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CHARACTERIZATIONS OF QUASIDISKS

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I. INTRODUCTION

A. Quasiconformal mappings

1. *Notation.* Throughout these lectures we will consider domains D in the euclidean complex plane R^2 and its one point compactification $\overline{R}^2 = R^2 \cup \infty$ equipped with the chordal metric

$$q(z,w) = \frac{2|z-w|}{\sqrt{|z|^2+1}\sqrt{|w|^2+1}} \qquad \text{for } z, w \in \overline{R}^2.$$

2. *Linear dilatation of a homeomorphism.* Suppose that D and D' are domains in $\overline{R}^2$ and that $f : D \to D'$ is a homeomorphism. For $z \in D \setminus \{\infty, f^{-1}(\infty)\}$ and $0 < r < \operatorname{dist}(z, \partial D)$ we let

$$l_f(z,r) = \min_{|z-w|=r} |f(z) - f(w)|, \quad L_f(z,r) = \max_{|z-w|=r} |f(z) - f(w)|$$

and call

$$H_f(z) = \limsup_{r \to 0} \frac{L_f(z,r)}{l_f(z,r)}$$

the linear dilatation of f at z.

3. *Geometric definition.* A sense preserving homeomorphism $f : D \to D'$ is a K-quasiconformal mapping where $1 \le K < \infty$ if

a. H_f is bounded in $D \setminus \{\infty, f^{-1}(\infty)\}$,
b. $H_f \le K$ a.e. in D.

4. *Class ACL.* A continuous real valued function u is ACL in a domain $D \subset \overline{R}^2$ if for each rectangle $[a,b] \times [c,d] \subset D$,

a. $u(x+iy)$ is absolutely continuous in x for a.e. $y \in [c,d]$,
b. $u(x+iy)$ is absolutely continuous in y for a.e. $x \in [a,b]$.

5. *Analytic definition.* A sense preserving homeomorphism $f : D \to D'$ is K-quasiconformal if and only if

a. f is ACL in D,
b. $\max_\theta |\partial_\theta f|^2 \leq K J_f$ a.e. in D.

Here $\partial_\theta f$ denotes the directional derivative of f taken in the direction θ and J_f the Jacobian of f.

6. *Remark.* If a homeomorphism f is ACL in D, then it has finite partial derivatives a.e. in D by measure theoretic arguments and hence a differential a.e. in D [24].

7. *Modulus and extremal length of a curve family.* Suppose that Γ is a family of curves in $\overline{R}^2$. We want to assign a number or modulus which measures the *size* of Γ and is conformally invariant. We say that ρ is *admissible* for Γ or in $\mathrm{adm}(\Gamma)$ if ρ is nonnegative and Borel measurable in R^2 and if

$$\int_\gamma \rho(z)|dz| \geq 1$$

for each locally rectifiable $\gamma \in \Gamma$. The *modulus* of the family Γ is then defined as

$$\mathrm{mod}(\Gamma) = \inf_\rho \int_{R^2} \rho(z)^2 \, dm,$$

where the infimum is taken over $\rho \in \mathrm{adm}(\Gamma)$, and the *extremal length* of Γ is given by

$$\lambda(\Gamma) = \frac{1}{\mathrm{mod}(\Gamma)}.$$

8. *Conformal invariance of the modulus.* If $f : D \to D'$ is conformal and if Γ is a family of curves in D, then $\mathrm{mod}(f(\Gamma)) = \mathrm{mod}(\Gamma)$.

9. *Proof.* We consider the case where $D, D' \subset R^2$. For each $\rho' \in \mathrm{adm}(f(\Gamma))$ let

$$\rho(z) = \begin{cases} \rho'(f(z))|f'(z)| & \text{if } z \in D, \\ 0 & \text{if } z \in R^2 \setminus D. \end{cases}$$

Then ρ is nonnegative and Borel measurable in R^2. If γ is locally rectifiable, then $f(\gamma) \in f(\Gamma)$ is locally rectifiable and

$$\int_\gamma \rho(z)|dz| = \int_\gamma \rho'(f(z))|f'(z)||dz| = \int_{f(\gamma)} \rho'(w)|dw| \geq 1.$$

Thus $\rho \in \mathrm{adm}(\Gamma)$,

$$\mathrm{mod}(\Gamma) \leq \int_{R^2} \rho(z)^2 dm = \int_D \rho'(f(z))^2 |f'(z)|^2 dm = \int_{D'} \rho'(w)^2 dm \leq \int_{R^2} \rho'(w)^2 dm$$

whence

$$\mathrm{mod}(\Gamma) \leq \inf_{\rho'} \int_{R^2} \rho'(w)^2 dm = \mathrm{mod}(f(\Gamma)).$$

Finally we obtain $\mathrm{mod}(\Gamma) = \mathrm{mod}(f(\Gamma))$ by applying the above argument to f^{-1}. ■

10. *Remark.* If the curves $\gamma \in \Gamma$ are disjoint arcs, we may think of them as homogeneous electric wires. Then the modulus $\mathrm{mod}(\Gamma)$ is a conformally invariant electrical transconductance for the family of wires γ and the extremal length $\lambda(\Gamma)$ is the total

electrical resistance of the system. In particular, $\operatorname{mod}(\Gamma)$ is big if the curves γ are short and plentiful and small if the curves are long or scarce.

11. *Modulus definition.* A sense preserving homeomorphism $f : D \to D'$ is K-quasiconformal if and only if

$$\frac{1}{K} \operatorname{mod}(\Gamma) \leq \operatorname{mod}(f(\Gamma)) \leq K \operatorname{mod}(\Gamma)$$

for each family Γ of curves in D.

12. *1-quasiconformal mappings.* f is a 1-quasiconformal mapping if and only if it is conformal, i.e. a homeomorphism which is analytic as a function of a complex variable in $D \setminus \{\infty, f^{-1}(\infty)\}$.

13. *Composition and inverse.* If $f : D \to D'$ is K-quasiconformal and $g : D' \to D''$ is K'-quasiconformal, then $gf = g \circ f$ is KK'-quasiconformal and f^{-1} is K-quasiconformal.

14. *Extension theorem.* If $f : D \to D'$ is quasiconformal and if D and D' are Jordan domains, then f has a homeomorphic extension which maps $\overline{D}$ onto $\overline{D'}$.

15. *Removable sets.* Suppose that $E \subset D$ is closed and contained in a countable union of rectifiable curves. If $f : D \to D'$ is a homeomorphism which is K-quasiconformal in each component of $D \setminus E$, then f is K-quasiconformal in D.

B. Quasidisks

1. *Definition.* A domain D is a K-*quasidisk* if it is the image of an open disk or half plane under a K-quasiconformal self mapping of $\overline{R}^2$. D is a *quasidisk* if it is a K-quasidisk for some K.

2. *Remark.* If D is a K-quasidisk, then ∂D is the image of a circle or line under a self homeomorphism of $\overline{R}^2$ which is differentiable a.e. Thus ∂D is a Jordan curve. Moreover ∂D is a circle or line whenever $K = 1$. Does ∂D have any nice analytic properties when $1 < K < \infty$? For example, is it locally rectifiable?

3. *Example* (Gehring-Väisälä [28]). We describe here an elementary example which shows that from the standpoint of euclidean geometry, the boundary of a quasidisk can be quite wild.

We say that a square is *oriented* if its sides are parallel to the coordinate axes and let Q and Q' denote the open squares

$$Q = Q' = \{z = x + iy : |x| < 1, |y| < 1\}.$$

Next set

$$z_1 = \frac{3}{4}, \quad z_2 = \frac{1}{4}, \quad z_3 = -\frac{1}{4}, \quad z_4 = -\frac{3}{4}$$

and

$$w_1 = \frac{1+i}{2}, \quad w_2 = \frac{-1+i}{2}, \quad w_3 = \frac{1-i}{2}, \quad w_4 = \frac{-1-i}{2},$$

and fix $0 < r < 1/2$ and $0 < s < 1$. Finally for $j = 1, 2, 3, 4$ let Q_j denote the open oriented square with center z_j and side length r and Q'_j the open oriented square with center w_j and side length s. Then we can choose a piecewise linear homeomorphism

$$f_0 : \overline{Q} \setminus \bigcup_{j=1}^{4} Q_j \to \overline{Q'} \setminus \bigcup_{j=1}^{4} Q'_j$$

such that f_0 is the identity on ∂Q and and such that f_0 is of the form $a_j z + b_j, a_j > 0$ on ∂Q_j with $f_0(\partial Q_j) = \partial Q'_j$. Then f_0 is K-quasiconformal in $Q \setminus \bigcup_j \overline{Q}_j$.

Next for each j choose oriented squares $Q_{j,k}$ in Q_j and $Q'_{j,k}$ in Q'_j in the same way as the squares Q_j and Q'_j were chosen in Q and Q', respectively. Then by scaling we can extend f_0 to obtain a piecewise linear homeomorphism

$$f_1 : \overline{Q} \setminus \bigcup_{j,k=1}^{4} Q_{j,k} \to \overline{Q'} \setminus \bigcup_{j,k=1}^{4} Q'_{j,k}$$

which is K-quasiconformal in $Q \setminus \bigcup_{j,k} \overline{Q}_{j,k}$.

