Quasicircles of dimension $1 + k^2$ do not exist

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Abstract

A well-known theorem of S. Smirnov states that the Hausdorff dimension of a k-quasicircle is at most $1 + k^2$. Here, we show the precise upper bound $D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{8/3-\varepsilon})$ where Σ^2 is the maximal asymptotic variance of the Beurling transform, taken over the unit ball of L^∞. The quantity Σ^2 was introduced in a joint work with K. Astala, A. Perälä and I. Prause where it was proved that $0.879 < \Sigma^2 \leq 1$, while recently, H. Hedenmalm discovered that surprisingly $\Sigma^2 < 1$. We deduce the asymptotic expansion of $D(k)$ from a more general statement relating the universal bounds for the integral means spectrum and the asymptotic variance of conformal maps. Our proof combines fractal approximation techniques with the classical argument of J. Becker and Ch. Pommerenke for estimating integral means.

1 Introduction

Let $D(k)$ denote the maximal Hausdorff dimension of a k-quasicircle, the image of the unit circle under a k-quasiconformal mapping of the plane. The first non-trivial bound (with the right growth rate) was given in 1987 by Becker and Pommerenke [7] who proved that $1+0.36\,k^2 \leq D(k) \leq 1+37\,k^2$ if k is small. In 1994, in his landmark work [2] on the area distortion of quasiconformal mappings, K. Astala suggested that the correct bound was

$$D(k) \leq 1 + k^2, \qquad 0 \leq k < 1. \tag{1.1}$$

Using a clever variation of Astala's argument, S. Smirnov [29] showed that the bound (1.1) indeed holds. A systematic investigation of the sharpness of (1.1) was initiated in [4] where the quantity

$$\Sigma^2 := \sup_{|\mu| \le \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu) \tag{1.2}$$

was introduced. Here,

$$\mathcal{S}\mu(z) = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{\mu(\zeta)}{(\zeta - z)^2} \, |d\zeta|^2 \tag{1.3}$$

denotes the *Beurling transform* of μ and

$$\sigma^2(g) = \lim_{R \to 1^+} \frac{1}{2\pi |\log(R-1)|} \int_{|z|=R} |g(z)|^2 \, |dz| \tag{1.4}$$

is the *asymptotic variance* of a Bloch function $g \in \mathcal{B}(\mathbb{D}^*)$ on the exterior unit disk. The motivation for studying Σ^2 comes from the work of McMullen [25] who showed that in dynamical cases (i.e. when μ is invariant under a co-compact Fuchsian group or eventually-invariant under a Blaschke product), one has an explicit connection between Hausdorff dimension and asymptotic variance:

$$2 \frac{d^2}{dt^2}\bigg|_{t=0} \operatorname{H.dim} w^{t\mu}(\mathbb{S}^1) = \sigma^2(\mathcal{S}\mu). \tag{1.5}$$

In [4], it was established that $\liminf_{k \to 0} (D(k) - 1)/k^2 \ge \Sigma^2$ and $0.879 \le \Sigma^2 \le 1$, while recently, H. Hedenmalm surprisingly proved that $\Sigma^2 < 1$, see [12]. Here, we complete this "trilogy" by showing:

Theorem 1.1.

$$D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{8/3-\varepsilon}), \qquad \textit{for any } \varepsilon > 0.$$

In particular, Smirnov's bound is not sharp for small k. István Prause informed me (private communication) that one can use the methods of [27, 29] to show that this implies that $D(k) < 1 + k^2$ for all $0 < k < 1$.

1.1 Integral means spectra

The aim of *geometric function theory* is to understand the geometric complexity of the boundary of a simply-connected domain $\Omega \subset \mathbb{C}$ in terms of the analytic complexity of the Riemann map $f : \mathbb{D} \to \Omega$. For domains with rough boundaries, the relationship between f and $\partial\Omega$ may be quantified using several geometric characteristics. One notable characteristic is the *integral means spectrum*

$$\beta_f(p) = \limsup_{r\to 1} \frac{\log \int_{|z|=r} |f'(z)^p| \, d\theta}{\log \frac{1}{1-r}}, \qquad p \in \mathbb{C}.$$

The importance of the spectrum $\beta_f(p)$ lies in the fact that it is Legendre-dual to the multifractal spectrum of harmonic measure [23, 8]. Taking the supremum of $\beta_f(p)$ over bounded simply-connected domains, one obtains the *universal integral means spectrum*

$$B(p) = \sup \beta_f(p).$$

Apart from various estimates [15, 18], not much is rigorously known about the qualitative features of $B(p)$. For instance, it is expected that $B(p) = B(-p)$ is an even function. However, while $B(2) = 1$ is an easy consequence of the area theorem, the statement "$B(-2) = 1$" is equivalent to the Brennan conjecture, which is a well-known and difficult open problem. Nor is it known whether $B(p) \in C^1$, let alone real-analytic. In this work, we are concerned with the quadratic behaviour of $B(p)$ near the origin.

It will be convenient for us to work with conformal maps defined on the exterior unit disk $\mathbb{D}^* = \{z : |z| > 1\}$. Unless stated otherwise, we assume that conformal maps are in *principal normalization*, satisfying $\varphi(z) = z + \mathcal{O}(1/|z|)$ near infinity.

Let $\mathbf{\Sigma}_k$ be the collection of conformal maps that admit k-quasiconformal extensions to the complex plane with dilatation at most k. Maximizing over $\mathbf{\Sigma}_k$, we obtain the spectra $B_k(p) := \sup_{\varphi\in\mathbf{\Sigma}_k} \beta_\varphi(p)$. We show:

Theorem 1.2. *For any* $0 < k < 1$,

$$\lim_{p\to 0} \frac{B_k(p)}{|p|^2/4} \; = \; \Sigma^2(k) \; := \; \sup_{\varphi\in\mathbf{\Sigma}_k} \sigma^2(\log \varphi'). \tag{1.6}$$

Theorem 1.3. *If* $k \to 0$ *and* $k|p| \to 0$*, then*

$$\lim \frac{B_k(p)}{k^2|p|^2/4} \; = \; \Sigma^2 \; := \; \sup_{|\mu|\le\chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu).$$

Remark. The Beurling transform $\mathcal{S}\mu$ is the infinitesimal analogue of $\log \varphi'$. Indeed, if $w^{t\mu} \in \mathbf{\Sigma}_k$ is the principal solution to the Beltrami equation $\overline{\partial} w = t\mu\, \partial w$, then $\log(w^{t\mu})' \approx t\mathcal{S}\mu$. Further remarks will be given in Section 2.

Theorem 1.4. (i) $\Sigma^2(k)/k^2$ *is a non-decreasing convex function on* $[0,1)$.
(ii) *Furthermore,* $\Sigma^2(k) = \Sigma^2 k^2 + \mathcal{O}(k^3)$ *as* $k \to 0$.

Together with Hedenmalm's estimate, Theorem 1.3 contradicts the very general conjecture

$$\text{“}B_k(p) = k^2p^2/4 \text{ for all } k \in [0,1) \text{ and } p \in [-2/k, 2/k]\text{”}$$

from [18, 27]. However, since we do not know whether or not $\lim_{k\to 1^-} \Sigma^2(k) \overset{?}{=} 1$, we cannot rule out Kraetzer's conjecture which asserts only that

$$\text{“}B(p) = p^2/4 \text{ for } p \in [-2,2].\text{”}$$

It is currently known that $0.93 < \lim_{k\to 1^-} \Sigma^2(k) < (1.24)^2$. We refer the reader to [4, Section 8] for the lower bound and to [14, 13] for the upper bound.

1.2 From integral means to dimensions of quasicircles

The implication (Theorem 1.3 $\Rightarrow$ Theorem 1.1) follows from the relation

$$\beta_\varphi(p) = p - 1 \quad \Longleftrightarrow \quad p = \mathrm{M.}\dim \varphi(\mathbb{S}^1), \qquad \varphi \in \mathbf{\Sigma}_k, \tag{1.7}$$

see [26, Corollary 10.18]. Here, two facts are tacitly being used: first, the work of Astala [1] shows that $D(k)$ may be characterized with Minkowski dimension in place of Hausdorff dimension. More precisely, Astala showed that when evaluating $D(k)$, it suffices to take the supremum over certain Ahlfors regular k-quasicircles for which the Hausdorff and Minkowski dimensions coincide.

Secondly, one can take a quasiconformal map that is conformal to one side and anti-symmetrize it in the spirit of [20, 29] to reduce its dilatation: a Jordan curve γ is a k'-quasicircle if and only if it can be represented as $\gamma = m(\varphi(\mathbb{S}^1))$ where m is a Möbius transformation and $\varphi \in \mathbf{\Sigma}_k$ with

$$k = \frac{2k'}{1 + (k')^2}. \tag{1.8}$$

This accounts for the discrepancy in the factor of 4 in Theorems 1.3 and 1.1.

