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Conformal welding and Koebe's theorem

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Abstract

It is well known that not every orientation-preserving homeomorphism of the circle to itself is a conformal welding, but in this paper we prove several results which state that every homeomorphism is "almost" a welding in a precise way. The proofs are based on Koebe's circle domain theorem. We also give a new proof of the well known fact that quasisymmetric maps are conformal weldings.

1. Introduction

Let $\mathbb{D} \subset \mathbb{R}^2$ be the open unit disk, let $\mathbb{D}^* = S^2 \backslash \overline{\mathbb{D}}$ and let $\mathbb{T} = \partial\mathbb{D} = \partial\mathbb{D}^*$ be the unit circle. Given a closed Jordan curve Γ, let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be conformal maps onto the bounded and unbounded complementary components of Γ respectively. Then $h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$ is a homeomorphism. Moreover, any homeomorphism arising in this way is called a conformal welding. The map $\Gamma \to h$ from closed curves to circle homeomorphisms is well known to be neither onto nor 1-to-1 (see Remarks 1 and 2), but in this paper we will show it is "almost onto" (every h is close to a conformal welding) and "far from 1-to-1" (there are h's which correspond to a dense set of Γ's).

We say that h is a generalized conformal welding on the set $E \subset \mathbb{T}$ if there are conformal maps $f : \mathbb{D} \to \Omega$, $g : \mathbb{D}^* \to \Omega^*$ onto disjoint domains such that f has radial limits on E, g has radial limits on $h(E)$ and these limits satisfy $f = g \circ h$ on E. Generalized conformal welding was invented by David Hamilton in [19] (see his papers [20] and [21] for applications to Kleinian groups and Julia sets). For $E \subset \mathbb{T}$, let $|E|$ denote its Lebesgue measure (normalized so that $|\mathbb{T}| = 1$) and $\mathrm{cap}(E)$ its logarithmic capacity (see §2).

Theorem 1. *Given any orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ and any $\varepsilon > 0$, there are a set $E \subset \mathbb{T}$ with $|E| + |h(E)| < \varepsilon$ and a conformal welding homeomorphism $H : \mathbb{T} \to \mathbb{T}$ such that $h(x) = H(x)$ for all $x \in \mathbb{T} \setminus E$.*

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In particular, every such h is a generalized conformal welding on a set E with Lebesgue measure as close to 1 *as we wish.*

The proof of Theorem 1 has two main steps. The first is the following.

THEOREM 2. *Any orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is a generalized conformal welding on $\mathbb{T} \setminus F$, where $F = F_1 \cup F_2$ and both F_1 and $h(F_2)$ have logarithmic capacity zero.*

Theorem 2 gives no information if h is "log-singular", i.e., $\mathbb{T} = F_1 \cup F_2$ with both F_1 and $h(F_2)$ of zero capacity. However, a different method shows that such a map is indeed a conformal welding, although in a radically nonunique way. We will say that a closed Jordan curve γ is flexible if two conditions hold. First, given any closed Jordan curve γ' and any $\varepsilon > 0$, there is a homeomorphism H of the sphere, conformal off γ, which maps γ to within ε of γ' in the Hausdorff metric. Second, given points z_1, z_2 in each component of $S^2 \setminus \gamma$, and points w_1, w_2 in each component of $S^2 \setminus \gamma'$, we can choose H above so that $H(z_1) = w_1$ and $H(z_2) = w_2$. Examples of such curves were constructed in [7] (although the second condition was not explicitly stated there, it does follow from the construction). Since γ and $H(\gamma)$ give the same conformal welding homeomorphism, we see that if h is the conformal welding associated to a flexible curve, then it is also associated to a set of curves which is dense in all closed curves.

THEOREM 3. *Suppose h is an orientation-preserving homeomorphism of the circle. Then h is the conformal welding of a flexible curve if and only if it is log-singular, i.e., if and only if there is a Borel set E such that both E and $h(\mathbb{T} \setminus E)$ have zero logarithmic capacity.*

Theorem 3 is proven by an explicit geometric construction. We can start with any two conformal maps f_0, g_0 onto smooth Jordan domains with disjoint closures. We then replace f_0 by a quasiconformal map f_1 which approximates f_0 except near the set E of zero capacity where we "push" the values closer to $g_0 \circ h$. Similarly we replace g_0 by a map g_1 which approximates it except near the zero capacity set $h(\mathbb{T} \setminus E)$ where we push the values closer to f_0. Thus for every point $x \in \mathbb{T}$, $f_1(x)$ is closer to $g_1(h(x))$ than $f_0(x)$ was to $g_0(h(x))$. Continuing by induction we obtain sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly to the desired maps f, g. By combining Theorems 3 and 2 we will obtain the proof of Theorem 1 in Section 8.

Note that Theorem 3 gives a condition for a homeomorphism h to be a welding in terms of h being sufficiently 'wild'. Previously known criteria say h is a welding if it is sufficiently 'nice' (e.g., h is quasisymmetric [30], [31], [36], or some weakening of quasisymmetric [15], [29]). As an illustration of our methods, in Section 4 we will give an elementary proof that quasisymmetric maps are conformal weldings (in [22] D. Hamilton refers to this as the "fundamental theorem of conformal welding").

Our approach to Theorem 2 is based on the following picture for conformal welding. Think of the homeomorphism h as mapping the unit circle $\mathbb{T}$ to $2\mathbb{T}$, the concentric circle of radius two. Now foliate the annulus $A = \{z : 1 < |z| < 2\}$ by curves which connect $x \in \mathbb{T}$ to $h(x) \in 2\mathbb{T}$ (for example, take the hyperbolic geodesic in A connecting these points). Now take the quotient space of the plane which collapses each of these curves to a point. By a theorem of R. L. Moore (see Remark 3) the result is the plane again, with the annulus A mapping to a closed curve Γ. Moreover, $\mathbb{D}$ and $2\mathbb{D}^*$ map to the complementary components of Γ with the boundary points x and $h(x)$ being identified. If these maps were also conformal we would be done, i.e., we would have a Γ corresponding to h. Although we know we can't always do this, our idea is to try to collapse as many of the curves in the foliation as possible, while keeping the maps on $\mathbb{D}$ and $2\mathbb{D}^*$ conformal. Our method for doing this is Koebe's circle domain theorem.