Continuing in this way, we obtain a homeomorphism $f : \overline{Q} \setminus E \to \overline{Q'} \setminus E'$ where E and E' are Cantor sets. Then f can be extended by continuity to give a K-quasiconformal mapping which maps $\overline{Q}$ onto $\overline{Q'}$ and is the identity on ∂Q.

Set $f(z) = z$ in $\overline{R}^2 \setminus \overline{Q}$. Then f is a K-quasiconformal self mapping of $\overline{R}^2$ which maps the upper half plane $H = \{z = x + iy : y > 0\}$ onto a quasidisk D whose boundary ∂D is not locally rectifiable. In fact for each $1 < a < 2$ we can choose $0 < s = s(a) < 1$ so that $\dim(\partial D) > a$ where dim denotes Hausdorff dimension.

4. *Remark.* We can use the analytic properties of quasiconformal mappings to show that $m(\partial D) = 0$ whenever D is a K-quasidisk, where m denotes plane measure. In fact a recent result due to Astala [5] shows that

$$\dim(\partial D) \le \frac{2K}{K+1}.$$

5. *Plan of remaining lectures.* Though quasidisks can be quite unruly domains, they occur very naturally in surprisingly many different branches of analysis and geometry. We consider here twenty six different properties of quasidisks which generalize corresponding properties of euclidean disks and which characterize the class of quasidisks. We then indicate how to establish a few of these properties. See also [19] and [21].

C. Characteristic properties of quasidisks

1. *Notation.* We will assume from now on that D is a simply connected proper subdomain of R^2 and that D^* is the exterior of D,

$$D^* = \overline{R}^2 \setminus \overline{D}.$$

Next for $z_0 \in R^2$ and $0 < r < \infty$ we let

$$B(z_0, r) = \{z \in R^2 : |z - z_0| < r\}, \quad B = B(0,1).$$

2. *Categories of properties.* The characteristic properties of a quasidisk D which we will discuss here fall into the following five categories.

a. Geometric properties.

   i. Reflection in ∂D.

ii. Inequalities.
iii. Local connectivity properties.
iv. Decomposition property.

b. Conformal invariants.

i. Relations between hyperbolic and euclidean geometry in D.
ii. Properties of harmonic measure in D.
iii. Relation between conjugate quadrilaterals in D and D^*.
iv. Extremal length of curve families in D and $\overline{R}^2$.

c. Injectivity criteria.

i. Analytic functions.
ii. Locally quasiconformal mappings.
iii. Local quasi-isometries.

d. Extension and continuity.

i. Functions of bounded mean oscillation.
ii. Functions with bounded Dirichlet integral.
iii. Quasiconformal mappings.
iv. Quasi-isometries.
v. Bloch functions.

e. Miscellaneous properties.

i. Homogeneity of D and of ∂D.
ii. Limit set of a quasiconformal group.
iii. Relation between Dirichlet integrals in D and D^*.
iv. Quasiconformal equivalence of $\overline{R}^3 \setminus \overline{D}$ and B^3.

3. *Remark.* A number of the properties for quasidisks which we discuss below can be used to characterize euclidean disks or half planes. We will indicate when this is the case. See also [31].

II. GEOMETRIC PROPERTIES

A. Reflection property

1. *Quasi-isometries.* Suppose that $E, E' \subset \overline{R}^2$. We say that $f : E \to E'$ is an L-*quasi-isometry* if

$$\frac{1}{L}\,|z_1 - z_2| \le |f(z_1) - f(z_2)| \le L\,|z_1 - z_2| \quad \text{for } z_1, z_2 \in E \setminus \{\infty\}$$

and $f(\infty) = \infty$ if $\infty \in E$.

2. *Remark.* If D is a half plane, then there exists a 1-quasi-isometry of $\overline{R}^2$ which maps D onto its exterior D^* and is the identity on ∂D.

3. *Reflection property.* D has this property if there exists an L-quasi-isometry of $\overline{R}^2$ which maps D onto D^* and is the identity on ∂D.

4. *Theorem* (Ahlfors [3]). If D is unbounded, then D is a quasidisk if and only if it has the reflection property.

5. *Remark.* D is a half plane if and only if it has the reflection property with $L = 1$.

B. Inequalities

1. *Remark.* If D is a disk or half plane, then for each pair of points $z_1, z_2 \in \partial D \setminus \{\infty\}$,

$$\min_{j=1,2} \operatorname{dia}(\gamma_j) \leq |z_1 - z_2|$$

where dia denotes the euclidean diameter and γ_1, γ_2 are the components of $\partial D \setminus \{z_1, z_2\}$.

2. *Two point inequality.* D satisfies this inequality if D is a Jordan domain and there exists a constant $a \geq 1$ such that for each pair of points $z_1, z_2 \in \partial D \setminus \{\infty\}$,

$$\min_{j=1,2} \operatorname{dia}(\gamma_j) \leq a\, |z_1 - z_2|$$

where γ_1, γ_2 are the components of $\partial D \setminus \{z_1, z_2\}$.

3. *Theorem* (Ahlfors [3]). D is a quasidisk if and only if it satisfies the two point inequality.

4. *Remark.* Suppose that D is a Jordan domain with $\infty \in \partial D$. Then D satisfies the two point inequality if and only if there exists a constant $b \geq 1$ such that each ordered triple of points $z_1, z_2, z_3 \in \partial D \setminus \{\infty\}$ satisfy the reversed triangle inequality

$$|z_1 - z_2| + |z_2 - z_3| \leq b\, |z_1 - z_3|,$$

in which case

$$\frac{|z_1 - z_2|}{|z_1 - z_3|} + \frac{|z_2 - z_3|}{|z_1 - z_3|} \leq b.$$

When $\infty \notin \partial D$, the ratios on the left hand side of the above inequality must be replaced by cross ratios and we are led to the following alternative Möbius invariant formulation of the two point inequality.

5. *Reversed triangle inequality.* D satisfies this if D is a Jordan domain and there exists a constant $b \geq 1$ such that

$$|z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1| \leq b\, |z_1 - z_3||z_2 - z_4|$$

for each ordered quadruple of points $z_1, z_2, z_3, z_4 \in \partial D \setminus \{\infty\}$.

6. *Lemma.* D satisfies the reversed triangle inequality if and only if it satisfies the two point inequality.

7. *Proof.* Suppose that D satisfies the two point inequality with constant a and choose $z_1, z_2, z_3, z_4 \in \partial D \setminus \{\infty\}$. By relabeling if necessary we may assume that

$$|z_1 - z_3| \leq |z_2 - z_4|.$$

Let γ_2 and γ_4 denote the components of $\partial D \setminus \{z_1, z_3\}$ which contain z_2 and z_4, respectively. Again by relabeling we may assume that

$$\operatorname{dia}(\gamma_2) \leq \operatorname{dia}(\gamma_4).$$

Then

$$|z_1 - z_2| \le \mathrm{dia}(\gamma_2) \le a\,|z_1 - z_3|, \quad |z_2 - z_3| \le \mathrm{dia}(\gamma_2) \le a\,|z_1 - z_3|$$

whence

$$|z_3 - z_4| \le |z_2 - z_3| + |z_2 - z_4| \le (a+1)|z_2 - z_4|$$

and

$$|z_4 - z_1| \le |z_1 - z_2| + |z_2 - z_4| \le (a+1)|z_2 - z_4|.$$

Thus

$$|z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1| \le b\,|z_1 - z_3||z_2 - z_4|$$

and D satisfies the reversed triangle inequality with constant $b = 2a(a+1)$.

Suppose next that D satisfies the reversed triangle inequality with constant b, fix $z_1, z_3 \in \partial D \setminus \{\infty\}$ and let γ_2 and γ_4 denote the components of $\partial D \setminus \{z_1, z_3\}$. If

$$\min_{j=2,4} \mathrm{dia}(\gamma_j) > 2b\,|z_1 - z_3|,$$

then we can choose $z_2 \in \gamma_2$ and $z_4 \in \gamma_4$ such that

$$|z_1 - z_2| > b\,|z_1 - z_3| \quad \text{and} \quad |z_1 - z_4| > b\,|z_1 - z_3|,$$

in which case

$$\begin{aligned} b\,|z_1 - z_3||z_2 - z_4| &\le b\,|z_1 - z_3|(|z_2 - z_3| + |z_3 - z_4|) \\ &= b\,|z_1 - z_3||z_3 - z_4| + b\,|z_1 - z_3||z_2 - z_3| \\ &< |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|, \end{aligned}$$

a contradiction. Hence D satisfies the two point inequality with constant $2b$. ∎

8. *Corollary.* D is a quasidisk if and only if it satisfies the reversed triangle inequality.

9. *Remark* (Ahlfors [1]). D is a disk or half plane if and only if it satisfies the reversed triangle inequality with $b = 1$.

C. Local connectivity properties

1. *Linear local connectivity.* A set $E \subset \overline{R}^2$ has this property if there exists a constant $c \ge 1$ such that for each $z_0 \in R^2$ and each $r > 0$

a. points in $E \cap \overline{B}(z_0, r)$ can be joined in $E \cap \overline{B}(z_0, cr)$,
b. points in $E \setminus B(z_0, r)$ can be joined in $E \setminus B(z_0, r/c)$.