1.3 A sketch of proofs

The proofs of Theorems 1.2 and 1.3 follow the argument of Becker and Pommerenke [7], except we use an L^2 bound for the non-linearity $n_\varphi = \varphi''/\varphi'$ instead of the L^∞ bound

$$\left|\frac{2n_\varphi}{\rho_*}(z)\right| \le 6k, \qquad \varphi \in \mathbf{\Sigma}_k\,. \tag{1.9}$$

Here, $\rho_*(z) = 2/(|z|^2-1)$ is the density of the hyperbolic metric on the exterior unit disk. By a *box* in $\mathbb{D}^*$, we mean an annular rectangle of the form

$$\square = \{z : r_1 < |z| < r_2,\ \theta_1 < \arg z < \theta_2\}, \qquad 1 < r_1 < r_2 < \infty.$$

We define the *hyperbolic width* of $\square$ as the hyperbolic length of its top side

$$\{r_2 e^{i\theta} : \theta_1 < \arg z < \theta_2\}$$

and the *hyperbolic height* as the hyperbolic length of its left or equivalently right side. The following theorem says that *upper bounds* on box averages yield *upper bounds* on integral means, while *lower bounds* on box averages yield *lower bounds* on integral means:

Theorem 1.5. *Suppose $\varphi \in \mathbf{\Sigma}_k$ is a conformal map which satisfies*

$$m \le \fint_\square \left|\frac{2n_\varphi}{\rho_*}(z)\right|^2 \rho_*(z)|dz|^2 \le M \tag{1.10}$$

for all sufficiently large boxes $\square$ in the exterior unit disk. Then,

$$m \le \liminf_{|p|\to 0} \frac{\beta_\varphi(p)}{|p|^2/4} \le \limsup_{|p|\to 0} \frac{\beta_\varphi(p)}{|p|^2/4} \le M.$$

Here, the notation $\fint_\square \dots \rho_*(z)|dz|^2$ suggests that we consider the average with respect to the measure $\rho_*(z)|dz|^2$, while a box is considered "large" if its hyperbolic width is at least 1 (any other positive constant will do) and its hyperbolic height is large. According to the above theorem, in order to maximize $\limsup_{|p|\to 0} \frac{\beta_\varphi(p)}{|p|^2/4}$ over the class $\mathbf{\Sigma}_k$, it suffices to construct a sequence of conformal mappings $\varphi_\varepsilon \in \mathbf{\Sigma}_k$, $\varepsilon > 0$, for which the average non-linearity (1.10) over large boxes is almost as large as possible, i.e. $> \Sigma^2(k) - \varepsilon$. This will be achieved using the quasiconformal fractal approximation technique of [4]. We require slightly more general considerations than those given in [4], so we examine these ideas in detail. These arguments take up Sections 4–6. In the course of this review, we will show:

Theorem 1.6. *Let M_{I} be the class of Beltrami coefficients that are* eventually-invariant *under $z \to z^d$ for some $d \geq 2$, i.e. satisfying $(z^d)^*\mu = \mu$ in some open neighbourhood of the unit circle. Then,*

$$\Sigma^2 = \sup_{\mu \in M_{\mathrm{I}},\ |\mu| \leq \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu), \tag{1.11}$$

$$\Sigma^2(k) = \sup_{\mu \in M_{\mathrm{I}},\ |\mu| \leq k \cdot \chi_{\mathbb{D}}} \sigma^2\big(\log(w^\mu)'\big). \tag{1.12}$$

The first statement (1.11) was proved explictly in [4], while the second statement (1.12) requires the locality of non-linearity (discussed in Section 6).

1.4 Applications to dynamical systems

Using Theorem 1.5, one can give an alternative (and perhaps more elementary) proof of McMullen's identity (1.5) that does not involve thermodynamic formalism. Suppose $\varphi = w^\mu$ is a conformal mapping of the exterior unit disk where μ belongs to one of the two classes of dynamical Beltrami coefficients below:

- $M_{\mathrm{B}} = \bigcup_f M_f(\mathbb{D})$ consists of Beltrami coefficients that are *eventually-invariant* under some finite Blaschke product

  $$f(z) = z \prod_{i=1}^{d-1} \frac{z - a_i}{1 - \overline{a_i} z}, \tag{1.13}$$

  i.e. Beltrami coefficients which satisfy $f^*\mu = \mu$ in some open neighbourhood of the unit circle.

- $M_{\mathrm{F}} = \bigcup_\Gamma M_\Gamma(\mathbb{D})$ consists of Beltrami coefficients that are invariant under some co-compact Fuchsian group Γ, i.e. $\gamma^*\mu = \mu$ for all $\gamma \in \Gamma$.

Taking advantage of the ergodicity of the geodesic flow on the unit tangent bundle T_1X (Fuchsian case) or Riemann surface lamination $\hat{X}_B$ (Blaschke case), it is not hard to show that the inequalities

$$\sigma^2(\log\varphi') - \varepsilon \,<\, \fint_{\square} \left|\frac{2n_\varphi}{\rho_*}(z)\right|^2 \rho_*|dz|^2 \,<\, \sigma^2(\log\varphi') + \varepsilon \tag{1.14}$$

hold for all sufficiently large boxes $\square$. This leads to the global form of McMullen's identity: $(1/2)\cdot\beta_\varphi''(0) = \sigma^2(\log\varphi')$. Considering $\varphi_t = w^{t\mu}$ and taking $t \to 0$ leads to the infinitesimal form (1.5).

Remark. To be honest, this argument does not show the true differentiability of the functions $p \to \beta_\varphi(p)$ and $t \to \operatorname{H.dim} w^{t\mu}(\mathbb{S}^1)$, only that $\beta_\varphi(p) = \sigma^2(\log\varphi')p^2/4 + o(p^2)$ and $\operatorname{H.dim} w^{t\mu}(\mathbb{S}^1) = \sigma^2(\mathcal{S}\mu)t^2/4 + o(t^2)$.

More generally, the equidistribution of boxes (1.14) holds for conformal maps to domains bounded by a wider class of fractals known as *Jordan repellers* (J, F). These subsume both the deformations of Blaschke products and Fuchsian groups considered above. A sketch of ideas related to equidistribution will be given in Appendix A.

1.5 Beltrami coefficients with sparse support

When studying thin regions of Teichmüller space, it is natural to consider Beltrami coefficients that are sparsely supported. For applications, see [9, 16, 24]. Suppose $\mu \in M(\mathbb{D})$ is a Beltrami coefficient supported on a "garden" $\mathcal{G} = \bigcup A_j$ where:

(1) Each A_j satisfies the *quasigeodesic property* – i.e. is located within hyperbolic distance S of a geodesic segment γ_j.

(2) *Separation property.* The hyperbolic distance $d_{\mathbb{D}}(\gamma_i, \gamma_j) > R$ is large.

Theorem 1.7. *If μ is a Beltrami coefficient with sparse support, then*

$$\operatorname{M.dim} w^{t\mu}(\mathbb{S}^1) \le 1 + C(S)e^{-R/2}|t|^2,$$

for $|t| < t_0(S, R)$ small.

1.6 Related results and open problems

After the first version of the paper was written, the author realized that the average non-linearity is related to the local variance of a dyadic martingale associated to a Bloch function, introduced by Makarov. These martingale arguments [17] give a quicker route to the main results of this paper as well as give additional characterizations of $\Sigma^2(k)$ in terms of the constant in Makarov's law of iterated logarithm and the transition parameter for the singularity of harmonic measure. Nevertheless, the Becker-Pommerenke argument is in some ways stronger: martingale techniques give a weaker error term in the expansion of $D(k)$.

In a recent work [11], Hedenmalm studied the notion of "asymptotic tail variance" of Bloch functions to show the estimate

$$B_k(p) \le (1+7k)^2 \cdot \frac{k^2|p|^2}{4}, \qquad |p| \le \frac{2}{k(1+7k)^2}. \tag{1.15}$$

However, the arguments of this paper are only effective when the product $k|p|$ is small. It would be natural to interpolate between Theorem 1.2 and (1.15) in the range $0 \le k|p| \le 2$. To conclude the introduction, we mention several open problems:

1. Is it true that $\lim_{p\to 0} 4\,B(p)/p^2 = \lim_{k\to 1^-} \Sigma^2(k)$?

2. For a fixed $k \in (0,1)$, is the function $B_k(p)$ differentiable on an interval $(-\varepsilon, \varepsilon)$?

3. Is it true that $D(k) = \Sigma^2 k^2 + a_3 k^3 + o(k^3)$ for some $a_3 \in \mathbb{R}$?

4. For a Bloch function $g \in \mathcal{B}(\mathbb{D}^*)$, let

   $$\sigma^2(g,R) = \frac{1}{2\pi|\log(R-1)|} \int_{|z|=R} |g(z)|^2\,|dz|$$

   and $\Sigma_R^2 := \sup_{|\mu|\le\chi_{\mathbb{D}}} \sigma^2(g,R)$. In [4], it was proved that

   $$|\Sigma_R^2 - \Sigma^2| \le C \cdot \log\frac{1}{R-1}, \qquad 1 < R < 2.$$

   Does one have "exponential mixing" $|\Sigma_R^2 - \Sigma^2| \le C \cdot \frac{1}{(R-1)^\gamma}$ for some $\gamma > 0$?

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2 Holomorphic families

In this section, we use the fractal approximation principle (Theorem 1.6) to show Theorem 1.4 which says that $\Sigma^2(k)/k^2$ is a convex function of $k \in [0, 1)$. We first recall the concept of a holomorphic family of conformal maps

$$\varphi_t \colon \mathbb{D}^* \to \mathbb{C}, \qquad \varphi_0(z) = z, \qquad \varphi_t(z) = z + \mathcal{O}(1/|z|), \qquad t \in \mathbb{D}.$$

According to the λ-lemma, each map φ_t admits a $|t|$-quasiconformal extension to the complex plane. Conversely, if $\varphi \in \mathbf{\Sigma}_k$ has a k-quasiconformal extension H, then it may be naturally included into a holomorphic family $\{\varphi_t,\, t \in \mathbb{D}\}$ with $\varphi = \varphi_k$. This is done by letting H_t be the principal homeomorphic solution to the Beltrami equation $\overline{\partial} H_t = t\,(\mu/k)\,\partial H_t$ and then restricting H_t to the exterior unit disk.

For a conformal mapping $\varphi \in \mathbf{\Sigma}$, we denote its *non-linearity* and *Schwarzian derivative* by

$$n_\varphi := \frac{\varphi''}{\varphi'} \qquad \text{and} \qquad s_\varphi := \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2. \tag{2.1}$$

For $\alpha \ge 0$, consider the function space $A^\infty_\alpha(\mathbb{D}^*)$ consisting of holomorphic functions on the exterior unit disk which satisfy

$$\|\phi\|_{A^\infty_\alpha(\mathbb{D}^*)} := \sup_{z \in \mathbb{D}^*} (|z|^2 - 1)^\alpha |\phi(z)| < \infty.$$

The following result is well known:

Lemma 2.1. *Given a standard holomorphic family $\{\varphi_t = w^{t\mu},\, t \in \mathbb{D}\}$ with $|\mu| \le \chi_{\mathbb{D}}$, the map $t \to b_{\varphi_t} = \log \varphi_t'$ is a Banach-valued holomorphic function from $\mathbb{D}$ to the Bloch space of the exterior unit disk. In particular, the mappings $t \to n_{\varphi_t} \in A^\infty_1(\mathbb{D}^*)$ and $t \to s_{\varphi_t} \in A^\infty_2(\mathbb{D}^*)$ are holomorphic.*

Proof. The holomorphy of n_{φ_t} and s_{φ_t} follow from the boundedness of $b \to b'$, $\mathcal{B}(\mathbb{D}^*) \to A_1^\infty(\mathbb{D}^*)$ and $\phi \to \phi' - \frac{1}{2} \cdot \phi^2$, $A_1^\infty(\mathbb{D}^*) \to A_2^\infty(\mathbb{D}^*)$. To see that b_{φ_t} is holomorphic, it suffices to check that it is weak-$*$ holomorphic.