We start with n equidistance points $\{x_k\}_1^n \subset \mathbb{T}$ and disjoint smooth curves $\{\gamma_n\}$ which connect these points to the points $2h(x_k) \in \{|z| = 2\}$ in the annulus $A = \{1 < |z| < 2\}$. Let $\Omega = \Omega_{n,\varepsilon}$ be the union of $\mathbb{D}$, $2\mathbb{D}^*$ and an ε-neighborhood of each γ_n, where ε is assumed to be so small that these neighborhoods are pairwise disjoint. By Koebe's circle domain theorem, any finitely connected plane domain can be conformally mapped to one bounded by circles and points. Thus our domain can be mapped to a domain whose complementary components are all disks. By taking $\varepsilon \to 0$ we obtain a closed chain of tangent circles, which divides the plane into two domains, Ω_n and Ω_n^*. See Figure 1. Assume that there is an $R < \infty$ so that the circle chain

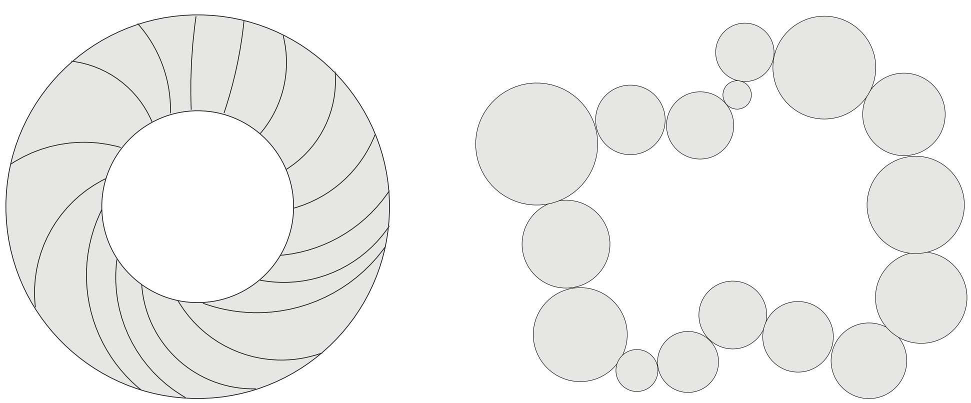

Figure 1: Using Koebe's theorem to build a welding

is contained in $\{z : 1 \leq |z| \leq R\}$ independent of n. Given this, it is easy to see that as $n \to \infty$ "most" of the disks collapse to points (at most $(R/\varepsilon)^2$ can remain larger than size ε), which implies that $|f_n(x) - g_n(h(x))| \to 0$ except at countably many points. In order to show there is an R with this property, we need to make an extra assumption about h. The precise statement we will prove is:

THEOREM 4. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism which is not log-singular (i.e., we assume that for any set $E \subset \mathbb{T}$ of zero logarithmic capacity, $h(\mathbb{T} \setminus E)$ has positive capacity). Then there are sequences of conformal maps $\{f_n\}$ on $\mathbb{D}$ and $\{g_n\}$ on $\mathbb{D}^*$ such that*

(1) $f_n(0) = 0$, $g_n(\infty) = \infty$.

(2) $\Omega_n = f_n(\mathbb{D})$ *and* $\Omega_n^* = g_n(\mathbb{D}^*)$ *are disjoint Jordan domains.*

(3) *There is an $R < \infty$ so that $S^2 \setminus (\Omega_n \cup \Omega_n^*) \subset \{z : 1 \le |z| \le R\}$ independent of n.*

(4) *There is a countable set $E \subset \mathbb{T}$ such that* $\lim_{n\to\infty} |f_n(x) - g_n(h(x))| = 0$ *for all $x \in \mathbb{T} \setminus E$.*

Note that our hypothesis on h is exactly complementary to the condition in Theorem 3. Thus these two results together imply

THEOREM 5. *Given any orientation-preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ there are nondegenerate sequences of conformal maps $f_n : \mathbb{D} \to \Omega_n$, $g_n : \mathbb{D}^* \to \Omega_n^*$ onto disjoint Jordan domains with $f_n(0) = 0$, $g_n(\infty) = \infty$ and such that $|f_n(x) - g_n(h(x))| \to 0$ for all $x \in \mathbb{T} \setminus E$, where E is a countable set.*

By "nondegenerate" sequence in Theorem 5 we mean that $f_n'(0)$ and $g_n'(\infty)$ are bounded away from zero uniformly. Equivalently, there is an $R < \infty$ such that $S^2 \setminus (\Omega_n \cup \Omega_n^*) \subset \{z : R^{-1} \le |z| \le R\}$, independent of n. From Theorem 5 we might expect that every homeomorphism is a generalized conformal welding except on a countable set. However, passage to the limit causes difficulties and we "lose control" on a set of zero logarithmic capacity, giving Theorem 2 instead. See Section 9 for some conjectures related to this.

Once we have Theorem 4 we will prove Theorem 2 using extremal length estimates. The idea is to pass to a subsequence such that $f_n \to f$ and $g_n \to g$ uniformly on compact sets. Since $|f_n - g_n \circ h| \to 0$ everywhere on $\mathbb{T}$ except for a countable set, the only way that $f(x) \ne g \circ h(x)$ off this set is for $f(x) \ne \lim_n f_n(x)$ or $g(x) \ne \lim_n g_n(x)$ (or for the limits not to exist). For a general sequence of maps this might happen on positive capacity (see Remark 5), but because all our map pairs are related by the same homeomorphism h, we can show this happens on at most one zero capacity set for each "side", which gives Theorem 2. As special cases of Theorem 2 we have

Corollary 6. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is a orientation-preserving homeomorphism such that E has zero logarithmic capacity if and only if $h(E)$ does. Then h is a generalized conformal welding on $\mathbb{T} \setminus F$, where F has zero logarithmic capacity.*

Corollary 7. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism that is log-regular (i.e., $\mathrm{cap}(F) = 0 \Rightarrow |h(F)| = |h^{-1}(F)| = 0$). Then h is a generalized conformal welding on a set of E such that both E and $h(E)$ have full Lebesgue measure.*

These results were conjectured by David Hamilton and Corollary 7 strengthens a result of his from [19]. We will refer to homeomorphisms which satisfy the conclusion of Corollary 7 as "almost everywhere weldings". The last step in the proof of Theorem 1 will be to convert a generalized conformal welding into an actual conformal welding using the following result.