Here *joined* means lie in a component of the specific set.

2. *Remark.* A set $E \subset R^2$ is *locally connected* in the usual sense at each $z_0 \in R^2$ if for each $s > 0$ there exists r, $0 < r < s$, such that points in $E \cap \overline{B}(z_0, r)$ can be joined in $E \cap \overline{B}(z_0, s)$. The property of *linear local connectivity* requires that, in addition,

a. the constant r is a fixed linear multiple of s and hence is independent of the point z_0,
b. the image of E under each Möbius transformation ϕ has the same property with c replaced by $g(c)$ where $g(1) = 1$. (Walker [65]).

Hence the property of being 1-linearly locally connected is invariant with respect to Möbius transformations.

3. *Theorem* (Gehring [17]). D is a quasidisk if and only if it is linearly locally connected.

4. *Theorem* (Langmeyer [39]). D is a disk or half plane if and only if it is linearly locally connected with $c = 1$.

5. *Remark.* If D is a disk or half plane, then ∂D is linearly locally connected with $c = 1$.

6. *Lemma.* D satisfies the two point inequality if and only if ∂D is linearly locally connected.

7. *Theorem* (Walker [65]). D is a quasidisk if and only if ∂D is linearly locally connected.

D. Decomposition property

1. *Remark.* D is a disk or half plane if and only if for each $z_1, z_2 \in D$ there exists a disk D' with $z_1, z_2 \in D' \subset D$.

2. *Decomposition property.* D is quasiconformally decomposable if there exists a constant $K \geq 1$ such that for each $z_1, z_2 \in D$ there exists a K-quasidisk D' with $z_1, z_2 \in D' \subset D$.

3. *Theorem* (Gehring-Osgood [26]). D is a quasidisk if and only if it is quasiconformally decomposable.

III. CONFORMAL INVARIANTS

A. Conformal invariants in a Jordan domain

1. *Configurations.* A Jordan domain D together with a finite number of points $z_1, \ldots, z_m \in D$ and $w_1, \ldots, w_n \in \partial D$ is said to be a *configuration* Σ. To determine if Σ is conformally equivalent to a second configuration Σ' consisting of a Jordan domain D' together with points $z'_1, \ldots, z'_m \in D'$ and $w'_1, \ldots, w'_n \in \partial D'$, it is sufficient to consider the case where $D = D' = B$. In this case Σ and Σ' are each determined by $2m + n$ real numbers. Since B has conformal self maps which carry an interior and a boundary point or three boundary points onto any other such pair or triple of points, the conformal type of Σ is determined by $N = 2m+n-3$ real numbers. The three cases when $N = 1$ together with a natural corresponding conformal invariant are as follows. See Ahlfors [4].

a. Two interior points z_1, z_2. The conformal invariant is the hyperbolic distance $h_D(z_1, z_2)$ between z_1 and z_2.
b. One interior point z_1 and two boundary points w_1, w_2. The conformal invariant is the harmonic measure $\omega(z_1, \alpha; D)$ where α is the boundary arc with endpoints w_1, w_2.
c. Four boundary points $w_1, \ldots, w_4$. The conformal invariant is the modulus of the quadrilateral $Q = D$ with vertices at $w_1, \ldots, w_4$.

We describe here how each of these invariants can be used to characterize the class of quasidisks.

B. Hyperbolic geometry

1. *Hyperbolic metric in* B. The *hyperbolic metric* in the unit disk B is defined by

$$\rho_B(z) = \frac{2}{1-|z|^2}$$

for $z \in B$. Next for each $z_1, z_2 \in B$ the *hyperbolic distance* between these points is given by

$$h_B(z_1, z_2) = \inf_\alpha \int_\alpha \rho_B(z)|dz|,$$

where the infimum is taken over all rectifiable curves α which join z_1 and z_2 in B. Then there exists a unique arc α such that

$$h_B(z_1, z_2) = \int_\alpha \rho_B(z)|dz|.$$

The arc α lies in the circular crosscut β of B which passes through z_1, z_2 and is orthogonal to ∂B. We call α the *hyperbolic segment* joining z_1 and z_2 and β the *hyperbolic line* which contains α. It is not difficult to show that

$$h_B(z_1, z_2) = \log\left(\frac{|1-\bar{z}_1 z_2| + |z_1 - z_2|}{|1-\bar{z}_1 z_2| - |z_1 - z_2|}\right).$$

2. *Hyperbolic metric in* D. We define the *hyperbolic metric* in D by

$$\rho_D(z) = \rho_B(g(z))|g'(z)|,$$

where g is any conformal mapping of D onto B. It follows from the Schwarz lemma that ρ_D is independent of the choice of g. Next we define the *hyperbolic distance* between $z_1, z_2 \in D$ by

$$h_D(z_1, z_2) = \inf_\alpha \int_\alpha \rho_D(z)|dz|,$$

where the infimum is taken over all rectifiable curves α which join z_1 and z_2 in D. Again there is a unique *hyperbolic segment* α in D for which

$$h_D(z_1, z_2) = \int_\alpha \rho_D(z)|dz|.$$

Then $h_D(z_1, z_2) = h_B(g(z_1), g(z_2))$ and g preserves the class of hyperbolic segments in D and B. Finally from the Schwarz lemma and the Koebe distortion theorem it follows that

$$\frac{1}{2\operatorname{dist}(z, \partial D)} \le \rho_D(z) \le \frac{2}{\operatorname{dist}(z, \partial D)}$$

for $z \in D$, where $\operatorname{dist}(z, \partial D)$ denotes the euclidean distance from z to ∂D.

3. *Remark.* We indicate here how a quasidisk D can be characterized in three different ways by comparing the euclidean and hyperbolic geometries in D.

a. The first bounds the hyperbolic distance between points in D in terms of the euclidean distance between the points and their distances from ∂D.

b. The second describes the euclidean length and position of hyperbolic segments in D.
c. The third states that up to a constant factor c, the endpoints of each hyperbolic segment in D minimize and maximize the euclidean distance between points of the segment and points not in D.

The first characterization makes use of the function

$$j_D(z_1, z_2) = \log \left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)} + 1 \right) \left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)} + 1 \right).$$

4. *Lemma.* j_D is a metric in D.

5. *Proof.* It suffices to show that

$$l = j_D(z_1, z_3) \leq j_D(z_1, z_2) + j_D(z_2, z_3) = r$$

for $z_1, z_2, z_3 \in D$. For convenience of notation let $d_i = \operatorname{dist}(z_i, \partial D)$, $i = 1, 2, 3$. Then from the euclidean triangle inequality and the inequalities

$$d_2 \leq |z_1 - z_2| + d_1, \quad d_2 \leq |z_2 - z_3| + d_3$$

we obtain

$$\begin{aligned} \exp(r) &= \frac{|z_1 - z_2| + d_1}{d_1} \frac{|z_1 - z_2| + d_2}{d_2} \frac{|z_2 - z_3| + d_2}{d_2} \frac{|z_2 - z_3| + d_3}{d_3} \\ &\geq \frac{|z_1 - z_2| + d_1}{d_1} \frac{|z_1 - z_3| + d_3}{|z_2 - z_3| + d_3} \frac{|z_1 - z_3| + d_1}{|z_1 - z_2| + d_1} \frac{|z_2 - z_3| + d_3}{d_3} \\ &= \frac{|z_1 - z_3| + d_1}{d_1} \frac{|z_1 - z_3| + d_3}{d_3} = \exp(l). \ \blacksquare \end{aligned}$$

6. *Remark.* The following two results indicate how j_D is related to the hyperbolic metric h_D.

7. *Lemma* (Gehring-Palka [27]). If $z_1, z_2 \in D$, then $j_D(z_1, z_2) \leq 4\, h_D(z_1, z_2)$.

8. *Proof.* Let α be the hyperbolic segment joining z_1 and z_2 in D. Then

$$2\, \rho_D(z) \geq \frac{1}{\operatorname{dist}(z, \partial D)} \geq \frac{1}{\operatorname{dist}(z_1, \partial D) + |z - z_1|} \quad \text{for } z \in \alpha$$

and thus

$$2\, h_D(z_1, z_2) \geq \int_\alpha \frac{d\,|z - z_1|}{\operatorname{dist}(z_1, \partial D) + |z - z_1|} \geq \log \left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)} + 1 \right).$$

Similarly

$$2\, h_D(z_1, z_2) \geq \log \left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)} + 1 \right)$$

and adding these two inequalities gives the desired conclusion. $\blacksquare$

9. *Lemma.* If D is a disk or half plane, then

$$h_D(z_1, z_2) \leq j_D(z_1, z_2) \quad \text{for } z_1, z_2 \in D.$$

10. *Proof.* Since each half plane can be written as the increasing union of disks, it is sufficient to consider the case where D is a disk. Next since h_D and j_D are both invariant

with respect to similarity mappings, we may further assume that $D = B$. In this case,

$$h_D(z_1, z_2) = \log\left(\frac{|1 - \bar{z}_1 z_2| + |z_1 - z_2|}{|1 - \bar{z}_1 z_2| - |z_1 - z_2|}\right) = \log\left(\frac{n}{d}\right).$$

Then

$$n = |1 - |z_2|^2 - z_2(\bar{z}_1 - \bar{z}_2)| + |z_1 - z_2| \le 1 - |z_2|^2 + (1 + |z_2|)|z_1 - z_2|$$

whence

$$n \le (1 - |z_2|^2)\left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)} + 1\right).$$

Similarly

$$n \le (1 - |z_1|^2)\left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)} + 1\right).$$

Next

$$nd = |1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$$

and thus

$$\frac{n}{d} = \frac{n^2}{nd} \le \left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)} + 1\right)\left(\frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)} + 1\right). \ \blacksquare$$

11. *Remark.* The conclusion of this lemma holds with equality whenever D is a disk and z_1, z_2 lie on a diameter and are separated by the center of D.

