (Why had the corresponding observation not been done in connection with Grötzsch mappings, in the thirties?)

For more details about the history of Beltrami equations and quasiconformal mappings in the plane we refer to the monograph Lehto-Virtanen [9].

The discovery of Bers immediately solved an open problem for quasiconformal mappings: They are absolutely continuous with respect to the Lebesgue area measure, because Morrey had proved it for his solutions as early as in 1938. This implies for a quasiconformal mapping f that $\partial f \neq 0$ a.e. It follows that the complex dilatation

$$\mu = \bar\partial f / \partial f$$

can be defined at almost all points.

The amalgamation of the two approaches to quasiconformal mappings, by way of complex analysis on the one hand and of Beltrami equations on the other hand, was quickly utilized. Complex dilatation started to play an increasingly important role in the general theory. Besides, its systematic use in the hands of Bers, Ahlfors and others soon brought about a striking new progress in the theory of Teichmüller spaces (see, e.g., Ahlfors [2], Chapter VI.)

In many cases problems dealt with in connection with Teicmüller spaces projected back to classical function theory. For instance, Interest was focused on univalent functions with quasiconformal extensions. From around 1970 on, they have been systematically studied by Kühnau and others, without any more connections with Riemann surfaces.

Also, Schwarzian derivative, introduced by H.A. Schwarz [14] in 1869, underwent a renaissance. On Riemann surfaces it is a quadratic differential, which explains its importance in the Teichmüller theory. In the plane it has revealed unsuspected links between quasiconformal mappings and classical complex analysis.

A third notion which has largely gained in importance in recent years is a quasidisc, i.e., the image of a disc under a quasiconformal mapping of the plane. Again, its importance was first recognized primarily in the Teichmüller theory and later in its own right. Gehring [6] has given a comprehensive exposition of the various geometric and analytic characterizations of quasidiscs which were known by 1982.

3. Quasiconformal Mappings in Several Dimensions

Around 1960, the relations between the various characterizations of quasiconformal mappings in the plane had more or less been clarified. Gehring and Väisälä then set off to do the same for quasiconformal mappings in higher dimensional euclidean spaces R^n. In spite of the fact that a good model existed in the case $n = 2$, they encountered many difficulties. For instance, conformal mappings offered less help than in the plane, because there are no others than Möbius transformations if $n \geq 3$. But in a few years, Gehring and Väisälä succeeded in proving a number of results which show that, by and large, the definitions holding for $n = 2$ can be generalized to higher dimensional cases. A solid foundation was thus laid for the theory of quasiconformal mappings in several dimensions, for which only scattered results had existed before. Lavrentjev seems to have been a pioneer, having studied quasiconformal mappings in R^3 in 1938. For a detailed account of the various definitions of quasiconformality in R^n we refer to the monograph Väisälä [16].

At an early stage it become clear that, in some respects, the theories are different in R^2 and in R^n, $n > 2$. A striking difference is the lack of the Riemann mapping theorem in higher dimensional spaces. In the plane, all simply connected domains bounded by a nondegenerate continuum are conformally equivalent. In contrast, already in R^3 it is a highly non-trivial question which topological spheres can be mapped onto each other quasiconformally. This problem was studied for the first time systematically by Gehring and Väisälä ([7]) in 1965.

Complex dilatation is a more complicated notion in R^n, $n > 2$, than it is in the plane. Partial differential equations can be utilized (Bojarski and Iwaniec [4]), but the necessity to do without the existence theorem of Beltrami equations has been bitterly felt on several occasions.

Let us conclude this article by some remarks on quasiconformal mappings which are not necessarily injective. If the mappings are not allowed to take on the value ∞, they are called quasiregular functions. In the plane, a natural definition for a quasiregular function f is that it be an L^2-solution of a Beltrami equation $\bar{\partial}w = \mu\partial w$, $\|\mu\|_\infty < 1$; f is K-quasiregular if $\|\mu\|_\infty \le (K - 1)/(K + 1)$. It is easy to prove that

$$f = g \circ h, \tag{3.1}$$

where h is a homeomorphic L^2-solution of the same equation (i.e., a quasiconformal homeomorphism) and g is an analytic function. Because of the very simple representation (3.1), quasiregular functions have not been found very interesting in the plane.

There is *no difficulty in generalizing* the definition of a quasiregular function for dimensions $n > 2$. Suppose f is continuous in a domain $A \subset R^n$ and belongs locally to the Sobolev space W_n^1 (i.e., f has generalized first derivatives which are locally L^n-integrable). The linear map $f'(x)\colon R^n \to R^n$ and the Jacobian $J_f(x) = \det f'(x)$ can then be defined for almost all $x \in A$. The function f is said to be K-quasiregular in A if

$$\|f'(x)\|^n \le KJ_f(x) \quad \text{a.e. in } A. \tag{3.2}$$

Here $\| \ \|$ denotes the sup norm. For $n = 2$, this definition agrees with the previous one. If f is injective in A, then f is a K-quasiconformal mapping of A.

At first glance, the theory of quasiregular functions in R^n,

$n > 2$, has nothing to do with complex analysis. However, a closer study has revealed surprising analogues with the classical theory of analytic functions.

The beginning for a systematic study of quasiregular functions in several dimensions was a paper of Rešetnjak [12] which appeared in 1966. Rešetnjak proved, among other things, that a non-constant quasiregular function is discrete and open. Inspired by Rešetnjak's paper, Martio, Rickman and Väisälä focused their interest on quasiregular functions and, from 1969 on, proved jointly a number of important results. (For a survey of the theory until 1978, see Väisälä [17].)

Rešetnjak's result that a non-constant quasiregular function is discrete and open is the same in all dimensions. In contrast, the branching properties of a quasiregular function are different in R^n from what they are in the plane. The branch set B_f is defined as the set of all points at which a non-constant quasiregular f is not locally homeomorphic. Zorič established in 1967 the striking result that if f is quasiregular in R^n, $n > 2$, and $B_f = \emptyset$, then f is a homeomorphism onto R^n. This is of course not true in the plane.

By Picard's classical theorem, a function which is non-constant and analytic in R^2 can omit only one finite value. At an early stage, it became a famous problem whether Picard's theorem holds for quasiregular functions in R^n. If $n = 2$, the answer is trivially affirmative, on the basis of the representation formula (3.1). An example constructed by Zorič shows that, as in the classical case, a non-constant quasiregular function can omit one value in R^n.

The first step towards a Picard theorem was the generalization of Liouville's theorm: Rešetnjak proved in 1968 that a non-constant quasiregular function in R^n is unbounded. Next, Rešetnjak and, independently, Martio, Rickman and Väisälä showed that the set of omitted values is of n-capacity zero. Finally, Rickman solved the problem in two steps. In 1978, he proved that the omitted set is finite. This still left open the exact number of exceptional values. Later Rickman showed that in R^3, the number of omitted values of a non-constant K-quasiregular function may tend to ∞ as $K \to \infty$.

Rickman's result indicates a marked similarity to the classical Picard theorem. More than that, Rickman has shown in recent years that it is possible to develop a value distribution theory for quasiregular functions along the classical lines. His theory exhibits a remarkable analogue with the classical Nevanlinna theory and the Ahlfors theory of covering surfaces (see Rickman [13].) In this sense, there is also in higher dimensions a connection between quasiregular functions and analytic functions of one complex variable, even though the connection is far less explicit than the one expressed by formula (3.1) in the plane.

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University of Helsinki
Department of Mathematics
Hallituskatu 15
SF-00100 Helsinki 10
Finland