Theorem 8. *Suppose $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ are conformal maps onto disjoint Jordan domains and let $E = f^{-1}(\partial\Omega \cap \partial\Omega^*)$. On E define $h = g^{-1} \circ f$. Then h can be extended from E to a conformal welding homeomorphism of $\mathbb{T}$ to itself.*

This result will be proven by an explicit geometric construction. We end this section with some remarks.

Remark 1. Even if we take $E = \mathbb{T}$, then generalized conformal welding on E is still weaker than the usual notion of conformal welding. Let K be the union of the graph γ of $\sin(1/x)$, $x \neq 0$, and the limiting vertical line segment $[-i, i]$. Let φ map the exterior of the segment conformally to the exterior of $[-1, 1]$ with $-i$ and i being identified at 0. The set $K' = [-1, 1] \cup \varphi(\gamma)$ divides the plane into a pair of simply connected domains so that the corresponding maps f, g each extend continuously to $\mathbb{T}$ except at one point where the radial limits both exist and equal 0. Thus $h = g^{-1} \circ f$ is a generalized conformal welding everywhere on $\mathbb{T}$. However, h is not a conformal welding map. If there were a closed Jordan curve giving the same homeomorphism, then we could map the two sides of K to the two sides of Γ with boundary values that match up on γ, and the image of γ would be Γ minus a point. Since smooth curves are removable for conformal maps, we get a conformal mapping from the complement of a line segment to the complement of a point, which is impossible by Liouville's theorem. Other examples of homeomorphisms which are not conformal weldings are given in [35], [42], [43] and [44].

Remark 2. It is already well known that mapping $\Gamma \to h$ is not 1-to-1 (even modulo Möbius transformations). One can build curves Γ and homeomorphisms $F : S^2 \to S^2$ which are conformal off Γ but not Möbius. Such curves are called nonremovable for conformal homeomorphisms, and clearly both Γ and $F(\Gamma)$ map to the same h. The simplest example is a curve with positive area; take a nonzero dilatation supported on Γ and solve the Beltrami equation to get a quasiconformal map which is conformal off Γ but not conformal everywhere. Other examples based on Fourier analysis are given by Kaufman in [25] (see also [26]) and further examples follow from the theory of null sets of Alhfors and Beurling [1], as described by Hamilton in [19] and [22].

Although nonremovable curves can have zero area (can even have Hausdorff dimension 1), they are always closely related to two dimensional curves as follows. Suppose F is conformal off Γ and fixes 0 and ∞. Then $G(z) = F(z)/z$ is bounded and continuous on the sphere and holomorphic of Γ. If $w \notin G(\Gamma)$ then G only takes this value finitely often and the argument principle implies $\#\{z \in \Omega : G(z) = w\} = -\#\{z \in \Omega^* : G(z) = w\} = 0$, i.e., $G(\Gamma) = G(S^2)$ (I learned this argument from A. Browder's book [9]). If F is not Möbius, then G is not constant, hence an open mapping on $S^2 \setminus \Gamma$. Thus $G(\Gamma)$ covers an open set and division by z converts F from a homeomorphism to a space filling curve.

Remark 3. Let us recall in more detail the result of R. L. Moore quoted earlier, starting with a few definitions. A decomposition of a compact set K is a collection of pairwise disjoint closed sets whose union is all of K. A collection $\mathcal{C}$ of closed sets in the plane is called upper semi-continuous if a collection of elements which converge in the Hausdorff metric must converge to a subset of another element. If $K = \mathbb{R}^2$ and all elements of $\mathcal{C}$ are continua which do not separate the plane we shall call $\mathcal{C}$ a Moore decomposition after R. L. Moore who proved

THEOREM 9 (Moore, [33]). *Suppose $\mathcal{C}$ is a Moore decomposition of $\mathbb{R}^2$. Then the quotient space formed by identifying each set to a point is homeomorphic to $\mathbb{R}^2$.*

Also see Daverman's book [14]. For an overview of Moore's life and work see [45] (reprinted in [16]) and [17]. For another application of Moore's topological work (i.e. the Moore triod theorem) to conformal mappings, see Pommerenke's paper [38].

Given a decomposition $\mathcal{C}$, let $\Omega(\mathcal{C})$ be the interior of the set of singletons and call $\mathcal{C}$ conformal if the quotient map in Moore's theorem can be chosen to be conformal on Ω (we call the quotient map a conformal collapsing). Not every Moore decomposition is conformal: if $\mathcal{C}$ is just $\{|z| \leq 1\}$ and singletons then we would get a conformal map from $\mathbb{D}^*$ to a punctured plane, which is impossible by Liouville's theorem. Which Moore decompositions are conformal? When is the quotient map unique up to Möbius transformations? These questions are probably too general to have neat answers, but our approach to conformal welding by collapsing arcs of a foliation can be viewed as a special case. If we understood conformal collapsing in general, there would be many applications to complex dynamics and Kleinian groups, where we know how to describe some dynamics topologically, but would like to know there is a consistent conformal structure (e.g., building a degenerate limit set from a Fuchsian group G by collapsing a G-invariant foliation of the disk).

We say a Moore decomposition is a Koebe decomposition if every element is either a closed disk or a point. Is every Moore decomposition conformally equivalent to a Koebe decomposition? If so, then there are only countably many sets that are not collapsed to points, and so this says that every Moore decomposition is almost conformal. This problem is probably also difficult (it contains the famous Koebe conjecture as a special case), but Theorem 5 might be seen as (weak) evidence in its favor. We will discuss related problems in Section 9.

Remark 4. We say a compact set $E \subset \mathbb{T}$ is an interpolation set for conformal maps if given any homeomorphism g of $\overline{\mathbb{D}}$ there is a conformal map f of the disk which extends continuously to E and equals g there. An earlier verison of the proof of Theorem 3 used a characterization of these sets as exactly the compact sets of zero logarithmic capacity. This result now appears in [6].