12. *Hyperbolic bound property.* D has this property if there exists a constant $c \ge 1$ such that

$$h_D(z_1, z_2) \le c\, j_D(z_1, z_2) \quad \text{for } z_1, z_2 \in D.$$

13. *Theorem* (Jones [35], Gehring-Hag [23]). D is a quasidisk if and only if it has the hyperbolic bound property.

14. *Conjecture.* D is a disk if and only if it is bounded and has the hyperbolic bound property with $c = 1$.

15. *Remark.* The above conjecture is true if, in addition, for each $z \in \partial D$ there exists a disk $D' = D'(z) \subset D$ with $z \in \partial D'$. See Gehring-Hag [23].

16. *Remark.* The second way of characterizing a quasidisk D in terms of euclidean properties of its hyperbolic segments is motivated by the following observation.

17. *Lemma.* If D is a disk or half plane, then for each hyperbolic segment α joining $z_1, z_2 \in D$ and each $z \in \alpha$,

a. $l(\alpha) \le \frac{\pi}{2}\, |z_1 - z_2|$,
b. $\min_{j=1,2} l(\alpha_j) \le \frac{\pi}{2} \operatorname{dist}(z, \partial D)$

where α_1, α_2 are the components of $\alpha \setminus \{z\}$ and l denotes euclidean length.

18. *Remark.* The constant $\frac{\pi}{2}$ cannot be replaced by a smaller number in either of these inequalities.

19. *Hyperbolic segment property.* D has this property if there exists a constant $c \ge 1$ such that for each hyperbolic segment α joining $z_1, z_2 \in D$ and each $z \in \alpha$,

a. $l(\alpha) \le c\, |z_1 - z_2|$,

b. $\min_{j=1,2} l(\alpha_j) \leq c\, \mathrm{dist}(z, \partial D)$,

where α_1, α_2 are the components of $\alpha \setminus \{z\}$.

20. *Theorem* (Gehring-Osgood [26]). D is a quasidisk if and only if it has the hyperbolic segment property.

21. *Uniform domain.* D is uniform if there exists a constant $c \geq 1$ such that each $z_1, z_2 \in D$ can be joined by an arc $\alpha \subset D$ where

a. $l(\alpha) \leq c\, |z_1 - z_2|$,
b. $\min_{j=1,2} l(\alpha_j) \leq c\, \mathrm{dist}(z, \partial D)$

for each $z \in \alpha$, where α_1, α_2 are the components of $\alpha \setminus \{z\}$.

22. *Theorem* (Martio-Sarvas [45]). D is a quasidisk if and only if it is a uniform domain.

23. *Remark.* The third way to characterize quasidisks in terms of the euclidean properties of their hyperbolic segments is suggested by the following result.

24. *Lemma* (Gehring-Hag [22]). If D is a disk or half plane and if α is a hyperbolic segment joining $z_1, z_2 \in D$, then

$$\frac{1}{\sqrt{2}} \min_{j=1,2} |z_j - w| \leq |z - w| \leq \sqrt{2} \max_{j=1,2} |z_j - w|$$

for each $z \in \alpha$ and $w \notin D$. The constant $\sqrt{2}$ cannot be replaced by a smaller number in either of these inequalities.

25. *Geodesic min-max property.* D has this property if there exists a constant $c \geq 1$ such that for each hyperbolic segment α joining $z_1, z_2 \in D$,

$$\frac{1}{c} \min_{j=1,2} |z_j - w| \leq |z - w| \leq c \max_{j=1,2} |z_j - w| \quad \text{for each } z \in \alpha \text{ and } w \notin D.$$

26. *Theorem* (Gehring-Hag [22]). D is a quasidisk if and only if it has the geodesic min-max property.

C. Harmonic measure

1. *Harmonic measure in B.* The *harmonic measure* of an open arc $\alpha \subset \partial B$ in B is defined by

$$\omega(z) = \omega(z, \alpha; B) = \int_\alpha P(z, \zeta) |d\zeta|$$

where $P(z, \zeta)$ is the *Poisson kernel*

$$P(z, \zeta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - \zeta|^2}.$$

Then $\omega(z)$ is the unique function which is bounded and harmonic in B with boundary values 1 in α and 0 in $\partial B \setminus \overline{\alpha}$. Note that

$$\omega(0) = \omega(0, \alpha; B) = \frac{l(\alpha)}{2\pi}.$$

2. *Harmonic measure in* D. Suppose that $D \subset \overline{R}^2$ is a Jordan domain and that g is a conformal mapping of D onto B. Then g has an extension which maps $\overline{D}$ homeomorphically onto $\overline{B}$. We define the *harmonic measure* of an open arc $\alpha \subset \partial D$ in D by

$$\omega(z) = \omega(z, \alpha; D) = \omega(g(z), g(\alpha); B).$$

Then $\omega(z)$ is independent of the choice of the mapping g.

3. *Remark.* The harmonic measure $\omega(z, \alpha; D)$ is a conformally invariant measure of a curvilinear angle subtended by the arc $\alpha \subset \partial D$ at the point $z \in D$. We give here two ways of describing quasidisks in terms of this measure. The first of these is based on the following observation.

4. *Lemma.* If D is a disk or half plane, then there exist points $z_0 \in D$ and $z_0^* \in D^*$ such that if α, β are adjacent open arcs in ∂D with

$$\omega(z_0, \alpha; D) = \omega(z_0, \beta; D),$$

then

$$\omega(z_0^*, \alpha; D^*) = \omega(z_0^*, \beta; D^*).$$

5. *Proof.* Suppose that D is a disk. By performing a preliminary similarity mapping we may assume that $D = B$. Then $g(z) = 1/z$ maps B^* conformally onto B and

$$\omega(\infty, \alpha; B^*) = \omega(0, g(\alpha); B) = \frac{l(g(\alpha))}{2\pi} = \frac{l(\alpha)}{2\pi} = \omega(0, \alpha; B)$$

for each open arc $\alpha \subset \partial B^* = \partial B$. This yields the desired result with $z_0 = 0$ and $z_0^* = \infty$. The case where D is a half plane follows similarly. ∎

6. *Harmonic symmetry property.* D has this property if D is a Jordan domain and if there exist points $z_0 \in D$, $z_0^* \in D^*$ and a constant $c \geq 1$ such that if α, β are adjacent open arcs in ∂D with

$$\omega(z_0, \alpha; D) = \omega(z_0, \beta; D),$$

then

$$\omega(z_0^*, \alpha; D^*) \leq c\, \omega(z_0^*, \beta; D^*).$$

7. *Theorem* (Krzyż [38]). D is a quasidisk if and only if it has the harmonic symmetry property.

8. *Remark.* D is a disk or half plane if and only if it has the harmonic symmetry property with $c = 1$.

9. *Complement of an arc.* Suppose that β is a closed arc with endpoints z_1 and z_2. Then $G = \overline{R}^2 \setminus \beta$ is a simply connected domain and there exists a conformal mapping g which maps G onto the right half plane H. Next since G is locally connected at z_1 and z_2, g has a continuous injective extension in $G \cup \{z_1, z_2\}$ [64] and we may choose g so that $g(z_1) = 0$ and $g(z_2) = \infty$. Let

$$b(z, \beta) = \max_{j=1,2}\, \omega(g(z), \alpha_j; H) = \frac{2}{\pi}\, |\arg(g(z))|,$$

where α_1 and α_2 are the positive and negative halves of the imaginary axis. Then $b(z, \beta) \to 1$ if and only if $z \to \beta \setminus \{z_1, z_2\}$. Hence the function $b(z, \beta)$ is a conformally invariant measure of the position of the point $z \in G$ with respect to the interior of β which attains its minimum 0 on the preimage of the positive real axis under g. The following observation suggests how this function may be used to characterize quasidisks.

10. *Remark.* If β is a closed subarc of a Jordan curve $C \subset \overline{R}^2$, then the function $b(z, \beta)$ measures how much the open arc $\gamma = C \setminus \beta$ *bends* towards β when measured with respect to harmonic measure in $\overline{R}^2 \setminus \beta$. In particular, if C is a circle or line, then for each closed arc $\beta \subset C$,

$$b(z, \beta) = 0 \quad \text{for } z \in \gamma = C \setminus \beta.$$

11. *Harmonic bending property.* D has this property if D is a Jordan domain and there exists a constant $0 \le c < 1$ such that for each closed arc $\beta \subset \partial D$,

$$b(z, \beta) \le c \quad \text{for } z \in \gamma = \partial D \setminus \beta.$$

12. *Theorem* (Fernández-Hamilton-Heinonen [14]). D is a quasidisk if and only if it has the harmonic bending property.