For simplicity of exposition, let us instead show that any norm-bounded, pointwise holomorphic function from the unit disk into $\mathcal{B}$ is weak-$*$ holomorphic. As is well-known [12, 30], the predual of $\mathcal{B}$ is the Bergman space A^1, with the dual pairing

$$\langle b, g \rangle = \lim_{r \to 1} \frac{1}{\pi} \int_{\mathbb{D}} b(z) \overline{g(rz)}\, |dz|^2, \qquad b \in \mathcal{B},\ g \in A^1.$$

Since the dilates $g_r(z) = g(rz)$ converge to $g(z)$ in A^1, the above limit converges uniformly in r as b ranges over bounded subsets of the Bloch space. Hence,

$$t \to \langle b_t, g \rangle = \lim_{r \to 1} \frac{1}{\pi} \int_{\mathbb{D}} b_t(z) \overline{g(rz)}\, |dz|^2$$

is a holomorphic function, being the uniform limit of holomorphic functions. $\square$

From the Neumann series expansion for principal solutions to the Beltrami equation [5, p. 165],

$$\varphi_t' = \partial \varphi_t = 1 + t\mathcal{S}\mu + t^2 \mathcal{S}\mu\mathcal{S}\mu + \dots, \qquad |z| > 1, \tag{2.2}$$

it follows that the derivative of the Bers embedding at the origin is just

$$(d/dt)\big|_{t=0} \log \varphi_t' = \mathcal{S}\mu. \tag{2.3}$$

In particular, $\mathcal{S}\mu \in \mathcal{B}(\mathbb{D}^*)$ and

$$\left\| \frac{\log \varphi_t'}{t} - \mathcal{S}\mu \right\|_{\mathcal{B}(\mathbb{D}^*)} = \mathcal{O}(|t|), \quad \text{for } |t| < 1/2. \tag{2.4}$$

Since the asymptotic variance is continuous in the Bloch norm [11], the function

$$u(t) = \sigma^2 \left(\frac{\log \varphi_t'}{t} \right) \tag{2.5}$$

extends continuously to $\sigma^2(\mathcal{S}\mu)$ at $t = 0$. Similarly to (2.3), $(\mathcal{S}\mu)' = (d/dt)\big|_{t=0} n_{\varphi_t}$ and $(\mathcal{S}\mu)'' = (d/dt)\big|_{t=0} s_{\varphi_t}$ are the infinitesimal forms of the non-linearity and the Schwarzian derivative respectively.

Proof of Theorem 1.4. Taking the supremum of (2.4) over all $|\mu| \le \chi_{\mathbb{D}}$ shows that $\lim_{k\to 0} \Sigma^2(k)/k^2 = \Sigma^2$, which is the statement (ii).

Part (i) uses a fractal approximation argument. According to Theorem 1.6, in the definition $\Sigma^2(k) = \sup_{\varphi \in \mathbf{\Sigma}_k} \sigma^2(\log \varphi')$, it suffices to take the supremum over conformal maps $\varphi = w^\mu$ that have "dynamically-invariant" quasiconformal extensions. According to [4, Section 8], in these fractal cases, the function $u(t)$ is subharmonic. In particular, this implies that $\sup_{|t|=k} u(t)$ is a non-decreasing convex function of $k \in [0,1)$. Therefore, $\Sigma^2(k)/k^2$ is also a non-decreasing convex function, being the supremum of such functions. □

3 Working on the upper half-plane

Since non-linearity is not invariant under $\mathrm{Aut}(\mathbb{D})$, it is convenient to work on the upper half-plane. This makes the computations quite a bit simpler. If $b \in \mathcal{B}(\mathbb{H})$ is a holomorphic function on $\mathbb{H}$ with

$$\|b\|_{\mathcal{B}(\mathbb{H})} = \sup_{z\in\mathbb{H}} 2y \cdot |b'(z)| \le \infty, \tag{3.1}$$

we define its *asymptotic variance* as

$$\sigma^2_{[0,1]}(b) = \limsup_{y\to 0^+} \frac{1}{|\log y|} \int_0^1 |b(x+iy)|^2 \, dx. \tag{3.2}$$

In [25, Section 6], McMullen showed that one can compute the asymptotic variance of Bloch functions by examining Césaro averages of integral means that involve higher order derivatives. Here we shall be content with the formula

$$\sigma^2_{[0,1]}(b) = \limsup_{h\to 0^+} \frac{1}{|\log h|} \int_h^1 \int_0^1 \left| \frac{2b'(x+iy)}{\rho_{\mathbb{H}}} \right|^2 \frac{|dz|^2}{y}. \tag{3.3}$$

Let $\mathbf{H}$ denote the class of conformal maps $f : \mathbb{H} \to \mathbb{C}$ which fix the points $0, 1, \infty$ and $\mathbf{H}_k \subset \mathbf{H}$ be the subclass consisting of conformal maps that admit a k-quasiconformal extension to the lower half-plane $\overline{\mathbb{H}}$. For $f \in \mathbf{H}$, the *integral means spectrum* is given by

$$\beta_f(p) = \limsup_{y\to 0^+} \frac{\log \int_0^1 |f'(x+iy)^p| \, dx}{|\log y|}, \qquad p \in \mathbb{C}.$$

For a Beltrami coefficient μ supported on $\overline{\mathbb{H}}$ with $\|\mu\|_\infty < 1$, let $\tilde{w}^\mu \in \mathbf{H}_k$ denote the *normalized* solution of the Beltrami equation $\overline{\partial} w = \mu\, \partial w$. (The notation w^μ is reserved for principal solutions defined for compactly supported coefficients.)

Since the formula for the Beurling transform (1.3) may not converge if μ is not compactly supported, we are obliged to work with a modified Beurling transform

$$\mathcal{S}^{\#}\mu(z) \;=\; -\frac{1}{\pi}\int_{\overline{\mathbb{H}}} \mu(\zeta)\left[\frac{1}{(\zeta-z)^2} - \frac{1}{\zeta(\zeta-1)}\right]|d\zeta|^2, \tag{3.4}$$

however, the formula for the derivative remains the same:

$$\text{“}(\mathcal{S}\mu)'(z)\text{”} \;:=\; (\mathcal{S}^{\#}\mu)'(z) \;=\; -\frac{2}{\pi}\int_{\overline{\mathbb{H}}} \frac{\mu(\zeta)}{(\zeta-z)^3}\,|d\zeta|^2. \tag{3.5}$$

Not surprisingly, $\mathcal{S}^{\#}\mu$ and $\log f'$ are Bloch functions. In fact,

$$\|\mathcal{S}^{\#}\mu\|_{\mathcal{B}(\mathbb{H})} \le \frac{8}{\pi}\cdot\|\mu\|_\infty, \qquad \|\log f'\|_{\mathcal{B}(\mathbb{H})} \le 6k, \tag{3.6}$$

have the same bounds as do Bloch functions on the disk. Furthermore, the universal bounds are also unchanged:

Lemma 3.1.

$$\Sigma^2 = \sup_{|\mu|\le\chi_{\overline{\mathbb{H}}}} \sigma^2_{[0,1]}(\mathcal{S}^{\#}\mu), \qquad \Sigma^2(k) = \sup_{f\in\mathbf{H}_k} \sigma^2_{[0,1]}(\log f'),$$

$$B_k(p) = \sup_{f\in\mathbf{H}_k} \beta_f(p).$$

3.1 Exponential transform

A convenient way to transfer results from the half-plane to the disk is by exponentiating. Let $\xi(w) = e^{-2\pi i w}$ be the exponential mapping which takes $\mathbb{H}\to\mathbb{D}^*$. For a normalized k-quasiconformal mapping f, define its *exponential transform* as

$$\mathcal{E}_f(w) = -\frac{1}{2\pi i}\cdot\log\circ f\circ\xi(w), \tag{3.7}$$

where the branch of logarithm is chosen so that $\log\mathcal{E}_f(1) = 1$. In terms of Beltrami coefficients, the dilatation $\operatorname{dil.}\mathcal{E}_f = \xi^*(\operatorname{dil.} f)$ is just the pullback of $\operatorname{dil.} f$, considered as a $(-1,1)$ form.

It is not difficult to see that $f \to \mathcal{E}_f(w)$ establishes a bijection between normalized k-quasiconformal mappings with ones satisfying the periodicity condition

$$\mathcal{E}_f(w+1) = \mathcal{E}_f(w) + 1. \tag{3.8}$$

Of interest to us, φ is conformal on $\mathbb{D}^* \iff \mathcal{E}_\varphi$ is conformal on $\mathbb{H}$.

Lemma 3.2. (i) *If $|\mu| \le \chi_{\mathbb{D}}$, then for $w \in \mathbb{H}$,*

$$\left| \left| \frac{(\mathcal{S}(\xi^*\mu))'}{\rho_{\mathbb{H}}}(w) \right| - \left| \frac{(\mathcal{S}\mu)'}{\rho_*}(\xi(w)) \right| \right| = o(1), \quad \textit{as } \operatorname{Im} w \to 0. \tag{3.9}$$

(ii) *If φ is a normalized k-quasiconformal mapping that is conformal on $\mathbb{D}^*$, then*

$$\left| \left| \frac{n_{\mathcal{E}_\varphi}}{\rho_{\mathbb{H}}}(w) \right| - \left| \frac{n_\varphi}{\rho_*}(\xi(w)) \right| \right| = o(1), \quad \textit{as } \operatorname{Im} w \to 0. \tag{3.10}$$

The above lemma follows from two observations about the exponential which imply that is does not change the asymptotic features of non-linearity:

I. ξ is approximately a local isometry on $\{w \in \mathbb{H} : \operatorname{Im} w < \delta\}$, when $\mathbb{H}$ and $\mathbb{D}^*$ are equipped with their hyperbolic metrics.