Remark 5. If $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}$ what can we say about the convergence of boundary values in general? It is not true that there is always a subsequence so that $\{f_n(x)\}$ converges for all x except in an exceptional set of zero logarithmic capacity. However, it is true that given any kernel function K which tends to ∞ faster than $\log \frac{1}{t}$, there is a subsequence that converges off an exceptional set of zero K-capacity. Both statements are proven in the 2005 Ph.D. thesis of Karyn Lundberg [32], the second strengthening an earlier result of David Hamilton.

Remark 6. Koebe's circle domain theorem and conformal welding had been previously linked via the theory of circle packings. Koebe's theorem can be used to prove the existence of finite circle packings with prescribed tangencies and He and Schramm [23] proved Koebe's conjecture for domains with countably many boundary components using circle packing techniques. Later, Williams [47], [46] used circle packing algorithms to compute conformal weldings, i.e., to compute h from Γ and Γ from h.

Remark 7. One cannot prove Theorem 1 by showing that any h agrees with a quasisymmetric map on large measure. If h maps a set E of positive Lebesgue measure to a set of zero Hausdorff dimension, then it cannot agree with any quasisymmetric map on any positive measure subset of E since quasisymmetric maps preserve sets of dimension zero.

Another way to look at this is to make the orientation-preserving homeomorphisms of the circle into a metric space by setting

$$d(f,g) = |\{x : f(x) \neq g(x)\}| + |\{x : f^{-1}(x) \neq g^{-1}(x)\}|.$$

Theorem 1 says conformal weldings are dense in this space, but one easily sees that quasisymmetric and log-singular homeomorphisms are each nowhere dense sets which are distance 1 apart. (It is standard to show this space is complete but nonseparable but a little more amusing to show it is path connected, but contains no nontrivial rectifiable paths.)

Remark 8. Several times in this paper we will use the well known observation that it suffices to take quasiconformal maps in the definition of conformal welding. More precisely, if $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ are K-quasiconformal with $h = g^{-1} \circ f$ on $\mathbb{T}$, then the measurable Riemann mapping theorem (e.g., [2]) implies there is a K-quasiconformal map Φ of the sphere so that $F = \Phi \circ f$ and $G = \Phi \circ g$ are both conformal. Since $G^{-1} \circ F = g^{-1} \circ f$ on $\mathbb{T}$, we see h is a conformal welding in the usual sense if it is a "quasiconformal welding".

Remark 9. We know that homeomorphisms h which satisfy the log-singularity condition of Theorem 3 exist because we know flexible curves exist (see [7]). A more direct inductive construction is as follows. Start with a linear mapping h_0 on an interval I^0. At the nth stage, assume we have divided I^0 into a finite number of subintervals $\{I_j^n\}$ and have a homeomorphism h_n of I^0 which is linear on each of these subintervals. Divide each I_j^n into n equal length subintervals. If I is one of these, divide I into two subintervals: the left one of length $\varepsilon_n|I|$ and the right one of length $(1-\varepsilon_n)|I|$. Define a homeomorphism h_{n+1} which is linear on every subinterval, so that $h_{n+1}(I) = h_n(I)$ and so that the right- hand interval of I maps to an interval of length $\varepsilon_n|h_n(I)|$. We choose ε_n so small that the union of all the left intervals has logarithmic capacity less than 2^{-n} and the union of the h_{n+1} images of the right- hand intervals also has capacity $\leq 2^{-n}$. It is easy to see that these maps converge to a homeomorphism h. If E is the set of points which are in infinitely many of the left-hand intervals, then clearly $\mathrm{cap}(E) = 0 = \mathrm{cap}(h(I^0 \setminus E))$ (by subadditivity of capacity).

Remark 10. It is interesting to compare Theorem 3 with results of A. Browder and J. Wermer for the disk algebra $A(\mathbb{D})$ (holomorphic functions on $\mathbb{D}$ which extend continuously to $\mathbb{T}$). Given a homeomorphism of the circle h they considered the set of functions $A_h = \{f \in A(\mathbb{D}) : f = g \circ h \text{ for some } g \in A(\mathbb{D}^*)\}$ and showed this collection was "large" if and only if h is singular, i.e., if and only if it maps some set of full Lebesgue measure to zero Lebesgue measure (e.g., [10], [11], [8], [5]). Large in their sense meant A_h is a Dirichlet algebra, i.e., the reals parts of functions in A_h are uniformly dense in all continuous real-valued functions on $\mathbb{T}$. Moreover, by the Rudin-Carleson theorem the compact boundary interpolation sets for the disk algebra are exactly the sets of zero Lebesgue measure, ([12], [40]), just as zero logarithmic capacity sets are for conformal maps (see Remark 4 and [6]). Are Theorem 3 and the Browder-Wermer theorem both special cases of a more general result?

The remaining sections of the paper are organized as follows.

Section 2. We recall the definition of logarithmic capacity and extremal length.

Section 3. We prove Theorem 4.

Section 4. We give a new, elementary proof that quasisymmetric homeomorphisms are conformal weldings.

Section 5. We prove Theorem 2

Section 6. We prove Theorem 8.

Section 7. We characterize flexible curves (Theorem 3).

Section 8. We prove Theorem 1.

Section 9. We state a generalization of the Koebe circle conjecture.

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2. Logarithmic capacity and extremal length

In this section we review some basic material on logarithmic capacity and extremal length. Experts may wish to skip it and refer back to it as needed.