13. *Remark* (Fernández-Hamilton-Heinonen [14]). D is a disk or half plane if and only if it has the harmonic bending property with $c = 0$.

D. Quadrilaterals

1. *Modulus of a quadrilateral.* A *quadrilateral* $Q = G(z_1, z_2, z_3, z_4)$ consists of a Jordan domain $G \subset \overline{R}^2$ together with four positively oriented vertices $z_1, z_2, z_3, z_4 \in \partial G$ which divide ∂G into four sides. Then Q can be mapped conformally onto a rectangle $R = R(0, m, m+i, i)$ so that the vertices and sides of Q and R correspond and the modulus of Q is given by $\mathrm{mod}(Q) = m$. If G^* is the exterior of G, then $Q^* = G^*(z_4, z_3, z_2, z_1)$ is also a quadrilateral, the *conjugate* of the quadrilateral Q.

2. *Remark.* If D is a disk or half plane and if Q and Q^* are conjugate quadrilaterals in D and D^* with $\mathrm{mod}(Q) = 1$, then $\mathrm{mod}(Q^*) = 1$.

3. *Conjugate quadrilateral inequality.* D has this property if D is Jordan and there exists a constant $c \ge 1$ such that if Q and Q^* are conjugate quadrilaterals in D and D^* with $\mathrm{mod}(Q) = 1$, then $\mathrm{mod}(Q^*) \le c$.

4. *Theorem* (Lehto-Virtanen [43], Pfluger [53]). D is a quasidisk if and only if it satisfies the conjugate quadrilateral inequality.

5. *Lemma.* D is a disk or half plane if and only if it satisfies the conjugate quadrilateral inequality with $c = 1$.

E. Extremal distance

1. *Remark.* Suppose that Γ is a family of curves. We observed earlier that $\mathrm{mod}(\Gamma)$ is a conformally invariant measure of Γ which is large if the curves in Γ are short and plentiful and small if the curves are long or scarce. We now use this quantity to compare distances between two continua $E, F \subset D$, as measured by the moduli of the families of curves

which join them in D and in $\overline{R}^2$, respectively. This leads to another characterization for quasidisks.

2. *Extremal distance.* Given continua $E, F \subset D$ we let Γ_D and Γ denote the families of all curves which join E and F in D and $\overline{R}^2$, respectively. We then call

$$\delta_D(E,F) = \mathrm{mod}(\Gamma_D) \quad \text{and} \quad \delta(E,F) = \mathrm{mod}(\Gamma)$$

the *extremal distances* between E and F in D and $\overline{R}^2$, respectively. Since $\Gamma_D \subset \Gamma$, it follows that $\delta_D(E,F) \le \delta(E,F)$.

3. *Extremal distance property.* D has this property if there exists a constant $c \ge 2$ such that

$$\delta(E,F) \le c\, \delta_D(E,F) \quad \text{for all continua } E, F \subset D.$$

The existence of such a constant c implies that D is not *bent* around part of its exterior D^* so that the *euclidean* distance between E and F in D is substantially larger than the distance in $\overline{R}^2$. This property cannot hold for any domain D with $c < 2$. See Yang [66].

4. *Theorem* (Gehring-Martio [25]). D is a quasidisk if and only if it has the extremal distance property.

5. *Theorem* (Yang [66]). D is a disk or half plane if and only if it has the extremal distance property with $c = 2$.

6. *Remark.* The following is an attractive application of the above characterization for quasidisks.

7. *Theorem* (Fernández-Heinonen-Martio [15]). Suppose that $f : G \to G'$ is a conformal mapping and that G' is a quasidisk. If $D \subset G$ is a quasidisk, then so is $D' = f(D)$.

8. *Proof.* Choose continua $E', F' \subset D'$, let $E = f^{-1}(E'), F = f^{-1}(F')$ and let c, c' be the extremal distance constants for D, G'. Then by the above theorem and the conformal invariance of extremal distance,

$$\delta(E',F') \le c'\, \delta_{G'}(E',F') = c'\, \delta_G(E,F) \le c'\, \delta(E,F) \le cc'\, \delta_D(E,F) = cc'\, \delta_{D'}(E',F').$$

Thus D' has the extremal distance property with constant cc'. ∎

9. *Subinvariance principle.* Suppose that $f : G \to G'$ is conformal where G' is a disk. According to this principle, if $E \subset G$ is *nice*, then so is $E' = f(E)$. Here are two examples.

a. *Invariance of quasidisks.* If E is a quasidisk, then so is E'.
b. *Level set problem.* If $E = G \cap L$ where L is a line, then $l(E') \le 2\, l(\partial G')$.

For this second example, see Hayman-Wu [32] and Øyma [50], [51].

IV. INJECTIVITY CRITERIA

A. Injectivity of analytic functions

1. *Remark.* Suppose that f is analytic in D. Then f is locally injective in D if and only if $f' \ne 0$ in D. We consider here two different criteria for the global injectivity of f involving

a. the Schwarzian derivative S_f,
b. the pre-Schwarzian or logarithmic derivative f''/f',

for the case where D is a disk of half plane. We then show how they can be used to characterize quasidisks.

2. *Schwarzian derivative.* The *Schwarzian derivative* of a function f, analytic and locally injective in D, is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Then $S_f = 0$ in D if and only if f is a Möbius transformation in which case f is injective in D. The following result shows that the size of the Schwarzian relative to the hyperbolic metric is related to the global injectivity of an analytic function.

3. *Theorem* (Lehto [41],[42]). If f is analytic and injective in D, then

$$|S_f| \leq 3\,\rho_D^2 \quad \text{in } D.$$

The constant 3 is sharp.

4. *Schwarzian radius of injectivity* $\sigma(D)$. For each D we let $\sigma(D)$ denote the supremum of the constants $a \geq 0$ such that f is injective in D whenever f is analytic with $f' \neq 0$ and

$$|S_f| \leq a\,\rho_D^2 \quad \text{in } D.$$

Then $\sigma(D) \leq \frac{1}{2}$ for all domains D. See Lehtinen [40].

5. *Theorem* (Ahlfors [3], Gehring [17]). D is a quasidisk if and only if $\sigma(D) > 0$.

6. *Theorem* (Nehari [48], Lehtinen [40]). D is a disk or half plane if and only if $\sigma(D) = \frac{1}{2}$.

7. *Remark.* Thus D is a quasidisk if and only if there exists a constant $a > 0$ such that f analytic in D is injective whenever

$$|S_f|\,\rho_D^{-2} \leq a \quad \text{in } D.$$

We consider next analogues of the above results for the pre-Schwarzian derivative.

8. *Pre-Schwarzian derivative* (Bers). For f analytic with $f' \neq 0$ we call

$$T_f = \frac{f''}{f'}$$

the *pre-Schwarzian* or *logarithmic* derivative of f. Then $T_f = 0$ in D if and only if f is a similarity mapping in which case it is injective.

9. *Theorem* (Osgood [49]). If f is analytic and injective in D, then

$$|T_f| \leq 4\,\rho_D \quad \text{in } D.$$

The constant 4 is sharp.

10. *Pre-Schwarzian radius of injectivity* $\tau(D)$. We let $\tau(D)$ denote the supremum of the constants $b \geq 0$ such that f is injective in D whenever f is analytic with $f' \neq 0$ and

$$|T_f| \leq b\,\rho_D \quad \text{in } D.$$

Then $\tau(D) \le \frac{1}{2}$ for all domains D. See Stowe [60].

11. *Theorem* (Astala-Gehring [6]). D is a quasidisk if and only if $\tau(D) > 0$.

12. *Theorem* (Becker-Pommerenke [7], [8]). If D is a disk or half plane, then $\tau(D) = \frac{1}{2}$.

13. *Remark* (Stowe [60]). The converse of Theorem 12 does not hold; there exists a domain D with $\tau(D) = \frac{1}{2}$ which is not a disk or half plane.

14. *Values of $\sigma(D)$ and $\tau(D)$.* Though the constants $\sigma(D)$ and $\tau(D)$ reflect the geometry of D, little is known about their values except for the following special cases.

a. If D is an angular sector of angle $k\pi$ where $0 < k \le 2$, then $\sigma(D) = \frac{1}{2}\min(k^2, 2k - k^2)$.
b. If D is a regular n-sided polygon, then $\sigma(D) = \frac{1}{2}\left(\frac{n-2}{n}\right)^2$.
c. If D is a rectangle with side ratio $c \in [1, 1.523\ldots]$, then $\sigma(D) = \frac{1}{8}$.

See Calvis [12], Lehtinen [40], Lehto [41] and Miller-Van Wieren [47].

B. Injectivity of locally quasiconformal mappings

1. *BMO norm.* Suppose that u is locally integrable in D. Then the *BMO norm* of u is given by

$$\|u\|_{BMO(D)} = \sup_{B_0} \frac{1}{m(B_0)} \int_{B_0} |u - u_{B_0}| dm,$$

where the supremum is taken over all disks B_0 with $\overline{B}_0 \subset D$ and

$$u_{B_0} = \frac{1}{m(B_0)} \int_{B_0} u \, dm.$$

2. *Remark.* The following observation suggests how this norm is related to the hyperbolic metric.