II. ξ is approximately linear on small balls $B(x,\delta)$ with $x \in \mathbb{R}$:

$$\left| \frac{1}{\xi'(x)} \cdot \frac{\xi(z) - \xi(w)}{z - w} - 1 \right| < \varepsilon, \qquad z, w \in B(x,\delta).$$

Both of these properties are consequences of Koebe's distortion theorem, where one takes into account that the exponential maps the real axis to the unit circle. The reader can consult [16, Section 2] for more details.

From Lemma 3.2, it is clear that

$$\sigma^2(\log \varphi') = \sigma^2_{[0,1]}(\log \mathcal{E}'_\varphi) \qquad \text{and} \qquad \beta_\varphi(p) = \beta_{\mathcal{E}_\varphi}(p),$$

The reader versed in the arguments of Sections 4–6 should have no trouble filling in the details in Lemmas 3.1 and 3.2.

3.2 Boxes and grids

By a *box* in the upper half-plane, we mean a rectangle whose sides are parallel to the coordinate axes, with the bottom side located above the real axis. We say two boxes are *similar* if they differ by an affine scaling $L(z) = az + b$ with $a > 0$, $b \in \mathbb{R}$. We use $\overline{\square} \subset \overline{\mathbb{H}}$ to denote the reflection of the box $\square$ with respect to the real line. Every box $\square$ is similar to a unique box of the form $[0, \alpha] \times [1/n, 1]$. In this case, we say that $\square$ is of type (n, α).

Boxes naturally arise in grids. By a *grid*, we mean a collection of similar boxes that tile $\mathbb{H}$. One natural collection of grids are the *n-adic grids* $\mathscr{G}_n$, defined for integer $n \geq 2$. An *n-adic interval* is an interval of the form $I_{j,k} = \bigl[j \cdot n^{-k}, (j+1) \cdot n^{-k}\bigr]$. To an n-adic interval I, we associate the *n-adic box*

$$\square_I = \Bigl\{w : \operatorname{Re} w \in I,\, \operatorname{Im} w \in \bigl[n^{-1}|I|,\, |I|\bigr]\Bigr\}.$$

It is easy to see from the construction that the boxes $\square_{I_{j,k}}$ with $j, k \in \mathbb{Z}$ have disjoint interiors and their union is $\mathbb{H}$.

Given two boxes $\square_1$ and $\square_2$, we say $\square_1$ *dominates* $\square_2$ if $\square_1 = [x_1, x_2] \times [y_1, y_2]$ and $\square_2 = [x_1, x_2] \times [\theta y_1, \theta y_2]$ for some $0 < \theta \leq 1$. In other words, $\square_2$ has the same hyperbolic height and is located strictly beneath $\square_1$. We let $\widehat{\mathscr{G}_n}$ denote the collection of boxes that are dominated by some box in $\mathscr{G}_n$ with $1/n < \theta \leq 1$. The advantage of the collection $\widehat{\mathscr{G}_n}$ is that any horizontal strip $\mathbb{R} \times [y/n, y] \subset \mathbb{H}$ of hyperbolic height $\log n$ can be tiled by boxes from $\widehat{\mathscr{G}_n}$. This property will be used in Section 7.

Suppose μ is a Beltrami coefficient supported on the lower half-plane. We say that μ is *periodic* with respect to a grid $\mathscr{G}$ (or rather with respect to $\overline{\mathscr{G}}$) if for any two boxes $\square_1, \square_2 \in \mathscr{G}$, we have $\mu|_{\overline{\square}_1} = L^*(\mu|_{\overline{\square}_2})$, where L is the affine map that takes $\square_1$ to $\square_2$. Given μ defined on a box $\square$, and a grid $\mathscr{G}$ containing $\square$, there exists a unique periodic Beltrami coefficient μ_{per} which agrees with μ on $\square$.

4 Locality of $(\mathcal{S}\mu)'/\rho_{\mathbb{H}}$

The technique of *fractal approximation* from [4] hinges on the local nature of the operator $\mu \to (\mathcal{S}\mu)'$. As a preliminary observation, we first show that $(\mathcal{S}\mu)'(z)$ is bounded as a 1-form:

Lemma 4.1. *Suppose μ is a Beltrami coefficient supported on the lower half-plane with $\|\mu\|_\infty \le 1$. Then, for $z \in \mathbb{H}$, $\big|(2(\mathcal{S}\mu)'/\rho_{\mathbb{H}})(z)\big| \le 8/\pi$.*

Proof. The proof is by direct calculation:

$$\begin{aligned}
|(\mathcal{S}\mu)'(z)| &\le \frac{2}{\pi}\int_{\overline{\mathbb{H}}} \frac{1}{|w-z|^3}\,|dw|^2, \\
&= \frac{2}{\pi}\int_{y_0}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^{3/2}}\,dxdy, \\
&= \frac{2}{\pi}\int_{y_0}^{\infty} \frac{x}{y^2\sqrt{x^2+y^2}}\bigg|_{-\infty}^{\infty} dy, \\
&= \frac{4}{\pi}\int_{y_0}^{\infty}\frac{dy}{y^2}, \\
&= \frac{4}{\pi y_0},
\end{aligned}$$

where $y_0 = \operatorname{Im} z$. Multiplying by 2 and dividing by the density of the hyperbolic metric gives the result. $\square$

If $z \in \mathbb{H}$ is far away from the reflection of the support of μ, one can give a better estimate. For a point $x + iy \in \mathbb{H}$, define its "square" neighbourhood as

$$Q_L(x+iy) := \big\{w : \operatorname{Re} w \in [x - e^L y, x + e^L y],\, \operatorname{Im} w \in [e^{-L}y, e^L y]\big\},$$

and let $Q_L(x - iy)$ be its reflection in the lower half-plane.

Lemma 4.2. *Under the assumptions of Lemma 4.1, if $\mu = 0$ on $Q_L(x - iy)$, then*

$$\left|\frac{2(\mathcal{S}\mu)'}{\rho_{\mathbb{H}}}(x+iy)\right| \le Ce^{-L}. \tag{4.1}$$

Proof. The lemma follows by estimating the contributions of the top, bottom, left and right parts of $\overline{\mathbb{H}} \setminus Q_L(x - iy)$ separately and adding them up. We leave the details to the reader. $\square$

Lemma 4.2 says that to determine the value of $(\mathcal{S}\mu)'/\rho_{\mathbb{H}}$ at a point $z \in \mathbb{H}$, up to small error, it suffices to know the values of μ in a neighbourhood of $\bar{z}$. In particular, if μ_1 and μ_2 are two Beltrami coefficients that agree on $Q_L(x-iy)$ with L large, then the difference $|((\mathcal{S}\mu_1 - \mathcal{S}\mu_2)'/\rho_{\mathbb{H}})(z)|$ is small.

Remark. It may seem more natural to work with round neighbourhoods of $x - iy$, i.e. to assume that μ vanishes on $\{w \in \overline{\mathbb{H}} : d_{\overline{\mathbb{H}}}(w, x - iy) < L\}$. However, in this case, the estimate (4.1) is only $\leq CLe^{-L}$.

We now come to the main result of this section.

Lemma 4.3. *Suppose μ_1 and μ_2 are two Beltrami coefficients on $\overline{\mathbb{H}}$ with $\|\mu_i\|_\infty \leq 1$, $i = 1, 2$. If $\mu_1 = \mu_2$ agree on an (n, α)-box $\overline{\square}$ with $\alpha \geq 1$, then*

$$\left| \fint_{\square} \left| \frac{2(\mathcal{S}\mu_1)'}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} - \fint_{\square} \left| \frac{2(\mathcal{S}\mu_2)'}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} \right| \leq \frac{C_1}{\log n}. \tag{4.2}$$

Proof. Without loss of generality, we may assume that $\square = [0, \alpha] \times [1/n, 1]$. From the elementary identity $\big| |a|^2 - |b|^2 \big| = |a - b| \cdot |a + b|$ and Lemma 4.1, it follows that the left hand side of (4.2) is bounded by

$$\frac{32}{\pi} \fint_{\square} \left| \frac{\big(\mathcal{S}(\mu_1 - \mu_2)\big)'}{\rho_{\mathbb{H}}}(z) \right| \frac{|dz|^2}{y}. \tag{4.3}$$

The claim now follows from Lemma 4.2 and integration. $\square$

5 Box Lemma

In this section, we show the infinitesimal version of the box lemma:

Lemma 5.1. (i) *For any Beltrami coefficient μ with $|\mu| \leq \chi_{\mathbb{H}}$ and (n, α)-box $\square \subset \mathbb{H}$ with $\alpha \geq 1$,*

$$\fint_{\square} \left| \frac{2(\mathcal{S}\mu)'}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} < \Sigma^2 + \frac{C}{\log n}. \tag{5.1}$$

(ii) *Conversely, for $n \geq 1$, there exists a Beltrami coefficient μ, periodic with respect to the n-adic grid, which satisfies*

$$\fint_{\square} \left| \frac{2(\mathcal{S}\mu)'}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} > \Sigma^2 - \frac{C}{\log n} \tag{5.2}$$

on every box $\square \in \widehat{\mathscr{G}_n}$.

Proof. (i) Assume for the sake of contradiction that there is a box $\square \subset \mathbb{H}$ and a Beltrami coefficient μ for which

$$\fint_{\square} \left| \frac{2(\mathcal{S}\mu)'}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} > \Sigma^2 + \frac{C}{\log n}, \tag{5.3}$$

with $C > C_1$ from Lemma 4.3. Let $\overline{\mathscr{G}} = \bigcup_{j=1}^{\infty} \overline{\square}_j$ be a grid containing $\overline{\square}$ and form the Beltrami coefficient μ_{per} by restricting μ to $\overline{\square}$ and periodizing with respect to $\overline{\mathscr{G}}$. According to Lemma 4.3, we would have

$$\fint_{\square_j} \left| \frac{2(\mathcal{S}\mu_{\text{per}})'}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} > \Sigma^2 + \varepsilon, \quad \text{for all } \square_j \in \mathscr{G}. \tag{5.4}$$

In view of (3.3), this implies $\sigma^2_{[0,1]}(\mathcal{S}^{\#}\mu_{\text{per}}) > \Sigma^2 + \varepsilon$, which contradicts the definition of Σ^2.