Suppose μ is a positive Borel measure on $\mathbb{R}^2$ and define its energy integral by

$$I(\mu) = \iint \log \frac{2}{|z-w|} d\mu(z) d\mu(w).$$

We put the "2" in the numerator so that the integrand is nonnegative when $z, w \in \mathbb{T}$ (in this paper we will only consider the capacity of subsets of $\mathbb{T}$). If $E \subset \mathbb{R}^2$ is Borel, let $\mathrm{Prob}(E)$ be the set of positive Borel measures with $\mu(E) = \|\mu\| = 1$ and define its logarithmic capacity as

$$\mathrm{cap}(E) = \frac{1}{\inf\{I(\mu) : \mu \in \mathrm{Prob}(E)\}}.$$

For subsets of the circle, cap is nonnegative, monotone and is countably subadditive ([13, p. 24, Lemma]; this is where we need the "2" in the definition of the energy integral). If $\mathrm{cap}(E) > 0$, there is a unique measure which minimizes the energy integral, which is called the equilibrium measure (it is also equal to the harmonic measure of $S^2 \setminus E$ with respect to infinity). An alternate version of logarithmic capacity is

$$\widetilde{\mathrm{cap}}(E) = \sup\{\exp(-I(\mu)) : \mu \in \mathrm{Prob}(E)\}.$$

The exponential in the definition is a technical convenience and gives it nice scaling; i.e., $\widetilde{\mathrm{cap}}(tE) = t \cdot \widetilde{\mathrm{cap}}(E)$. The two versions of logarithmic capacity are related by the equations

$$\widetilde{\mathrm{cap}} = \exp(-1/\mathrm{cap}), \quad \mathrm{cap} = (\log \widetilde{\mathrm{cap}}^{-1})^{-1}.$$

Note that if $E \subset \mathbb{T}$ then $\widetilde{\mathrm{cap}}(E) = 0$ if and only if $\mathrm{cap}(E) = 0$. Thus we may speak of sets of positive or zero capacity without specifying which definition we mean and we will use both versions throughout the paper.

Logarithmic capacity is closely related to the usual Robin constant γ_E defined by

$$\gamma_E = \inf\{I(\mu) - \log 2 : \mu \in \mathrm{Prob}(E)\} = \frac{1}{\mathrm{cap}(E)} - \log 2.$$

The $\log 2$ enters because we put a "2" in our energy integral, whereas the usual definition does not.

If $f : \mathbb{D} \to \Omega$ is conformal and $E \subset \partial\Omega$ then we will call $\mathrm{cap}(f^{-1}(E))$ the capacity of E with respect to Ω (the value depends on the choice of f, but whether or not it is zero is independent of f).

In this paper, we shall only use a few well known facts about logarithmic capacity. The proof of the following is easy and left to the reader.

Lemma 10. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is a bi-Hölder homeomorphism, i.e., there are a $C < \infty$ and $\alpha > 0$ such that*

$$\frac{1}{C}|x - y|^{1/\alpha} \le |h(x) - h(y)| \le C|x - y|^{\alpha}.$$

Then $\mathrm{cap}(E) = 0$ *if and only if* $\mathrm{cap}(h(E)) = 0$.

We will need the fact that all Borel sets are capacitable, i.e., if E is Borel, then

$$\operatorname{cap}(E) = \inf\{\operatorname{cap}(U) : E \subset U, U \text{ open } \}.$$

The exact form we will use is contained in:

LEMMA 11. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is a homeomorphism. Then the following are equivalent.*

(1) *For any $\varepsilon > 0$ there is a finite union of closed intervals $E \subset \mathbb{T}$ such that both $\operatorname{cap}(E)$ and $\operatorname{cap}(h(\mathbb{T} \setminus E))$ are $\leq \varepsilon$.*

(2) *For any n there is a compact set $E_n \subset \mathbb{T}$ such that both $\operatorname{cap}(E_n)$ and $\operatorname{cap}(h(\mathbb{T} \setminus E_n))$ are $\leq 1/n$.*

(3) *There is a Borel set E such that both E and $h(\mathbb{T}\setminus E)$ have zero logarithmic capacity.*

Proof. Trivially, (1) $\Rightarrow$ (2). To prove (2) $\Rightarrow$ (3), let $E = \cap_n \cup_{k>n} E_{2^k}$. Clearly E has zero capacity (since cap is countably subadditive). Its complement is contained in $\cap_n \cup_{k>n} (\mathbb{T} \setminus E_{2^k})$ whose h image also has zero capacity and we are done.

Finally, to prove (3) $\Rightarrow$ (1) we use Theorem 7 on page 24 of Carleson's book [13] which says that for any Borel E and any $\varepsilon > 0$ there is an open set U containing E such that $\operatorname{cap}(U) \leq \operatorname{cap}(E) + \varepsilon$. Applying this to the sets E and $h(\mathbb{T} \setminus E)$ in condition (3) we obtain open sets U_1 and U_2 so that $\mathbb{T} \subset U_1 \cup U_2$ and both U_1 and $h(U_2)$ have capacity $\leq \varepsilon$. Since $\mathbb{T}\setminus U_2$ is compact and covered by the components of U_1 there is a finite collection of these components which also covers. Let the closure of the union of these components be F, which clearly has capacity $\leq \varepsilon$. The complement of F is contained in U_2 and hence the capacity of $h(\mathbb{T} \setminus F)$ is also less than ε. $\square$

It is convenient to estimate logarithmic capacity in terms of extremal length, so we start by recalling the definition. Suppose $\mathcal{P}$ is a family of rectifiable paths in a domain Ω and suppose ρ is a nonnegative function on Ω such that $\int_\gamma \rho ds \geq 1$ for every $\gamma \in \mathcal{P}$. We define the modulus of the family to be

$$\operatorname{mod}(\mathcal{P}) = \inf_\rho \iint_\Omega \rho^2 dxdy,$$

and the extremal length

$$\lambda(\mathcal{P}) = \frac{1}{\operatorname{mod}(\mathcal{P})}.$$

See Ahlfors' book [2]. It is easy to see that modulus and extremal length are conformal invariants. An important example is the path family connecting the two boundary components of an annulus $\{z : a < |z| < b\}$. Standard arguments show the extremal length of this family is $2\pi \log(b/a)$. The connection to logarithmic capacity is given by the following result, Pfluger's Theorem, e.g., Theorem 9.17 of [39],

LEMMA 12. *Suppose $E \subset \mathbb{T}$ is compact, $K \subset \mathbb{D}$ is compact and connected and $\mathcal{P}$ is the path family in $\mathbb{D}$ connecting K to E. Then $\widetilde{\mathrm{cap}}(E) \simeq \exp(-\pi\lambda(\mathcal{P}))$, with constants that depend only on K.*

One particular consequence we will use is the following.