3. *Lemma.* If B_0 is a disk in D with center z_0, then

$$\frac{1}{m(B_0)} \int_{B_0} h_D(z, z_0) dm \le 2.$$

4. *Proof.* The left hand side of the above inequality is an integral average of a function which is invariant with respect to conformal mappings. Hence this term is invariant with respect to similarity mappings and we may assume that B_0 is the unit disk B. Then

$$\int_B h_D(z, 0) dm \le \int_B h_B(z, 0) dm = \int_B \log\left(\frac{1 + |z|}{1 - |z|}\right) dm = 2\, m(B). \ \blacksquare$$

5. *Lemma.* If u is harmonic in D, then

$$\frac{1}{2}\, \|u\|_{BMO(D)} \le \sup_D |\text{grad } u|\, \rho_D^{-1} \le 6\, \|u\|_{BMO(D)}.$$

6. *Proof.* It suffices to show for each disk $B_0 = B(z_0, d)$ with $\overline{B}_0 \subset D$ that

$$|\text{grad } u(z_0)|\, 2d \le \frac{6}{m(B_0)} \int_{B_0} |u(z) - u(z_0)| dm,$$

whence

$$|\text{grad } u(z_0)|\, \rho_D(z_0)^{-1} \le |\text{grad } u(z_0)|\, 2\text{dist}(z_0, \partial D) \le 6\, \|u\|_{BMO(D)}$$

by III.B.2, and that

$$\frac{1}{m(B_0)}\int_{B_0}|u(z)-u(z_0)|dm \le 2\sup_D |\text{grad } u|\ \rho_D^{-1}.$$

If $0<r<d$, then we obtain

$$|\text{grad } u(z_0)|\ \pi r^2 \le \int_0^{2\pi} |u(re^{i\theta}+z_0)-u(z_0)|r d\theta$$

from differentiating the Poisson integral and

$$\begin{aligned}|\text{grad } u(z_0)|\ 2d &\le \frac{6}{m(B_0)}\int_0^d |\text{grad } u(z_0)|\pi r^2 dr\\ &\le \frac{6}{m(B_0)}\int_{B_0}|u(z)-u(z_0)|dm \le 6\ \|u\|_{BMO(D)}\end{aligned}$$

since $u(z_0)=u_{B_0}$.

Next if α is a hyperbolic geodesic joining z_0 to z in D, then

$$|u(z)-u(z_0)| \le c\int_\alpha \rho_D|dz| = c\ h_D(z,z_0),$$

where $c=\sup_D |\text{grad } u|\ \rho_D^{-1}$, and

$$\frac{1}{m(B_0)}\int_{B_0}|u(z)-u(z_0)|dm \le \frac{c}{m(B_0)}\int_{B_0} h_D(z,z_0)dm \le 2c$$

by Lemma IV.B.3. ∎

7. *Corollary* (Astala-Gehring [6]). If f is analytic with $f'\neq 0$ in D, then

$$\frac{1}{4}\ \|\log(J_f)\|_{BMO(D)} \le \sup_D |T_f|\ \rho_D^{-1} \le 3\ \|\log(J_f)\|_{BMO(D)}.$$

8. *Remark.* The above Corollary shows that the BMO-norm of $\log(J_f)$ is a natural alternative for the pre-Schwarzian derivative T_f when considering injectivity results for locally conformal mappings. Moreover the following quasiconformal analogue of the Theorems of Lehto and Osgood above suggests that this norm offers a way to extend results on the injectivity of analytic functions to the class of locally quasiconformal mappings.

9. *Theorem* (Reimann [55]). If f is K-quasiconformal in D with $f(D)\subset R^2$, then

$$\|\log(J_f)\|_{BMO(D)} \le m$$

where $m=m(K)<\infty$.

10. *Locally quasiconformal mappings. f* is *locally K-quasiconformal* in D if each point of D has a neighborhood in which f is K-quasiconformal.

11. *Theorem* (Astala-Gehring [6]). Suppose that D is a disk or half plane. Then for each $1\le K<2$ there exists a constant $c=c(K)>0$ with the following property. If f is locally K-quasiconformal in D with $f(D)\subset R^2$ and

$$\|\log(J_f)\|_{BMO(D)} \le c,$$

then f is injective in D. The constant 2 is sharp, i.e., no such constant c exists if $K\ge 2$.

12. *Quasiconformal injectivity property.* D has this property if for some $K > 1$ there exists a constant $c > 0$ such that f is injective whenever f is locally K-quasiconformal in D with $f(D) \subset R^2$ and

$$\| \log(J_f) \|_{BMO(D)} \leq c.$$

13. *Theorem* (Astala-Gehring [6]). D is a quasidisk if and only if it has the quasiconformal injectivity property.

14. *Remark.* If f is conformal in D with $f(D) \subset R^2$, then we see from Lemma 5 that the function

$$u = \log(J_f) = 2\ \mathrm{Re}(\log(f'))$$

is harmonic and has finite BMO-norm in D. When do these properties characterize the Jacobian of a conformal mapping? The answer yields still another description of a quasidisk.

15. *Theorem* (Astala-Gehring [6]). D is a quasidisk if and only if there exists a constant $c > 0$ such that each function u which is harmonic in D with $\|u\|_{BMO(D)} \leq c$ can be written in the form $u = \log(J_f)$ where f is conformal in D with $f(D) \subset R^2$.

C. Injectivity of local quasi-isometries

1. *Local quasi-isometries.* We say that f is a *local L-quasi-isometry* in $E \subset R^2$ if each point of E has a neighborhood U such that

$$\frac{1}{L}|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq L|z_1 - z_2| \quad \text{for } z_1, z_2 \in E \cap U.$$

2. *Remark.* Suppose that f is a local L-quasi-isometry in D. The following results show that whether or not f is injective in D depends on L and D.

3. *Lemma.* If f is a local 1-quasi-isometric in D, then f is injective in D.

4. *Example.* For each $L > 1$

$$f(z) = \frac{|z|}{L} \exp(iL^2 \arg(z)), \quad |\arg(z)| < \pi$$

is a local L-quasi-isometry in $D = R^2 \setminus \{z = -x : 0 \leq x < \infty\}$ which is not injective.

5. *Theorem* (John [34]). If D is a disk or half plane and if f is a local L-quasi-isometry in D with $L \leq 2^{1/4}$, then f is injective in D.

6. *Proof.* Suppose otherwise. Because f is a local homeomorphism, we can choose a disk U with $\overline{U} \subset D$ and points $z_1, z_2 \in \partial U$ such that f is injective in U with $f(z_1) = f(z_2)$.

Let α be the circular arc in $\overline{U}$ which is orthogonal to ∂U at z_1 and z_2 and let E denote the component of $U \setminus \alpha$ whose image $E' = f(E)$ is enclosed by $\alpha' = f(\alpha)$. Then

$$l(\alpha') \leq L\, l(\alpha)$$

because f is a local L-quasi-isometry. Next the fact that f is injective in U implies that f^{-1} is a local L-quasi-isometry in $U' = f(U)$. Hence

$$m(E) \leq L^2\, m(E').$$

Finally by the isoperimetric inequality,

$$m(E') \le \frac{l(\alpha')^2}{4\pi}$$

and from elementary geometry we obtain

$$\frac{l(\alpha)^2}{2\pi} < m(E) \le L^2\, m(E') \le L^2\, \frac{l(\alpha')^2}{4\pi} \le L^4\, \frac{l(\alpha)^2}{4\pi}$$

whence $L^4 > 2$, a contradiction. ■

7. *Remark.* The constant $2^{1/4}$ in Theorem 3 is not sharp. It is conjectured that $2^{1/2}$ is the right bound, a constant which would be sharp. The problem of determining the sharp bound has been open for the past 35 years. See, for example, Gevirtz [29].

8. *Rigid domain.* We let $L(D)$ denote the supremum of the numbers $d \ge 1$ such that f is injective whenever f is a local L-quasi-isometry in D with $L \le d$. We say that D is *rigid* if $L(D) > 1$.

9. *Theorem* (Gehring [18], Martio-Sarvas [45]). D is a quasidisk if and only if it is rigid.

10. *Theorem* (Gehring [18]). If D is a quasidisk and if f is a local L-quasi-isometry in D with $L < L(D)$, then f is injective in D and has an M-quasi-isometric extension to $\overline{R}^2$ where M depends only on L and $L(D)$.

11. *Sketch of proof.* Let $D' = f(D)$ and suppose that g is a local L'-quasi-isometry in D' with

$$1 \le L' < \frac{L(D)}{L}.$$

Then $h = gf$ is a local LL'-quasi-isometry in D with $LL' < L(D)$. Thus h is injective in D, g is injective in D',

$$L(D') \ge \frac{L(D)}{L} > 1$$

and D' is a quasidisk. Next the fact that D and D' are uniform allows one to show that f is an L''-quasi-isometry in D and hence has a homeomorphic extension $f^* : \overline{D} \to \overline{D'}$.