(ii) Conversely, suppose ν is a Beltrami coefficient with

$$|\nu| \le \chi_{\overline{\mathbb{H}}} \quad \text{and} \quad \sigma^2_{[0,1]}(\mathcal{S}^{\#}\nu) \ge \Sigma^2 - \varepsilon.$$

Consider the n-adic grid $\mathscr{G}_n$. By the pigeon-hole principle, there exists an n-adic box $\square$ for which the integral in (5.1) is at least $\Sigma^2 - \varepsilon$. Restricting ν to $\overline{\square}$ and periodizing over n-adic boxes produces a Beltrami coefficient ν_{per} which satisfies

$$\fint_{\square_j} \left| \frac{2(\mathcal{S}\nu_{\text{per}})'}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} > \Sigma^2 - \frac{C}{\log n}, \quad \text{for all } \square_j \in \mathscr{G}_n. \tag{5.5}$$

A careful inspection reveals that the above estimate holds for all boxes in the collection $\widehat{\mathscr{G}_n}$. This completes the proof of Lemma 5.1. $\square$

Remark. Given a periodic Beltrami coefficient μ_{per} on $\overline{\mathbb{H}}$ from (ii), we may multiply it by the characteristic function of the strip $S = \{z \in \overline{\mathbb{H}} : |\operatorname{Im} z| < 1\}$ to get a Beltrami coefficient $\mu_{\text{per}} \cdot \chi_S$ on $\overline{\mathbb{H}}$ that is periodic under $z \to z+1$. By construction, $\mu_{\text{per}} \cdot \chi_S$ descends to a Beltrami coefficient on the unit disk via the exponential mapping, which is eventually-invariant under $z \to z^n$. Clearly, the asymptotic variance is unchanged in this process. This proves the statement (1.11) from Theorem 1.6.

6 Locality of $n_f/\rho_{\mathbb{H}}$

Lemma 6.1. (i) *Fix $0 < k < 1$. Given $\varepsilon > 0$, if $n \geq n(\varepsilon)$ is sufficiently large, then for any (n,α)-box $\square \subset \mathbb{H}$ with $\alpha \geq 1$ and any conformal map $f \in \mathbf{H}_k$,*

$$\fint_{\square} \left| \frac{2n_f}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} \;<\; \Sigma^2(k) + \varepsilon. \tag{6.1}$$

(ii) *Conversely, for any $\varepsilon > 0$, there exists a conformal map $f = \tilde{w}^{\mu} \in \mathbf{H}_k$, whose dilatation $\mu = \operatorname{dil.} f := \overline{\partial} f / \partial f$ is periodic with respect to the n-adic grid for some $n \geq 1$, and which satisfies*

$$\fint_{\square} \left| \frac{2n_f}{\rho_{\mathbb{H}}}(z) \right|^2 \frac{|dz|^2}{y} \;>\; \Sigma^2(k) - \varepsilon \tag{6.2}$$

on every box $\square \in \widehat{\mathscr{G}_n}$.

Since we do not require a quantitative estimate, it suffices to give a simple compactness argument. In view of the arguments from the previous section, it is enough to show:

Lemma 6.2. *Suppose $\tilde{w}^{\mu_1}$ and $\tilde{w}^{\mu_2} \in \mathbf{H}_k$ are two k-quasiconformal mappings that are conformal on the upper half-plane. For any $\varepsilon > 0$, there exists R sufficiently large so that if $\mu_1 = \mu_2$ on $B_{\text{hyp}}(\overline{w}_0, R) = \{w \in \overline{\mathbb{H}} : d_{\mathbb{H}}(w, \overline{w}_0) > R\}$, $w_0 \in \mathbb{H}$, then*

$$\left| \frac{n_{\tilde{w}^{\mu_1}} - n_{\tilde{w}^{\mu_2}}}{\rho_{\mathbb{H}}}(w_0) \right| < \varepsilon. \tag{6.3}$$

Proof. Since non-linearity is invariant under compositions with affine maps $z \to az + b$, $a > 0$, $b \in \mathbb{R}$, it suffices to prove the lemma with $w_0 = i$.

To the contrary, suppose that one could find sequences of Beltrami coefficients $\{\mu_n\}$ and $\{\nu_n\}$, with $\mu_n = \nu_n$ on $B_n = B_{\text{hyp}}(-i, n)$ and

$$\|\tilde{w}^{\mu_n} - \tilde{w}^{\nu_n}\|_{L^\infty(B)} > 1/n, \quad \text{where } B = \{w \in \mathbb{H} : d_{\mathbb{H}}(w, i) < 1\}. \tag{6.4}$$

Since the collection of normalized quasiconformal mappings with dilatation bounded by k forms a normal family, we can extract a convergent subsequence $\tilde{w}^{\mu_{n_j}} \to \tilde{w}$.

Stoilow factorization allows us to write $\tilde{w}^{\nu_n} = H_n \circ \tilde{w}^{\mu_n}$, where H_n is a normalized quasiconformal map with $\operatorname{supp}(\operatorname{dil.} H_n) \subseteq w^{\mu_n}(\mathbb{H} \setminus B_n)$ and $\|\operatorname{dil.} H_n\|_\infty < \frac{2k}{1+k^2}$. Since the supports of $\operatorname{dil.} H_{n_j}$ shrink to $\tilde{w}(\mathbb{R})$ which has measure 0, the only possible limit of H_{n_j} is the identity. This rules out (6.4), thus proving the lemma. □

Proceeding like in the previous section proves the second statement (1.12) from Theorem 1.6.

7 Estimating integral means

In this section, we review the Becker-Pommerenke argument which gives the bound $D(k) \le 1 + 36\,k^2 + \mathcal{O}(k^3)$. We then modify the argument to take advantage of the box lemma, allowing us to replace 36 with Σ^2.

7.1 Becker-Pommerenke argument

For convenience, we work in the subclass $\mathbf{H}_k^1 \subset \mathbf{H}_k$ of maps that satisfy the periodicity condition $f(z+1) = f(z) + 1$. In view of the exponential transform (3.7), we are secretly working with a conformal map of the exterior unit disk. Define

$$I_p(y) := \int_0^1 |f'(x+iy)^p|\,dx, \qquad p \in \mathbb{C}. \tag{7.1}$$

Since $\frac{\partial}{\partial y} = i(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}})$, differentiation under the integral sign shows

$$I_p'(y) \;=\; -p \int_0^1 |f'(x+iy)^p|\; \operatorname{Im}\biggl(\frac{f''}{f'}\biggr) dx. \tag{7.2}$$

In view of the normalization, $f' \to 1$, $f'' \to 0$ as $y \to \infty$, in which case $I_p'(y) \to 0$. To estimate I_p, we use a variant of Hardy's identity on the upper half-plane which says that for a holomorphic function $g(z)$ with $g(z+1) = g(z)$,

$$\frac{d^2}{dy^2} \int_0^1 |g(x+iy)|\,dx = \int_0^1 \Delta|g(re^{i\theta})|\,dx. \tag{7.3}$$

Indeed, $\frac{d^2}{dx^2}\int_0^1 |g(x+iy)|\,dx = 0$ by periodicity. Applying Hardy's identity to $f'(z)^p$ gives

$$I_p''(y) \;=\; |p|^2 \int_0^1 |f'(x+iy)^p| \left|\frac{f''}{f'}\right|^2 dx. \tag{7.4}$$

In particular, $I_p'(y)$ is increasing as $y \to \infty$ which shows that $I_p'(y) \le 0$. Replacing non-linearity by its supremum bound, we obtain

$$I_p''(y) \le \frac{9\,k^2|p|^2}{y^2}\, I_p(y). \tag{7.5}$$

From the differential inequality (7.5) together with $I_p(y) \ge 0$, $I_p'(y) \le 0$, it follows that

$$I_p(y) \le C \cdot y^{-9k^2|p|^2}, \qquad k \in [0,1), \quad p \in \mathbb{C},$$

see Lemma 7.1(i) below. In other words, $\beta_f(p) \le 9\,k^2|p|^2$. Anti-symmetrization (1.8) and the equation $\beta_f(\mathrm{M.}\dim\partial\Omega) = \mathrm{M.}\dim\partial\Omega - 1$ yield the dimension bound $D(k) \le 1 + 36\,k^2 + \mathcal{O}(k^3)$.

7.2 A differential inequality

To make use of (7.5), we used an elementary fact about differential inequalities. If necessary, the reader may consult [26, Proposition 8.7].

Lemma 7.1. (i) *Suppose $u(y)$ is a C^2 function on $(0, y_0)$ with*

$$u \ge 0 \qquad \textit{and} \qquad u' \le 0$$

satisfying

$$u''(y) \le \frac{\alpha u}{y^2}, \tag{7.6}$$

for some constant $\alpha > 0$. Then,

$$u(y) \;\le\; v(y) \;=\; Cy^{-\beta}, \quad \textit{where } \beta^2 + \beta = \alpha,$$

when $C > 0$ is sufficiently large so that $u(y_0) \le v(y_0)$ and $|u'(y_0)| \le |v'(y_0)|$.

(ii) *Conversely, if (7.6) is replaced by*

$$u''(y) \ge \frac{\alpha u}{y^2}, \tag{7.7}$$

then

$$u(y) \;\geq\; v(y) \;=\; cy^{-\beta},$$

when $c > 0$ *is sufficiently small so that* $u(y_0) \geq v(y_0)$ *and* $|u'(y_0)| \geq |v'(y_0)|$.

Remark. When $\alpha > 0$ is small, $\beta = \alpha - \mathcal{O}(\alpha^2)$, in which case $\alpha \approx \beta$.