COROLLARY 13. *If f is a conformal map on $\mathbb{D}$ and takes the boundary value 0 at every point of $E \subset \mathbb{T}$, then* $\mathrm{cap}(E) = 0$

Proof. Suppose $K \subset \mathbb{D}$ is compact and choose r so small that $D(0,r) \cap f(K) = \emptyset$. Then the extremal length of the path family connecting K to E in $\mathbb{D}$ is greater than for the family crossing the annulus $\{z : \varepsilon < |z| < r\}$ in Ω. Taking $\varepsilon \to 0$ and using the estimate for annuli discussed above proves the result. $\square$

Using Lemma 18, which we will prove later, one can show that it suffices to assume f has radial limit 0 on E in Lemma 13.

Suppose $\partial\Omega$ is bounded in $\mathbb{R}^2$ and $f : \mathbb{D} \to \Omega$ is conformal. For $0 < r < 1$, let

$$a_f(r) = \mathrm{area}(\Omega \setminus f(D(0,r))).$$

Since $\partial\Omega$ is compact it is easy to see that this tends to zero as $r \to 1$.

LEMMA 14. *There is a $C < \infty$ so that the following holds. Suppose $f : \mathbb{D} \to \Omega$ and $\frac{1}{2} \le r < 1$. Let $E = \{x \in \mathbb{T} : |f(sx) - f(rx)| \ge \delta$ for some $r < s < 1\}$. Then the extremal length of the path family $\mathcal{P}$ connecting $D(0,r)$ to E is bounded below by $\delta^2/Ca(r)$.*

Proof. Suppose $z, w \in \Omega$, suppose γ is the hyperbolic geodesic connecting z and w and suppose $\tilde{\gamma}$ is any path in Ω connecting these points. By the Gehring-Hayman inequality [18], there is a universal $C < \infty$ such that $\ell(\gamma) \le C\ell(\tilde{\gamma})$ (here $\ell(\gamma)$ denotes the length of γ). In other words, up to a constant, the hyperbolic geodesic has the shortest Euclidean length amongst all curves in Ω connecting the two points.

Now suppose we apply this with $z = f(sx)$ and $w \in f(D(0,r))$. Then the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them. But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe distortion theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ. Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and equal 0 elsewhere. Then ρ is admissible for $f(\mathcal{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$. Thus $\lambda(\mathcal{P}) \geq \frac{\delta^2}{C^2 a(r)}$. □

If f has radial limits on $E \subset \mathbb{T}$ then the previous lemma is still valid for $s = 1$. For subsets of the circle it is known that

$$\widetilde{\text{cap}}(E) \geq |E|/C, \tag{2.1}$$

(e.g., XI.2.E in [41]). Combining this with Lemmas and 12 and 14 gives

Corollary 15. *If $f : \mathbb{D} \to \Omega$ is a conformal map onto a bounded domain then for any $\delta > 0$,*

$$|\{x \in \mathbb{T} : |f(x) - f(rx)| \geq \delta\}| \to 0,$$

as $r \to 1$.

Lemma 16. *Suppose $E_1, \dots, E_n$ is a finite collection of compact sets and let $\mathcal{C}_k$ be the path family connecting $D(0,r)$ to E_k, $k = 1, \dots, n$. Suppose each of these families has extremal length $\geq c > 0$. Then the path family connecting $E = \cup E_k$ to $D(0,r)$ has extremal length $\geq c/n^2$.*

Proof. If $\{\rho_i\}$ are admissible metrics for $\{\mathcal{C}_k\}$ respectively then $\rho = \sum_k \rho_k$ is admissible for $\mathcal{C}$ and by Cauchy-Schwarz $\int \rho^2 \leq n \int \sum \rho_k^2$. Taking the infimum over admissible metrics gives the result. □

Lemma 17. *Suppose $f : \mathbb{D} \to \Omega$ is conformal and for $R \geq 1$,*

$$E = \{x \in \mathbb{T} : |f(x)| \geq R \operatorname{dist}(f(0), \partial\Omega)\}.$$

Then $\text{cap}(E) \leq CR^{-1/2}$ *(with C independent of Ω).*

Proof. Assume $f(0) = 0$ and $\operatorname{dist}(0, \partial\Omega) = 1$ and let $\rho(z) = |z|^{-1}/\log R$ for $z \in \Omega \cap \{1 < |z| < R\}$. Then ρ is admissible for the path family connecting $D(0, 1/2)$ to $\partial\Omega \setminus D(0,R)$ and $\iint \rho^2 dxdy \leq 2\pi/\log R$. By the Koebe distortion theorem $f^{-1}(D(0,1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω. The result follows by Lemma 12. □

Lemma 17 also follows from a stronger result of Balogh and Bonk in [3].

Given a compact set $E \subset \mathbb{T}$ we will now define the associated "sawtooth" region W_E and a 2-quasiconformal map between W_E and $\mathbb{D}$ which keeps E fixed pointwise. Suppose $\{I_n\}$ are the connected components of $\mathbb{T} \setminus E$ and for each n let $\gamma_n(\theta)$ be the circular arc in $\mathbb{D}$ with the same endpoints as I_n makes angle θ with I_n (so $\gamma_n(0) = I_n$ and $\gamma_n(\pi/2)$ is the hyperbolic geodesic with the same endpoints as I_n). Let $C_n(\theta)$ be the region bounded by I_n and $\gamma_n(\theta)$, and let $W_E(\theta) = \mathbb{D} \setminus \cup_n C_n(\theta)$. See Figure 2.

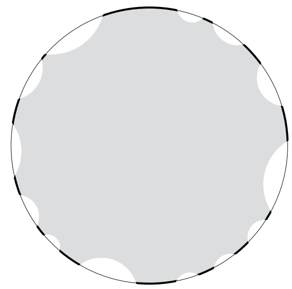

Figure 2: The sawtooth domain W_E

For the rest of the paper we will let $W_E = W_E(\pi/8)$ (and let $W_E^* \subset \mathbb{D}^*$ be its reflection across $\mathbb{T}$). We can map $\mathbb{D}$ to W_E by a 2-quasiconformal map f as follows. First let f be the identity on $W_E(\pi/2)$. Then map $U_n = C_n(\pi/2)\backslash C_n(\pi/4)$ (which is a crescent of angle $\pi/4$) to $V_n = C_n(\pi/2)\backslash C_n(3\pi/8)$ (which is a crescent of angle $\pi/8$) as follows: map U_n to the cone $\{z : 0 < \arg(z) < \pi/4\}$ by a Möbius transformation, then to $\{z : 0 < \arg(z) < \pi/8\}$ by halving the angle and then to V_n by another Möbius transformation. Finally, map $C_n(\pi/4)$ to $C_n(3\pi/8) \setminus C_n(\pi/8)$ by a Möbius transformation. See Figure 3. It is easy to check that these maps can be chosen to match up along the common boundaries and hence define a 2-quasiconformal map.