If D is unbounded, then so is D' and the quasidisk reflection property implies that f^* has an M-quasi-isometric extension to $\overline{R}^2$. If D is bounded, then we can choose an auxiliary Möbius transformation ϕ so that $\phi(D)$ and $\phi(D')$ are unbounded and complete the proof as above. ■

12. *Physical interpretation.* Think of D as an elastic plane body, let f denote the deformation of D under a force field and let

$$L_f(z) = \limsup_{h \to 0}\ \max\left(\frac{|f(z+h) - f(z)|}{|h|},\ \frac{|h|}{|f(z+h) - f(z)|}\right)$$

denote the *strain* in D at the point z caused by the force field. Then f is a local quasi-isometry if and only if $L_f(z)$ is bounded and $L(D)$ is the supremum of the allowable strains before D collapses. The above theorem says that if

$$\sup_{z \in D} L(z) < L(D),$$

then the shape of the deformed body $f(D)$ is roughly the same as that of the original body D.

13. *Remark.* It would be interesting to know what sort of analogue for the above result holds in higher dimensions.

V. EXTENSION AND CONTINUITY

A. Extension of functions with bounded mean oscillation

1. *The class BMO.* Suppose that u is locally integrable in a domain $G \subset R^2$. We say that u has *bounded mean oscillation* or is in $BMO(G)$ if

$$\|u\|_{BMO(G)} = \sup_{B_0} \frac{1}{m(B_0)} \int_{B_0} |u - u_{B_0}| dm < \infty,$$

where as in IV.B.1 the supremum is taken over all disks B_0 with $\overline{B}_0 \subset G$ and

$$u_{B_0} = \frac{1}{m(B_0)} \int_{B_0} u \, dm.$$

2. *Example.* Functions with bounded mean oscillation arise very naturally in many parts of analysis. For example if $z_1 \in D$, then the hyperbolic distance $u(z) = h_D(z, z_1)$ is in $BMO(D)$ with $\|u\|_{BMO(D)} \leq 4$.

3. *Proof.* If B_0 is any disk with center z_0 and $\overline{B}_0 \subset D$, then

$$\begin{aligned} |u_{B_0} - u(z_0)| &\leq \frac{1}{m(B_0)} \int_{B_0} |h_D(z, z_1) - h_D(z_0, z_1)| \, dm \\ &\leq \frac{1}{m(B_0)} \int_{B_0} h_D(z, z_0) \, dm \leq 2 \end{aligned}$$

whence

$$\frac{1}{m(B_0)} \int_{B_0} |u(z) - u_{B_0}| \, dm \leq \frac{1}{m(B_0)} \int_{B_0} (|u(z) - u(z_0)| + 2) \, dm \leq 4$$

by Lemma IV.B.3. ■

4. *Remark.* If $v \in BMO(R^2)$ and if u is the restriction of v to D, then $u \in BMO(D)$ with

$$\|u\|_{BMO(D)} \leq \|v\|_{BMO(R^2)}.$$

The converse is not true. For example, if $u(z) = h_D(z, 1)$ where

$$D = \{z = x + iy : \ 0 < x < \infty, \ |y| < 1\},$$

then u is in $BMO(D)$ but u has no BMO-extension to R^2.

5. *Theorem* (Reimann-Rychener [56]). If D is a disk or half plane and if $u \in BMO(D)$, then u has an extension $v \in BMO(R^2)$ with

$$\|v\|_{BMO(R^2)} \leq c \, \|u\|_{BMO(D)}$$

where $c \geq 1$ is an absolute constant.

6. *BMO extension domain.* D is such a domain if there exists a constant $c \geq 1$ such that each $u \in BMO(D)$ has an extension $v \in BMO(R^2)$ with

$$\|v\|_{BMO(R^2)} \leq c\, \|u\|_{BMO(D)}.$$

7. *Theorem* (Jones [35]). D is a quasidisk if and only if it is a BMO-extension domain.

B. Extension of functions with bounded Dirichlet integral

1. *The class* L^2_1. Suppose that u is locally integrable in a domain $G \subset R^2$. We say that u has a bounded Dirichlet integral or is in $L^2_1(G)$ if u is ACL in G with

$$E_G(u) = \int_G |\text{grad } u|^2 \, dm < \infty.$$

2. *Remark.* If D is a disk or half space and if $u \in L^2_1(D)$, then u has an extension $v \in L^2_1(R^2)$ with

$$E_{R^2}(v) \leq 2\, E_D(u).$$

3. L^2_1 *extension domain.* D is such a domain if there exists a constant $c \geq 1$ such that each function $u \in L^2_1(D)$ has an extension $v \in L^2_1(R^2)$ with

$$E_{R^2}(v) \leq c\, E_D(u).$$

4. *Theorem* (Gol'dstein-Vodop'janov [30], Jones [36]). D is a quasidisk if and only if it is an L^2_1-extension domain.

C. Extension of quasiconformal mappings

1. *Remark.* If D is a disk or half plane, then each K-quasiconformal self mapping f of D can be extended by reflection to yield a K-quasiconformal self mapping g of $\overline{R}^2$.

2. *Quasiconformal extension domain.* D is such a domain if there exists a constant $c \geq 1$ such that each K-quasiconformal self mapping f of D has a cK-quasiconformal extension g to $\overline{R}^2$.

3. *Lemma.* D is a disk or half plane if and only if it is a quasiconformal extension domain with $c = 1$.

4. *Theorem* (Rickman [57]). D is a quasidisk if and only if it is a quasiconformal extension domain.

D. Extension of quasi-isometries

1. *Schoenflies theorem.* Suppose that D is a disk, that $C = \partial D$ and that $f : C \to C'$ is a homeomorphism. Then C' is the boundary of a Jordan domain D'. A well known theorem of Schoenflies asserts that f has a homeomorphic extension g to D which maps D onto D'. What is the analogue of this result for quasi-isometries?

2. *Quasi-isometric extension property.* We say D has this property if D is a Jordan domain with $C = \partial D$ and if there exists a constant $c \geq 1$ such that each L-quasi-isometry $f : C \to C'$ has a cL-quasi-isometric extension $g : D \to D'$.

3. *Example.* For each $L > 1$ there exists a bounded Jordan domain D with $C = \partial D$ and an L-quasi-isometry f on C which has no M-quasi-isometric extension to D for any constant $M < \infty$.

4. *Proof.* Fix $L > 1$ and let

$$D = \{z = x + iy : |x| < 1,\ a(|x|^{1/2} - 1) < y < 1\}$$

where $a = (L-1)/2L$. Then D has an outward directed cusp with tip at $z = -ia$ and

$$f(x + iy) = x + i|y|$$

is L-quasi-isometric on $C = \partial D$. Next if g is an M-quasi-isometric extension of f to D and if

$$\alpha = \alpha(t) = \{z = x + iy \in D :\ y = t\}$$

for $-a < t < 0$, then $g(\alpha) \subset D$ and

$$M \geq \frac{l(g(\alpha))}{l(\alpha)} \geq \frac{a^2}{a+t} \to \infty$$

as $t \to -a$, a contradiction. ■

5. *Theorem* (Gehring [20], Tukia [63]). If D is bounded, then D is a quasidisk if and only if it has the quasi-isometric extension property.

6. *Remark.* The necessity also holds in the above theorem when D is unbounded. However the sufficiency fails in this case. For example, the half strip

$$D = \{z = x + iy :\ 0 < x < \infty,\ |y| < 1\}$$

has the quasi-isometric extension property but D is not a quasidisk since it does not satisfy the two point inequality in II.B.2.

E. Continuity of Bloch functions

1. *Bloch functions.* A function f analytic in D is said be a *Bloch* function or is in $B(D)$ if

$$\|f\|_{B(D)} = \sup_D |f'(z)|\ \rho_D(z)^{-1} < \infty.$$

2. *Example.* Bloch functions play an important role in complex analysis. For example, if g is conformal in D, then $f = \log(g')$ is in $B(D)$ with $\|f\|_{B(D)} \leq 4$. See also, for example, Bonk [10] and Liu-Minda [44].

3. *Remark.* If f is in $B(D)$ where D is a disk or half plane, then

$$|f(z_1) - f(z_2)| \leq \|f\|_{B(D)}\ j_D(z_1, z_2) \quad \text{for } z_1, z_2 \in D,$$

where j_D is the metric defined in III.B.3. This continuity property for Bloch functions holds precisely when D is a quasidisk.

4. *Theorem* (Langmeyer [39]). D is a quasidisk if and only if there exists a constant c such that

$$|f(z_1) - f(z_2)| \leq c\ \|f\|_{B(D)}\ j_D(z_1, z_2) \quad \text{for } f \text{ in } B(D) \text{ and } z_1, z_2 \in D.$$

VI. MISCELLANEOUS PROPERTIES

A. Homogeneity properties

1. *Homogeneous sets.* A set $E \subset \overline{R}^2$ is homogeneous with respect to a family F of mappings if for each $z_1, z_2 \in E$ there exists an $f \in F$ such that

$$f(E) = E, \quad f(z_1) = z_2.$$

2. *Remark.* If D is a disk or half plane, then D and ∂D are both homogeneous with respect to the family of Möbius transformations of $\overline{R}^2$.

3. *Class $QC(K)$.* This is the family of all K-quasiconformal self mappings of $\overline{R}^2$. Hence $QC(1)$ is simply the family of Möbius transformations in $\overline{R}^2$.

4. *Theorem* (Brechner-Erkama [13], [11]). D is a quasidisk if and only if ∂D is homogeneous with respect to the family $QC(K)$ for some fixed K.