7.3 Averaging over annuli

Using the box lemma, we can give an improvement in the argument of Becker and Pommerenke. Given an integer $n \geq 1$, let $A(y)$ denote the rectangle $[0,1] \times [y/n, y]$ and $R = \log n$ be its hyperbolic height. Consider the function

$$u(y) \;:=\; \int_{y/n}^{y} I_p(h)\,\frac{dh}{h} \;=\; \int_{1/n}^{1} I_p(yh)\cdot\frac{dh}{h}. \tag{7.8}$$

Since $\|\log f'\|_{\mathcal{B}(\mathbb{H})} \leq 6$, we have $u(y) \asymp I_p(y)$, which allows us to compute the integral means spectrum of f by measuring the growth of $u(y)$ as $y \to 0^+$. Differentiating (7.8) twice gives

$$u''(y) \;=\; \int_{y/n}^{y} I_p''(h)\,hdh \;=\; \frac{|p|^2}{4y^2}\int_{A(y)} |f'(z)^p| \left|\frac{2n_f}{\rho_{\mathbb{H}}}\right|^2 \frac{dh}{h}\cdot dx. \tag{7.9}$$

Lemma 7.2. *Suppose that the average non-linearity of* $f \in \mathbf{H}^1_k$ *over any box* $\square \in \widehat{\mathscr{G}_n}$ *with* $\square \subset [0,1] \times [0, y_0]$ *is bounded above by* M*. Then,*

$$u''(y) \leq M\exp(CRk|p|)\cdot\frac{|p|^2}{4y^2}\cdot u(y), \qquad y \in (0, y_0), \tag{7.10}$$

Proof. We partition $A(y)$ into $\widehat{\mathscr{G}_n}$-boxes $B_1, B_2, \dots, B_{N(y)}$. (The number of boxes is roughly proportional to the hyperbolic length of the segment $\{x + iy : 0 \leq x \leq 1\}$, but we will not use this.) Since f has a k-quasiconformal extension to the plane, $\|\log f'\|_{\mathcal{B}} \leq 6k$. As the hyperbolic diameter of a box in $\widehat{\mathscr{G}_n}$ is comparable to R, the multiplicative oscillation of $|f'(z)^p|$ in each box B_j is at most

$$\operatorname{osc}_{B_j}|f'(z)^p| := \sup_{z_1,z_2\in B_j}\log\frac{|f'(z_1)^p|}{|f'(z_2)^p|} \leq C_1 Rk|p|.$$

In other words, if $k|p|$ is small, $|f'(z)^p|$ is essentially constant on boxes, i.e.

$$\int_{B_j} |f'(z)^p| \left|\frac{2n_f}{\rho_{\mathbb{H}}}\right|^2 \frac{|dz|^2}{y} \approx |f'(c_j)^p| \int_{B_j} \left|\frac{2n_f}{\rho_{\mathbb{H}}}\right|^2 \frac{|dz|^2}{y}, \tag{7.11}$$

where c_j is an arbitrary point in B_j. Hence,

$$\int_{B_j} |f'(z)^p| \left|\frac{2n_f}{\rho_{\mathbb{H}}}\right|^2 \frac{|dz|^2}{y} \le \exp(C_2 Rk|p|) \cdot M \int_{B_j} |f'(z)^p| \frac{|dz|^2}{y}.$$

Summing over all the boxes that make up $A(y)$ gives (7.10). $\square$

The same argument shows:

Lemma 7.3. *Suppose that the average non-linearity of $f \in \mathbf{H}_k^1$ over any box $\square \in \widehat{\mathscr{G}_n}$ with $\square \subset [0,1] \times [0, y_0]$ is bounded below by m. Then,*

$$u''(y) \ge m \exp(-CRk|p|) \cdot \frac{|p|^2}{4y^2} \cdot u(y), \qquad y \in (0, y_0). \tag{7.12}$$

7.4 Applications

We now prove Theorem 1.2 which says that for a fixed $k \in (0,1)$,

$$\lim_{p \to 0} \frac{B_k(p)}{|p|^2/4} = \Sigma^2(k). \tag{7.13}$$

Proof of Theorem 1.2. According to Lemma 6.1, we may choose $n \ge 1$ sufficiently large so that the average non-linearity of $f \in \mathbf{H}_k$ over any box in $\widehat{\mathscr{G}_n}$ is at most $\Sigma^2(k) + \varepsilon/3$. Lemma 7.2 implies the differential inequalities

$$u''(y) \le \Big(\Sigma^2(k) + 2\varepsilon/3\Big) \cdot \frac{|p|^2}{4y^2} \cdot u(y), \qquad \text{for } p \text{ small.}$$

Applying Lemma 7.1(i) gives

$$u(y) \le C \cdot y^{-|p|^2(\Sigma^2(k)+\varepsilon)/4}.$$

An analogous bound holds for $I_p(y)$ since $I_p(y) \asymp u(y)$. This proves the upper bound in (7.13). The lower bound is similar, but uses the converse part of Lemma 6.1 and Lemmas 7.3 and 7.1(ii). Note that we can assume that $\mu = \text{dil.}\, f$ is supported in the strip $S = \{z \in \overline{\mathbb{H}} : |\operatorname{Im} z| < 1\}$ and is invariant under the translation $z \to z + 1$ to ensure that $f = \tilde{w}^\mu \in \mathbf{H}_k^1$ (see the remark following the proof of Lemma 5.1). $\square$

For small k, we can give a more precise estimate. Combining Lemma 5.1 with (2.4) shows that the average non-linearity over a box in $\widehat{\mathscr{G}_n}$ is bounded by

$$\fint_{B_j} \left| \frac{2n_f}{\rho_{\mathbb{H}}} \right|^2 \frac{|dz|^2}{y} \le (\Sigma^2 + C/R)k^2 + Ck^3. \tag{7.14}$$

Putting this into Lemma 7.2 gives

$$u''(y) \le \exp(CRk|p|) \cdot \left[(\Sigma^2 + C/R)k^2 + Ck^3\right] \cdot \frac{|p|^2}{4y^2} \cdot u(y). \tag{7.15}$$

Choosing $R \asymp 1/\sqrt{k|p|}$, we get the error term of $\mathcal{O}\big((k|p|)^{5/2}\big)$ in (7.15). We conclude that for any conformal map $f \in \mathbf{H}^1_k$,

$$u(y) \le C \cdot y^{-k^2|p|^2/4\left(\Sigma^2 + C\sqrt{k|p|}\right)}.$$

Repeating the argument for the special conformal maps provided by Lemma 6.1(ii), we obtain the estimate in the other direction. As mentioned in the introduction, this entails $D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{5/2})$.

8 The $\mathcal{O}(k^{8/3-\varepsilon})$ error term

In this section, we show that the maximal Hausdorff dimension of a k-quasicircle satisfies

$$D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{8/3-\varepsilon}), \qquad \text{for any } \varepsilon > 0. \tag{8.1}$$

We focus on the upper bound in (8.1) and leave the lower bound to the reader. Suppose $f : \mathbb{H} \to \mathbb{C}$ is a conformal mapping of the upper half-plane which admits a k-quasiconformal extension with $0 < k < 1/2$. Let

$$B = [0,1] \times \left[1/e^R, 1\right], \qquad R = k^{-\gamma}, \quad \gamma > 0,$$

be a box in $\mathbb{H}$ and z_B be the midpoint of its top edge. Our objective is to slightly improve the argument of Lemma 7.2 by showing that:

Lemma 8.1. *Suppose $p \in [1,2)$. For any $\gamma \in (0, 2/3)$,*

$$\mathcal{Q}(B) := \frac{\int_B |f'(z)^p| \left|\frac{2n_f}{\rho_{\mathbb{H}}}\right|^2 \frac{|dz|^2}{y}}{\int_B |f'(z)^p| \frac{|dz|^2}{y}} \le \Sigma^2 k^2 + \mathcal{O}_\gamma(k^{2+\gamma}). \tag{8.2}$$

From the scale-invariance of the problem, the same estimate must also hold on any box similar to B. The arguments of the previous section now give the upper bound in (8.1). Here, we remind the reader that in view of (1.7), the range of exponents $p \in [1, 2)$ is sufficient for applications to Minkowski dimension.

In the proof of Lemma 7.2, we made the assumption that Rkp was small in order to guarantee that $|f'(z)^p|$ was approximately constant in B. To be able to take $R = k^{-\gamma}$ with $\gamma > 1/2$, we introduce the *exceptional set*

$$\mathscr{E} := \big\{ z \in B : |\log f'(z) - \log f'(z_B)| > k^\gamma \big\}. \tag{8.3}$$

Lemma 8.2. *Suppose $p \in [1, 2)$ and $\gamma \in (0, 2/3)$. Then,*

$$\int_{\mathscr{E}} \frac{|dz|^2}{y} < C \cdot k^\gamma \cdot \int_B \frac{|dz|^2}{y} \tag{8.4}$$

and

$$\int_{\mathscr{E}} |f'(z)^p| \frac{|dz|^2}{y} < C \cdot k^\gamma \cdot |f'(z_B)^p| \int_B \frac{|dz|^2}{y}. \tag{8.5}$$

With the above lemma, the proof of Lemma 8.1 runs as follows:

Proof of Lemma 8.1 assuming Lemma 8.2. Since

$$\left| \frac{2n_f}{\rho_{\mathbb{H}}} \right|^2 < 36k^2 \qquad \text{and} \qquad \mathrm{osc}_B \, |f'(z)^p| < 2,$$

the bound (8.5) gives

$$\mathcal{Q}(B) \;=\; \frac{\int_B |f'(z)^p| \left|\frac{2n_f}{\rho_{\mathbb{H}}}\right|^2 \frac{|dz|^2}{y}}{\int_B |f'(z)^p| \frac{|dz|^2}{y}} \;=\; \frac{\int_{B\setminus\mathscr{E}} |f'(z)^p| \left|\frac{2n_f}{\rho_{\mathbb{H}}}\right|^2 \frac{|dz|^2}{y}}{\int_{B\setminus\mathscr{E}} |f'(z)^p| \frac{|dz|^2}{y}} + \mathcal{O}(k^{2+\gamma}).$$

From the definition of the exceptional set (8.3),

$$1 - Ck^\gamma \;\le\; \left| \frac{f'(z)}{f'(z_B)} \right|^p \;\le\; 1 + Ck^\gamma, \qquad z \in B \setminus \mathscr{E},$$

we obtain

$$\mathcal{Q}(B) = ⨍_{B\setminus\mathscr{E}} \left| \frac{2n_f}{\rho_{\mathbb{H}}} \right|^2 \frac{|dz|^2}{y} + \mathcal{O}(k^{2+\gamma}).$$

According to (7.14), the average non-linearity over B is bounded by $(\Sigma^2 + Ck^\gamma)k^2$. Combining with (8.4) shows that the average non-linearity over $B \setminus \mathscr{E}$ is also at most $(\Sigma^2 + Ck^\gamma)k^2$. This completes the proof. $\square$

Let $L_S = \{z \in B : \operatorname{Im} z = 1/e^S\}$ be the line segment consisting of points in B for which the hyperbolic distance to the top edge is S and define $\mathscr{E}_S := \mathscr{E} \cap L_S$. Set $g(z) = \log f'(z) - \log f'(z_B)$. Since $\| \log f' \|_{\mathcal{B}(\mathbb{H})} \le 6k$,

$$\Big| \Big\{ z \in L_S : | \operatorname{Re} g(z)| > \eta \Big\} \Big| \le \exp\bigg(-c_0 \cdot \frac{\eta^2}{k^2 S} \bigg), \tag{8.6}$$

where $c_0 > 0$ is a universal constant. This follows from the sub-Gaussian estimate for martingales with bounded increments [22, Equation (2.9)]. Alternatively, for an analytic proof of (8.6), the reader may consult [11].