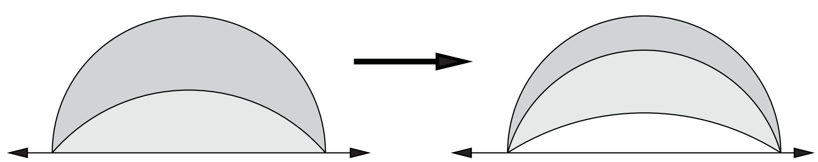

Figure 3: Mapping the disk to W_E

If $f : \mathbb{D} \to \Omega$ and $0 < r < 1$, then define

$$d_f(r) = \sup\{|f(z) - f(w)| : |z| = |w| = r \text{ and } |z - w| \le 1 - r\}.$$

If $\partial\Omega$ is bounded in the plane, then it is easy to see that this goes to zero as $r \nearrow 1$, since otherwise any neighborhood of $\partial\Omega$ would contain infinitely many disjoint disks of a fixed, positive size.

Lemma 18. *Suppose $f : \mathbb{D} \to \Omega \subset S^2$ is conformal. Then for any $\varepsilon > 0$ there is a compact set $E \subset \mathbb{T}$ with* $\text{cap}(\mathbb{T} \setminus E) < \varepsilon$ *such that f is continuous on $\overline{W_E}$.*

Proof. By applying a square root and a Möbius transformation, we may assume that $\partial\Omega$ is bounded in the plane. Given $r < 1$ let

$$E(\varepsilon, r) = \{x \in \mathbb{T} : |f(sx) - f(tx)| > \varepsilon \text{ for some } r < s < t < 1\}$$

and note that by Lemmas 12 and 14

$$\widetilde{\text{cap}}(E(\varepsilon, r)) \leq \exp(-\pi\varepsilon^2/Ca(r)).$$

Moreover, this set is open since f is continuous at the points sx and tx. So if we take $\varepsilon_n = 2^{-n}$, and use the relationship between cap and $\widetilde{\text{cap}}$ we can choose r_n so close to 1 that $\text{cap}(E_n) \equiv \text{cap}(E(\varepsilon_n, r_n)) \leq \varepsilon 2^{-n}$. If we define $E = \mathbb{T} \setminus \cup_{n>1} E_n$, then E is closed and $\mathbb{T} \setminus E$ has capacity $\leq \varepsilon$ by subadditivity.

To show that f is continuous at every $x \in \overline{W_E}$, we want to show that $|x - y|$ small implies $|f(x) - f(y)|$ is small. We only have to consider points $x \in \partial W_E \cap \mathbb{T}$. First suppose $y \in \partial W_E \cap \mathbb{T}$. Choose the maximal n so that $s = |x - y| \leq 1 - r_n$. Then $x, y \notin E_n$, and so

$$|f(x) - f(y)| \leq |f(x) - f(sx)| + |f(sx) - f(sy)| + |f(sy) - f(y)|.$$

The first and last terms on the right are $\leq \varepsilon_{n-1}$ by the definition of E. The middle term is at most $d_f(1-s)$ (which tends to 0 as $s \to 0$). Thus $|f(x)-f(y)|$ is small if $|x - y|$ is.

Now suppose $x \in \partial W_E \cap \mathbb{T}$, $y \in \partial W_E \setminus \mathbb{T}$. From the definition of W_E it is easy to see there is a point $w \in \partial W_E \cap \mathbb{T}$ such that $|w-y| \leq 2(1-|y|) \leq 2|x-y|$. For the point w we know by the argument above that $|f(x) - f(w)|$ is small. On the other hand, if $t = 1 - |y|$, then

$$|f(y) - f(w)| \leq |f(y) - f(tw)| + |f(tw) - f(w)|.$$

The first term is bounded by $Cd_f(1 - t)$ and the second is small since $w \notin E_n$. Thus $|f(x) - f(y)|$ is small depending only on $|x - y|$. Hence f is continuous on $\overline{W_E}$. $\square$

3. Welding via Koebe's theorem: proof of Theorem 4

We define a circle chain $\mathcal{C}$ to be a finite union of closed disks $\{D_k\}_1^n$ in $\mathbb{R}^2$ which have pairwise disjoint interiors and such that D_k is tangent to D_{k+1} for $k = 1, \ldots, n-1$, D_n is tangent to D_1 and there are no other tangencies. We also assume the disks are numbered in counterclockwise order. The complement, $X = S^2 \setminus \cup_k D_k$, of a circle chain consists of two disjoint Jordan domains. We shall denote the bounded component by Ω and the unbounded component by Ω^*. Let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be Riemann maps. We shall call (f, g) a normalized circle chain pair if $f(0) = 0$, $g(\infty) = \infty$ and $\text{dist}(0, \partial\Omega) = 1$. Clearly, given a circle chain, we can always obtain a normalized pair by composing with a Möbius transformation.

LEMMA 19. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism and suppose $\{x_k\}_1^n \subset \mathbb{T}$ is a finite collection of distinct points listed in counterclockwise order. Let $I_k = (x_k, x_{k+1}), k = 1, \dots, n$ (modulo n). Then there is a normalized circle chain pair so that for each k,*

$$f(I_k) = \partial D_k \cap \partial\Omega,$$
$$g(h(I_k)) = \partial D_k \cap \partial\Omega^*.$$

We will say that any circle chain that satisfies this conclusion corresponds to h. Another way of stating the lemma is that given any finite positive sequences $\{a_k\}$ and $\{b_k\}$ such that $\sum_{k=1}^n a_k = \sum_{k=1}^n b_n = 1$ we can find a circle chain so that the harmonic measure of each disk in the chain satisfies

$$\omega(D_k, 0, \Omega) = a_k, \quad k = 1, \dots n,$$
$$\omega(D_k, \infty, \Omega^*) = b_k, \quad k = 1, \dots n.$$

It is a fact that this circle chain is unique up to Möbius transformations, but we will not need this here. One can prove uniqueness by considering two chains corresponding to the same data. By taking conformal maps between the complements of two such chains and repeatedly extending them by reflection, we can show these maps extend to a homeomorphism of the sphere which is conformal except on a Jordan curve which is the limit set of the Kleinian group generated by reflections in the elements of our circle chain. It is known such a curve is a quasicircle (see Section 4) and hence is removable for conformal maps. Thus the maps extend to be conformal on the whole sphere, i.e., Möbius.

