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ON THE AREA DISTORTION BY QUASICONFORMAL MAPPINGS

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ABSTRACT. We give the sharp constants in the area distortion inequality for quasiconformal mappings in the plane.

Astala [1] proved the following theorem conjectured by Gehring and Reich in [3]:

Theorem A. *Let f be a K-quasiconformal mapping of $D = \{z : |z| < 1\}$ onto itself with $f(0) = 0$. Then for any measurable $E \subset D$ we have*

$$|f(E)| \leq C(K)|E|^{1/K},$$

where $|\cdot|$ stands for the area.

The first author [2] obtained a shorter proof which did not make use of the elaborate Thermodynamic Formalism and Holomorphic Motion Theory of the original proof of Astala. In late 1992 the second author [4] circulated a minimal proof which gives sharp bounds for the constants under the normalization $f \in \Sigma(K)$, i.e. f is a K-quasiconformal mapping of the plane which is conformal on $C\backslash\overline{D}$ and $f(z) = z + o(1)$ near ∞. In the interests of having a short sharp proof we combined our efforts.

Usually in what follows Δ is the closed unit disk $\{z : |z| \leq 1\}$, but any compact set of transfinite diameter 1 will do (and this is important in our proof). We note that this normalization implies that for any $E \subset \Delta$ the area of E and $f(E)$ is bounded by π.

Theorem 1. *Let f be a K-quasiconformal mapping of the plane which is conformal on $\mathbf{C}\backslash\Delta$, where Δ is a compact set of transfinite diameter 1, and $f(z) = z + o(1)$ near ∞.*

(i) *If f is conformal on $E \subset \Delta$ (i.e., f has dilatation $\mu = 0$ a.e. on E), then*

$$|f(E)| \leq \pi^{1-1/K}|E|^{1/K}.$$

(ii) *If $E \subset \Delta$ with f conformal on $\mathbf{C}\backslash E$, then*

$$|f(E)| \leq K|E|.$$

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(iii) *Hence in general for* $E \subset \Delta$

$$|f(E)| \le K\pi^{1-1/K}|E|^{1/K}.$$

Remarks. Theorem A follows from Theorem 1 via standard distortion estimates for quasiconformal mappings. The constants in Theorem 1 are best possible. Part (ii) is essentially due to Gehring and Reich. Part (i) gives sharp bounds for a conjectured inequality for the singular integral transform

$$Tg(\zeta) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int\int_{|z-\zeta|>\varepsilon} \frac{g(z)\,dx\,dy}{(z-\zeta)^2},$$

i.e., for every $E \subset \Delta$ we have

$$\int\int_{\Delta\setminus E} |T(\chi_E)|\,dx\,dy \le |E|\log\frac{\pi}{|E|}.$$

Lemma 1. *Let* $a_1, \dots, a_n$ *be positive functions in the unit disk, such that* $\log a_j$ *are harmonic and*

$$\sum_{j=1}^{n} a_j(\lambda) \le 1, \qquad |\lambda| < 1. \tag{1}$$

Then

$$\sum_{j=1}^{n} a_j(\lambda) \le \left(\sum_{j=1}^{n} a_j(0)\right)^{(1-|\lambda|)/(1+|\lambda|)}, \qquad |\lambda| < 1.$$

The proof is based on the following "Variational Principle" from statistical mechanics which was also used by Astala.

Lemma A. *Let* $p_j > 0$ *and* $q_j > 0$ *be probability distributions on the set* $\{1, \dots, n\}$. *Then*

$$-\sum_{j=1}^{n} p_j \log q_j + \sum_{j=1}^{n} p_j \log p_j \ge 0.$$

Proof. The left side of the inequality is equal to $\sum q_j\phi(p_j/q_j)$, where $\phi(x) = x\log x$. This function ϕ is convex, so

$$\sum q_j\phi\left(\frac{p_j}{q_j}\right) \ge \phi\left(\sum q_j\frac{p_j}{q_j}\right) = \phi(1) = 0.$$

Proof of Lemma 1. For $|\lambda| < 1$ and $|z| < 1$ define the probability distributions

$$p_j = \frac{a_j(\lambda)}{\sum a_j(\lambda)} \quad \text{and} \quad q_j = \frac{a_j(z)}{\sum a_j(z)}.$$

Now fix λ and set

$$H(z) = -\sum p_j \log a_j(z) + \sum p_j \log p_j.$$

Observe that H is harmonic in z. By Lemma A and hypothesis (1)

$$H(z) \ge -\log\sum a_j(z) \ge 0.$$

Thus by Harnack's inequality

$$H(z) \ge \frac{1-|z|}{1+|z|}H(0).$$

Putting $z = \lambda$ and using Lemma A again we obtain

$$H(\lambda) = -\log \sum a_j(\lambda) \geq \frac{1-|\lambda|}{1+|\lambda|}\left(-\sum p_j \log a_j(0) + \sum p_j \log p_j\right)$$
$$\geq \frac{1-|\lambda|}{1+|\lambda|}\left(-\log \sum a_j(0)\right),$$

which proves Lemma 1.