5. *Theorem* (Sarvas [58]). D is a quasidisk if and only if D is a Jordan domain which is homogeneous with respect to the family $QC(K)$ for some fixed K.

6. *Example* (Palka [52]). There exists a domain D which is not a quasidisk but which is homogeneous with respect to the family $QC(K)$ for a fixed $K > 1$; hence the hypothesis that D be a Jordan domain is necessary in the above theorem. On the other hand, the following result shows that this hypothesis is not necessary when $K = 1$.

7. *Theorem* (Kimel'fel'd [37]). D is a disk or half plane if and only if it is homogeneous with respect to the family $QC(1)$.

B. Limit set of a quasiconformal group

1. *Limit set of a group of homeomorphisms.* Suppose that G is a group of self homeomorphisms g of $\overline{R}^2$, i.e. a family which is closed under composition and taking inverses. We say that w_0 is in the *limit set* $L(G)$ of G if there exist distinct $g_j \in G$ and a point $z_0 \in \overline{R}^2$ such that $w_0 = \lim_{j\to\infty} g_j(z_0)$.

2. *Remark.* If D is a disk or half plane, then there exists a finitely generated group G of Möbius transformations or mappings in QC(1) with ∂D as its limit set. For if H is the upper half plane, then the modular group

$$G = \langle g, h\rangle, \quad \text{where} \quad g(z) = z + 1,\ h(z) = -1/z,$$

has $L(G) = \partial H$. Next if f maps H conformally onto D, then the group

$$fGf^{-1} = \langle fgf^{-1}, fhf^{-1}\rangle$$

has $\partial D = f(\partial H)$ as its limit set.

3. *Theorem* (Maskit [46], Sullivan [61], Tukia [62]). D is a quasidisk if and only if ∂D is a Jordan curve which is the limit set of a finitely generated group of mappings in $QC(K)$ for some fixed K.

C. Comparable Dirichlet integrals

1. *Remark.* If D is a disk or half plane and if u and u^* are harmonic in D and D^*, respectively, with continuous and equal boundary values, then

$$\int_D |\text{grad } u|^2 dm = \int_{D^*} |\text{grad } u^*|^2 dm.$$

2. *Comparable Dirichlet integral property.* D and D^* have this property if they are Jordan domains and there exists a constant $c \geq 1$ such that

$$\frac{1}{c} \int_D |\text{grad } u|^2 dm \leq \int_{D^*} |\text{grad } u^*|^2 dm \leq c \int_D |\text{grad } u|^2 dm$$

for each pair of functions u and u^* which are harmonic in D and D^*, respectively, with continuous and equal boundary values.

3. *Theorem* (Ahlfors [2], Springer [59]). D is a quasidisk if and only if D and D^* have the comparable Dirichlet integral property.

D. Quasiconformal equivalence of $\overline{R}^3 \setminus \overline{D}$ and B^3

1. *Linear dilatation of a homeomorphism in $\overline{R}^n$.* Suppose that G and G' are domains in $\overline{R}^n$ and that $f : G \to G'$ is a homeomorphism. For $x \in G \setminus \{\infty, f^{-1}(\infty)\}$ and $0 < r < \text{dist}(x, \partial G)$ we let

$$l_f(x,r) = \min_{|x-y|=r} |f(x) - f(y)|, \quad L_f(x,r) = \max_{|x-y|=r} |f(x) - f(y)|$$

and call

$$H_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)}$$

the linear dilatation of f at x.

2. *Quasiconformal mappings in $\overline{R}^n$.* A sense preserving homeomorphism $f : G \to G'$ is a K-quasiconformal mapping where $1 \leq K < \infty$ if

a. H_f is bounded in $G \setminus \{\infty, f^{-1}(\infty)\}$,
b. $H_f \leq K$ a. e. in G.

3. *Remark.* If D is a disk or half plane in R^2, then $G = \overline{R}^3 \setminus \overline{D}$ can be mapped 2-quasiconformally onto the unit ball B^3 in R^3.

4. *Proof.* Since D is a disk or half plane, we can choose a Möbius transformation $h : \overline{R}^2 \to \overline{R}^2$ which maps D onto the right half plane

$$D' = \{(x_1, x_2) \in R^2 : 0 < x_1 < \infty, \ |x_2| < \infty\}.$$

Then h has an extension h^* which is a Möbius transformation in $\overline{R}^3$ and which maps $G = \overline{R}^3 \setminus \overline{D}$ onto $G' = \overline{R}^3 \setminus \overline{D}'$.

Next there exists a 2-quasiconformal mapping g^* which unfolds G' around the x_2-axis onto the upper half space

$$H^3 = \{(x_1, x_2, x_3) \in R^3 : |x_1| < \infty, \ |x_2| < \infty, \ 0 < x_3 < \infty\}.$$

Finally there exists a second Möbius transformation f^* in $\overline{R}^3$ which carries H^3 onto B^3 and $f^* g^* h^*$ maps G 2-quasiconformally onto B^3. ■

5. *Theorem* (Gehring [16]). D is a quasidisk if and only if $G = \overline{R}^3 \setminus \overline{D}$ can be mapped quasiconformally onto B^3.

VII. TABLE OF IMPLICATIONS

A. Summary. We conclude with a sketch for establishing a few of the characterizations which we have discussed earlier together with very brief comments on what is involved in establishing each step.

B. First circle of implications

1. *A quasidisk D has the hyperbolic segment property.* This proof stems from an argument due to Jerison and Kenig [33]. Suppose that α is a hyperbolic segment in a K-quasidisk D and that f is a conformal mapping of the upper half plane H normalized so that the imaginary axis is mapped onto the hyperbolic line which contains α. Then f has an extension g which is a K^2-quasiconformal in $\overline{R}^2$. Standard distortion properties for conformal and quasiconformal mappings plus integration then allow one to show that α has the desired properties.

2. *D is uniform if it has the hyperbolic segment property.* This is an immediate consequence of the definition of uniform domain.

4. *A uniform domain is linearly locally connected.* This is a relatively elementary argument using only the properties involved in these two properties.

5. *A linearly locally connected domain satisfies the two point inequality.* If D is linearly locally connected, then it is locally connected at each point of ∂D and hence Jordan. The desired two point inequality then follows from the two inequalities in the definition of linear local connectivity.

6. *D is a quasidisk if it satisfies the two point inequality.* This argument is due to Ahlfors [3]. Suppose that f and f^* are conformal mappings of D and D^* onto the upper and lower half planes H and H^* normalized so that $f^{-1}(\infty) = f^{*-1}(\infty)$. Then the two point condition implies that

$$\phi = f^*(f^{-1}) : \partial H \to \partial H^*$$

is quasisymmetric. Hence by a theorem due to Beurling and Ahlfors [9], ϕ has a quasiconformal extension to $\overline{R}^2$ and D is a quasidisk.

C. Second circle of implications

1. *A quasidisk has the BMO extension property.* This argument is due to Jones [35]. If D is a quasidisk, then there exists a quasiconformal self mapping f of $\overline{R}^2$ which maps D onto a disk or half plane D' so that $f(\infty) = \infty$. Next $u' = u(f^{-1})$ is BMO in D' by a theorem due to Reimann [55], a simple construction yields a BMO extension v' of v to R^2 and $v = v'(f)$ is BMO in R^2 again by Reimann's theorem.

2. *The BMO extension property implies the hyperbolic bound.* This follows from setting $u(z) = h_D(z, z_0)$ where $z, z_0 \in D$.

3. *The hyperbolic bound implies the segment property.* This follows from a simplified version, due to Gehring and Osgood [26], of an argument of Jones [35].

D. Third circle of implications

1. $\sigma(D) > 0$, $\tau(D) > 0$ *and* $L(D) > 1$ *if* D *is uniform.* This argument is due to Martio and Sarvas [45]. Suppose that D is uniform with constant $c \geq 1$ and that f is analytic and locally injective with

$$|T_f| \leq \frac{1}{4c^2}\, \rho_D$$

in D. Then for each $z_1, z_2 \in D$, integration yields

$$\left| \frac{f(z_1) - f(z_2)}{f'(z_0)} - (z_1 - z_2) \right| < |z_1 - z_2|$$

where z_0 is the midpoint of the curve α joining z_1, z_2 which corresponds to the fact that D is uniform. Thus f is injective and

$$\tau(D) \geq \frac{1}{4c^2} > 0.$$

This and the fact that

$$S_f = T_f' - \frac{1}{2}\, T_f^2$$

then allow one to conclude that

$$\sigma(D) \geq \frac{1}{16c^3} > 0.$$

The argument for local quasi-isometries is similar.

2. D *is linearly locally connected if* $\sigma(D) > 0$, $\tau(D) > 0$ *or* $L(D) > 1$. Suppose that D is not linearly locally connected. Then for each $a > 0$ and $b > 0$ one can construct explicitly functions f and g which are analytic and locally, but not globally, injective in D such that

$$|S_f| \leq a\, \rho_D^2 \quad \text{and} \quad |T_g| \leq b\, \rho_D$$

in D. See Gehring [17] and Astala-Gehring [6]. A similar construction yields for each $L > 1$ a local L-quasi-isometry h in D which is not injective. See Gehring [18].

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