Proof of Lemma 8.2. To show (8.4), it suffices to demonstrate that $|\mathscr{E}_S| < C \cdot k^\gamma$ for any $0 \le S \le R$. Putting $\eta = k^\gamma$ and $S \le R = k^{-\gamma}$ in (8.6) gives

$$|\mathscr{E}_S| \le \exp(-c_0 \cdot k^{3\gamma - 2}). \tag{8.7}$$

In order to get any kind of decay, we must have $3\gamma - 2 < 0$. Of course, any $\gamma < 2/3$ gives an *exponential* decay rate, which is more than sufficient. The second statement (8.5) follows from (8.4) and the fact that $\operatorname{osc}_B |f'(z)^p| < 2$. $\square$

9 Sparse Beltrami Coefficients

In this section, we prove Theorem 1.7 which gives a stronger bound for the dimension of quasicircle $\tilde{w}^\mu([0, 1])$ if the dilatation μ has small support. We assume that μ is a Beltrami coefficient on $\overline{\mathbb{H}}$ for which $\operatorname{supp} \mu \subseteq \mathcal{G} = \bigcup_{j \in J} A_j$ is a union of "crescents" that satisfy the quasigeodesic and separation properties. That is, we assume each crescent lies within a hyperbolic distance S from a geodesic arc $\gamma_j \subset \overline{\mathbb{H}}$ and that the hyperbolic distance between any two crescents is bounded below by R.

We are interested in studying the quadratic behaviour of $t \to \operatorname{M.dim} \tilde{w}^{t\mu}([0, 1])$ when S is fixed and R is large. In the dynamical setting, this was considered in [16], where it was sufficient to measure the length of the intersection of $\mathcal{G}$ with horizontal lines. The non-dynamical case was examined by Bishop [9], but without the quadratic behaviour. The proof of Theorem 1.7 follows from the Becker-Pommerenke argument and the estimate:

Theorem 9.1. *For any Beltrami coefficient μ with $\|\mu\|_\infty \le 1$ and $\operatorname{supp}\mu \subseteq \mathcal{G}$, we have*

$$\fint_L \left| \frac{2(\mathcal{S}\mu)'}{\rho_{\mathbb{H}}}(z) \right|^2 dx \le Ce^{-R/2}, \tag{9.1}$$

where L is a horizontal line segment which has hyperbolic diameter R.

The crucial feature of hyperbolic geometry that we use is that *a horocycle connecting two points is exponentially longer than the geodesic.* Thus, L is extremely long: its length (as measured in the hyperbolic metric) is comparable to $e^{R/2}$.

For convenience, let us denote the horizontal lines in $\mathbb{C}$ by

$$\ell_y = \{z : \operatorname{Im} z = y\}, \qquad \text{and} \qquad \bar{\ell}_y = \{z : \operatorname{Im} z = -y\}.$$

We need an elementary lemma, which is an exercise in Fubini's theorem:

Lemma 9.1. *Suppose μ is a Beltrami coefficient on $\overline{\mathbb{H}}$ with $\|\mu\|_\infty \le 1$, such that the length of the intersection of any horizontal line $\bar{\ell}_y$ with $\operatorname{supp}\mu$ is finite. Then, for $y > 0$,*

$$\int_{\ell_y} \left| \frac{(\mathcal{S}\mu)'}{\rho_{\mathbb{H}}}(z) \right| dx \lesssim \int_0^\infty \frac{|\operatorname{supp}\mu \cap \bar{\ell}_h|}{(y+h)^3} \cdot dh.$$

Proof of Theorem 9.1. From the scale-invariance of the problem, we may assume that $L = \big[i, i + |L|\big]$, where $|L| \asymp e^{R/2}$. Let V be the vertical strip $\{z : 0 < \operatorname{Re} z < |L|\}$. We may write $\mu = \mu_1 + \mu_2$ where $\mu_1 = \mu \cdot \chi_V$ and $\mu_2 = \mu \cdot \chi_{\overline{\mathbb{H}} \setminus V}$. Set $\mathcal{G}_1 = \operatorname{supp}\mu_1$ and $\mathcal{G}_2 = \operatorname{supp}\mu_2$. Expanding the square and using Lemma 4.1, we get

$$\int_L \left| \frac{2(\mathcal{S}\mu)'}{\rho_{\mathbb{H}}}(z) \right|^2 dx \le \int_L \left| \frac{2(\mathcal{S}\mu_2)'}{\rho_{\mathbb{H}}}(z) \right|^2 dx + \frac{96}{\pi} \int_L \left| \frac{(\mathcal{S}\mu_1)'}{\rho_{\mathbb{H}}}(z) \right| dx. \tag{9.2}$$

To complete the proof, we need to show that both summands in (9.2) are $\mathcal{O}(1)$. For the first summand, this follows from Lemma 4.2. For the second summand, we appeal to Lemma 9.1, where we use the bound $|\mathcal{G}_1 \cap \bar{\ell}_h| = \mathcal{O}(1)$ for $h < 1$ and $|\mathcal{G}_1 \cap \bar{\ell}_h| = \mathcal{O}(h)$ for $h \ge 1$. To explain this bound, we recall the logic from [16, Section 6]: the hyperbolic length of the intersection $A_j \cap \bar{\ell}_h$ is $\mathcal{O}(1)$, but as soon as soon as $\bar{\ell}_h$ intersects a crescent A_j, a segment of hyperbolic length $\mathcal{O}(e^{R/2})$ must be disjoint from the other crescents. □

A Equidistribution in dynamics

In this appendix, we show that if a conformal mapping is "dynamically defined" then its non-linearity is approximately equidistributed over large boxes, see (1.14). We first examine the case of Blaschke products, the case of general Jordan repellers is similar.

A.1 Blaschke case

Suppose B is a Blaschke product with an attracting fixed point at the origin (1.13) and $\mu \in M_B(\mathbb{D})$ is a Beltrami coefficient on the unit disk which is eventually-invariant under B with $\|\mu\|_\infty < 1$. We view $\varphi = w^\mu(z)$ as a confomal mapping of the exterior unit disk. Then, $F = \varphi \circ B \circ \varphi^{-1}$ is holomorphic in an open neighbourhood of $\varphi(\mathbb{S}^1)$. For purposes of equidistribution, the crucial property of $n_\varphi = \varphi''/\varphi'$ is:

Lemma A.1. *The function $h = |n_\varphi/\rho_*|^2$ is almost-invariant under the dynamics of $B(z)$.*

Recall from [25] that a continuous function $h : \mathbb{D}^* \to \mathbb{C}$ is *almost-invariant* if for any $\varepsilon > 0$, there exists $R(\varepsilon) > 1$, so that for any orbit $z \to B(z) \to \cdots \to B^{\circ k}(z)$ contained in $\{z : 1 \le |z| < R\}$, we have $|h(z) - h(B^{\circ k}(z))| < \varepsilon$. Almost-invariant functions naturally extend to the Riemann surface lamination $\hat{X}_B$ which is defined as the set of tails of all inverse orbits $(z_0 \leftarrow z_1 \leftarrow z_2 \leftarrow \dots)$ with $z_i \in \mathbb{D}^*$, where the constant inverse orbit of the attracting fixed point $(\infty \leftarrow \infty \leftarrow \dots)$ is excluded. For more on Riemann surface laminations, we refer the reader to [25, Section 10].