Actually we require the continuous version of Lemma 1. Namely $a(z, \lambda)$ is to be defined on $E \times D$ and $\log a(z, \lambda)$ is harmonic in λ. If

$$\int\int_E a(z, \lambda) dx\, dy \leq 1, \qquad z = x + iy, |\lambda| < 1,$$

then we have

$$\int\int_E a(z, \lambda) dx\, dy \leq \left(\int\int_E a(z, 0) dx\, dy\right)^{(1-|\lambda|)/(1+|\lambda|)}$$

The application to Theorem 1 is immediate. Suppose that f has complex dilatation μ supported on Δ. Without loss of generality we may assume that μ is smooth (a uniform bound for the smooth case yields the general uniform bound since the smooth case is dense). Define the function $f_\lambda \in \sum(K_\lambda)$, $K_\lambda = (1+|\lambda|)/(1-|\lambda|)$, with dilatation

$$\mu_\lambda(z) = \lambda \frac{K+1}{K-1}\mu(z), \qquad |\lambda| < 1.$$

This is done by the standard solution of the Beltrami equation:

$$f_\lambda(z) = z + S\mu_\lambda + S\mu_\lambda T\mu_\lambda + S\mu_\lambda T\mu_\lambda T\mu_\lambda + \cdots,$$

where S is the complex Cauchy transform. Now f_λ has Jacobian

$$J_\lambda(z) = |\partial_z f_\lambda(z)|^2(1 - |\mu_\lambda(z)|^2).$$

As the dilatations are smooth this is everywhere nonzero. If f is conformal on E define

$$a(z, \lambda) = \frac{1}{\pi}|\partial_z f_\lambda(z)|^2.$$

By the Holomorphic Dependence of Parameter Theorem for the Beltrami equation (see, for example, [5]) $\partial_z f_\lambda$ is holomorphic in λ. Thus $\log a(z, \lambda)$ is harmonic for $|\lambda| < 1$, $z \in E$. By the classical Area Theorem for a conformal mapping as $f_\lambda(z) = z + o(1)$, $z \to \infty$,

$$\int\int_\Delta J_\lambda(z) dx\, dy \leq \pi.$$

Thus $a(z, \lambda)$ satisfies the continuous version of Lemma 1 giving

$$\int\int_E J_\lambda(z)\frac{dx\, dy}{\pi} \leq \left(\frac{|E|}{\pi}\right)^{(1-|\lambda|)/(1+|\lambda|)}$$

Setting $\lambda = (K-1)/(K+1)$ gives $\mu_\lambda = \mu$ and thus

$$|f(E)| \leq \pi^{1-1/K}|E|^{1/K},$$

completing the first part of the proof.

To prove part (ii) and the bound for T we sketch the arguments of Gehring and Reich. This begins with the observation that for any set G

$$\int\int_G |T(\chi_G)|dx\,dy \le |G|$$

(by Cauchy-Schwarz as T is a unitary transformation of $L^2(\mathbf{C})$). Hence for any function ρ supported on G as T is also (almost) self-adjoint

$$\left|\int\int_G T(\rho)dx\,dy\right| \le \|\rho\|_\infty |G|. \tag{2}$$

Finally for any function μ, $\|\mu\|_\infty = 1$, supported on E we define $\mu_t(z) = t\mu(z)$ and the corresponding family of normalized maps f_t, $0 < t < 1$, $f_0(z) = z$ and $f_{|\lambda|} = f$. This can be realised as a deformation family of quasiconformal maps

$$\frac{\partial f_t}{\partial t} = g_t \circ f_t, \qquad g_t(z) = z + S\rho_t,$$
$$\rho_t = \frac{\mu \circ f_t^{-1}}{1 - t^2|\mu \circ f_t^{-1}|^2} e^{2i\arg(\partial_z f_t^{-1})}, \qquad f_0(z) = z,$$

by the composition formula for dilatations. Now as $\partial S = T$

$$\frac{d|f_t(E)|}{dt} = 2\Re \int\int_{f_t(E)} T(\rho_t)dx\,dy.$$

Thus by (2)

$$\frac{d|f_t(E)|}{dt} \le 2\frac{|f_t(E)|}{1-t^2},$$

so by integration

$$|f_t(E)| \le \frac{1+t}{1-t}|E|,$$

which proves the result.

The third part follows by writing $f = g \circ h$ where h is conformal on E and g is conformal on $\mathbf{C}\backslash h(E)$. Thus h has dilatation $\mu(z)$ on $\Delta\backslash E$, zero elsewhere, and g has dilatation $\mu(h^{-1}(z))$ on $h(E)$, zero elsewhere. We see that h is normalized and so is g as $h(\Delta)$ has transfinite diameter 1.

The bound on T is also proved by holomorphic deformation. For any function μ, $\|\mu\|_\infty < 1$, supported on $\Delta\backslash E$ we define $\mu_\lambda(z) = \lambda\mu(z)$ and the corresponding family of normalized maps f_λ. This time we let $\lambda \to 0$ to find that

$$\begin{aligned}|f_\lambda(E)| &= |E| + 2\Re\left(\lambda \int\int_E T(\mu)dx\,dy\right) + o(\lambda)\\ &\le \pi^{2\lambda+o(\lambda)}|E|^{1-2\lambda+o(\lambda)} = |E| + 2|\lambda|\,|E|\log\frac{\pi}{|E|} + o(\lambda)\end{aligned}$$

by part (i) of Theorem 1. Hence we obtain

$$\left|\int\int_E T(\mu)dx\,dy\right| \le |E|\log\frac{\pi}{|E|}$$

and so as in the proof of (ii) for all μ supported on $\Delta\backslash E$ and bounded by 1

$$\left|\int\int_{\Delta\backslash E} T(\chi_E)\overline{\mu}(z)dx\,dy\right| \leq |E|\log\frac{\pi}{|E|}.$$

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