Proof. It suffices to show that $|n_\varphi/\rho_*|$ is almost-invariant. Differentiating the conjugacy relation $F^{\circ k} \circ \varphi = \varphi \circ B^{\circ k}$, one discovers

$$n_{F^{\circ k}}(\varphi(z))\varphi'(z) + n_\varphi(z) = n_\varphi(B^{\circ k}(z))(B^{\circ k})'(z) + n_{B^{\circ k}}(z).$$

By Koebe's distortion theorem, if $z \in \mathbb{D}^*$ is close to the unit circle,

$$|(B^{\circ k})'(z)| = \frac{\rho_*(z)}{\rho_*(B^{\circ k}(z))} + \varepsilon(R),$$

where the term $\varepsilon(R) \to 0$ as $R \to 1^+$. To handle the error terms, note that $|n_{B^{\circ k}}(z)|$ and $|n_{F^{\circ k}}(\varphi(z))|$ are bounded while $|\varphi'(z)| \le C(|z|-1)^k$ since $\varphi \in \mathbf{\Sigma}_k$. This can be seen from Cauchy's estimates and the Lehto majorant principle, see [26, Chapter 5.6]. Putting these estimates together, we obtain

$$|n_\varphi/\rho_*(z)| \;=\; \frac{|n_\varphi(B^{\circ k}(z))(B^{\circ k})'(z)|}{\rho_*(z)} + \varepsilon(R) \;=\; \frac{|n_\varphi(B^{\circ k}(z))|}{\rho_*(B^{\circ k}(z))} + \varepsilon(R)$$

as desired. $\square$

We now explain how to use the almost-invariance of h to show the approximate equidistribution

$$\sigma^2(\log\varphi') - \varepsilon \;<\; \fint_{\square} \left|\frac{2n_\varphi}{\rho_*}(z)\right|^2 \rho_*|dz|^2 \;<\; \sigma^2(\log\varphi') + \varepsilon, \tag{A.1}$$

where $\square \subset \{z : 1 < |z| < 2\}$ is any box of hyperbolic width ≥ 1 whose hyperbolic height is sufficiently large. As discussed in [25, Theorem 10.2], the ergodicity of the geodesic flow in the Riemann surface lamination $\hat{X}_B$ implies that

$$\frac{1}{|\log(R-1)|}\int_R^2 h(re^{i\theta})\cdot\rho_*(re^{i\theta})dr \;\to\; \sigma^2(\log\varphi'), \quad \text{a.e. } \theta \in [0,2\pi). \tag{A.2}$$

Averaging over $\theta \in [a,b]$, one obtains equidistribution for any sufficiently large "basic" box whose top edge is contained in $\{z : |z| = 2\}$ and whose hyperbolic width $(2/3)\cdot|b-a| > 1$:

$$\left|\frac{1}{(b-a)|\log(R-1)|}\int_R^2\int_a^b h(z)\cdot\rho_*|dz|^2 \;-\; \sigma^2(\log\varphi')\right| < \varepsilon(R).$$

It remains to extend the approximate equidistribution to general boxes. Given a large box $\square$, the dynamics of B can be used to blow it up to definite size – for instance we can consider the maximal index k for which $\square \to B^{\circ k}(\square)$ is univalent. The argument is now concluded by the following two simple observations:

I. By Koebe's distortion theorem and the almost-invariance of h, the average values of h on $\square$ and $B^{\circ k}(\square)$ are approximately equal.

II. Up to small symmetric difference, $B^{\circ k}(\square)$ is a basic box.

A.2 Jordan repellers

We now discuss equidistribution in the context of Jordan repellers. These are certain dynamically-invariant quasicircles defined by the following conditions:

(i) The set J is a Jordan curve, presented as a union of closed arcs $J = J_1 \cup J_2 \cup \dots \cup J_n$, with pairwise disjoint interiors.

(ii) For each $i = 1, 2, \dots, n$, there exists a univalent function $F_i : U_i \to \mathbb{C}$, defined on a neighbourhood $U_i \supset J_i$, such that F_i maps J_i bijectively onto the union of several arcs, i.e.

$$F_i(J_i) = \bigcup_{j \in \mathcal{A}_i} J_j,$$

(iii) Additionally, we want each map F_i to preserve the complementary regions $\Omega_\pm$ in $\mathbb{S}^2 \setminus J$, i.e. $F_i(U_i \cap \Omega_\pm) \subset \Omega_\pm$.

(iv) We require that the *Markov map* $F : J \to J$ defined by $F|_{J_i} = F_i$ is *mixing*, that is, for a sufficiently high iterate, we have $F^{\circ N}(J_i) = J$.

(v) Finally, we want the dynamics of F to be *expanding*, i.e. for some $N \geq 1$, we have $\inf_{z \in J} |(F^{\circ N})'(z)| > 1$. (At the endpoints of the arcs and their inverse orbits under F, we consider one-sided derivatives.)

We can assume that $\infty \in \Omega_+$ does not belong to J. Let $\varphi : \mathbb{D}^* \to \Omega_+$ be the conformal map which belongs to the class $\mathbf{\Sigma}$. Pulling back the Markov map F by φ and using the Schwarz reflection principle, we obtain a Markov map B on $\mathbb{S}^1 = I_1 \cup I_2 \cup \dots \cup I_n$ which expresses the unit circle as a Jordan repeller.

The proof of the almost-invariance of $|n_\varphi/\rho_*|$ only uses the conjugacy relation $F \circ \varphi = \varphi \circ B$ and hence goes through without modification. The blowing up argument also proceeds verbatim since the dynamics of B are asymptotically affine. Perhaps the only step that needs explanation is the proof of equidistribution along a.e. radial ray (A.2).

The unit circle $\mathbb{S}^1$ carries a unique absolutely continuous probability measure m which is invariant under the Markov map B. The solenoid $\hat{\mathbb{S}}^1_B$ is defined as the set of all inverse orbits $(x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \dots)$. The solenoid comes equipped with a natural measure $\hat{m}$ which projects to m under any coordinate function. Since the Lebesgue measure m is ergodic on $\mathbb{S}^1$ for the action of B, $\hat{m}$ is ergodic under the shift map

$(x_i) \to (x_{i+1})$ on the solenoid. One can define the Riemann surface lamination $\hat{X}_B$ as the set of tails of inverse orbits $(z_0 \leftarrow z_1 \leftarrow z_2 \leftarrow \dots)$ with $z_i \in \mathbb{D}^*$, $B_{j(i)}(z_i) = z_{i-1}$ for which $z_i/|z_i| \in I_j$. Like in the Blaschke case, almost-invariant functions naturally extend to the Riemann surface lamination $\hat{X}_B$. Up to a set of measure 0, $\hat{X}_B$ can also be represented as the suspension of $\hat{\mathbb{S}}^1$ by the roof function $\rho(x) = \log|B'(x_0)|$, that is,

$$\hat{X}_B := (\hat{\mathbb{S}}^1_B \times \mathbb{R}) / \big((x_i), t\big) \sim \big((x_{i+1}), t + \rho(x_0)\big).$$

The ergodicity of $\hat{m}$ implies the ergodicity of the product measure $\hat{m} \times dt$ on the suspension. With the above definitions, the reader may examine the arguments of McMullen [25, Section 10] to verify (A.2).

References

[1] K. Astala, *Calderón's problem for Lipschitz classes and the dimension of quasicircles*, Rev. Mat. Iberoamericana 4 (1988), no. 3-4, 469–486.

[2] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. 173 (1994), no. 1, 37–60.

[3] K. Astala, F. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space*, J. d'Analyse Math. 46 (1986), no. 1, 16–57.

[4] K. Astala, O. Ivrii, A. Perälä, I. Prause, *Asymptotic variance of the Beurling transform*, Geom. Funct. Anal., 25 (2015), no. 6, 1647–1687.

[5] K. Astala, T. Iwaniec, G. J. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton University Press, 2009.

[6] R. Bañuelos, C. N. Moore, *Mean growth of Bloch functions and Makarov's law of the iterated logarithm*, Proc. Amer. Math. Soc. 112 (1991), 851–854.

[7] J. Becker, C. Pommerenke, *On the Hausdorff dimension of quasicircles*, Ann. Acad. Sci. Fenn. Ser. A I Math. 12 (1987), 329–333.

[8] I. Binder, *Harmonic measure and rotation spectra of planar domains*, preprint, 2008.

[9] C. J. Bishop, *Big deformations near infinity*, Illinois J. Math. 47 (2003), no. 4, 977–996.

[10] L. Carleson, P. W. Jones, *On coefficient problems for univalent functions and conformal dimension*, Duke Math. J., 66 (1992), no. 2, 169–206.

[11] H. Hedenmalm, *Bloch functions and asymptotic tail variance*, preprint, arXiv:1509.06630, 2015.

[12] H. Hedenmalm, *Bloch functions, asymptotic variance, and geometric zero packing*, preprint, arXiv:1602.03358, 2016.

[13] H. Hedenmalm, I. R. Kayumov, *On the Makarov law of the iterated logarithm*, Proc. Amer. Math. Soc. 135 (2007), 2235–2248.

[14] H. Hedenmalm, S. Shimorin, *On the universal integral means spectrum of conformal mappings near the origin*, Proc. Amer. Math. Soc. 135 (2007), no. 7, 2249–2255.

[15] H. Hedenmalm, A. Sola, *Spectral notions for conformal maps: a survey*, Comput. Meth. Funct. Th. 8 (2008), no. 2, 447–474.

[16] O. Ivrii, *The geometry of the Weil-Petersson metric in complex dynamics*, preprint, arXiv:1503.02590, 2015.

[17] O. Ivrii, *On Makarov's principle in conformal mapping*, preprint, 2016.

[18] P. W. Jones, *On scaling properties of harmonic measure*, Perspectives in analysis, Mathematical Physics Studies 27 (Springer, Dordrecht, 2005) 73–81.

[19] I. R. Kayumov, *The law of the iterated logarithm for locally univalent functions*, Ann. Acad. Sci. Fenn. Math. 27 (2002), no. 2, 357–364.

[20] R. Kühnau, *Möglichst konforme Spiegelung an einer Jordankurve*, Jahresber. Deutsch. Math.-Verein. 90 (1988), no. 2, 90–109.

[21] T. H. N. Le, M. Zinsmeister, *On Minkowski dimension of Jordan curves*, Ann. Acad. Sci. Fenn. Math. 39 (2014), no. 2, 787–810.

[22] N. G. Makarov, *Probability methods in the theory of conformal mappings*, (Russian) Algebra i Analiz 1 (1989), no. 1, 3–59; translation in Leningrad Math. J. 1 (1990), no. 1, 1–56.

[23] N. G. Makarov, *Fine structure of harmonic measure*, St. Petersburg Math. J. 10 (1999), no. 2, 217–268.

[24] C. T. McMullen, *Cusps are dense*, Annals of Math. 133 (1991), 217–247.

[25] C. T. McMullen, *Thermodynamics, dimension and the Weil-Petersson metric*, Invent. Math. 173 (2008), no. 2, 365–425.

[26] C. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften 299, Springer-Verlag, 1992.

[27] I. Prause, S. Smirnov, *Quasisymmetric distortion spectrum*, Bull. Lond. Math. Soc. 43 (2011), 267–277.

[28] F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps II*, Studia Math. 97 (1991), 189–225.

[29] S. Smirnov, *Dimension of quasicircles*, Acta Math. 205 (2010), no. 1, 189–197.

[30] K. Zhu. *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics 226, Springer, 2005.