Proof of Lemma 19. The Koebe circle domain theorem ([27], [28]; also see [23] and its references) states that given any finitely connected domain Ω there is a conformal map $f : \Omega \to \tilde{\Omega}$ onto a domain bounded by circles and points. We shall apply this to a domain $\Omega = \Omega_\varepsilon$ constructed as follows. Given n points $\{x_k\}$ on the unit circle $\mathbb{T}$, let $y_k = 2h(x_k) \in 2\mathbb{T} = \{z : |z| = 2\}$. Let γ_n be disjoint smooth Jordan arcs which connect x_k to y_k in the annulus $A = \{z : 1 \leq |z| \leq 2\}$, e.g., the hyperbolic geodesics in A connecting these points. Let $\{I_k\} \subset \mathbb{T}$ be the arcs bounded by the points $\{x_k\}$ and let $\{J_k\}$ be the corresponding arcs on $2\mathbb{T}$. Thus J_k has harmonic measure $|h(I_k)|$ with respect to ∞. Let $\delta = \inf_k |h(I_k)|$ be the smallest of these harmonic measures.

Our domain Ω is the union of $\mathbb{D}$, $2\mathbb{D}^* = \{z : |z| > 2\}$ and an ε-neighborhood of each γ_n, where ε is assumed to be so small that these neighborhoods are pairwise disjoint and $\partial\Omega$ has n components.

Let $f_\varepsilon : \Omega_\varepsilon \to \Omega_\varepsilon^*$ be the map given by Koebe's theorem. Normalizing by Möbius transformation we may assume $f(0) = 0$, $f(\infty) = \infty$ and $\text{dist}(0, \partial\Omega_\varepsilon) = 1$.

We claim that the n circles in the complement of Ω_ε^*, are all contained in some disk $D(0, R)$ with R independent of ε (but R may depend on h and n). To see this, suppose the union of closed disks satisfies $\cup_k D_k \subset \{1 \le |z| \le R\}$ and that it hits both boundary components. Let Ω_1 be the connected component of $f_\varepsilon(\Omega_\varepsilon \cap D(0,3/2))$ containing 0. Then for ε small enough, each interval I_k has harmonic measure $\ge 1/2n$ in Ω_1 and hence has capacity in Ω_1 which is bounded away from zero depending only on n. Thus by Lemma 17, every disk must hit $\{|z| \le M_1\}$, for some M_1 depending only on n. Similarly for Ω_2 (the connected component of $f_\varepsilon(\Omega \cap \{|z| > 3/2\})$ containing ∞); i.e., there is an M_2 depending only on δ such that every disk must hit $\{|z| = R/M_2\}$. If R is so large that $R/M_2 > 2M_1$, then every disk in our chain hits both $\{|z| = M_1\}$ and $\{|z| = 2M_1\}$. For large n this contradicts the following simple fact:

Lemma 20. *At most six disjoint disks can hit both* $\{|z| = 1\}$ *and* $\{|z| = 2\}$.

Proof. Each such disk has a subdisk of diameter 1 contained in the annulus $\{1 \le |z| \le 2\}$. Each of these intersects the circle $\{|z| = \sqrt{3}/2\}$ in an arc of angle measure $\pi/3$, and hence there can be at most six of them. □

Now we can pass to the limit as $\varepsilon \to 0$, passing to a subsequence where each disk converges, and we are done. □

Now that we have the finite approximations, we want to show they stay bounded as $n \to \infty$. The argument is similar to what we have just done. We will say a circle chain has ε-links if every disk has harmonic measure $\le \varepsilon$ with respect to both 0 and ∞.

Lemma 21. *Suppose* $h : \mathbb{T} \to \mathbb{T}$ *is an orientation-preserving homeomorphism such that for every set* E *of zero logarithmic capacity,* $h(\mathbb{T} \setminus E)$ *has positive logarithmic capacity. Then there is an* $R < \infty$ *and an* $\varepsilon > 0$ (*each depending only on* h) *so that for any normalized circle chain corresponding to* h *with* ε*-links,*

$$X = S^2 \setminus (\Omega \cup \Omega^*) \subset \{z : 1 \le |z| \le R\}.$$

Proof. Fix $R > 1$ and consider a normalized circle chain such that $X = S^2 \setminus (\Omega \cup \Omega^*) \subset A(1, R)$ and X intersects both boundary components of this annulus. Divide the (closed) disks in the circle chain into three collections: $\mathcal{C}_1$ are the disks which lie inside $D(0, \sqrt{R})$, $\mathcal{C}_2$ are the disks that lie outside $D(0, \frac{1}{2}\sqrt{R})$ and $\mathcal{C}_3$ are all the rest. By Lemma 20 there are at most six elements in $\mathcal{C}_3$. For $i = 1, 2, 3$, let $E_i = f^{-1}(\cup_{D \in \mathcal{C}_i} \partial\Omega_1 \cap D)$. Then E_2 has small logarithmic capacity depending only on R by Lemma 17, and E_3 has small capacity since it is a union of at most six intervals each of length $\le \varepsilon$. Similarly, $h(E_1)$ has small capacity depending only on R.

By choosing ε small enough and R large enough we could find such sets where $E_1 \cup E_2 \cup E_3 = \mathbb{T}$ and $\operatorname{cap}(E_2 \cup E_3) + \operatorname{cap}(h(E_1))$ is as small as we wish. But by Lemma 11, this contradicts our assumption on h, and so R must remain bounded as $\varepsilon \to 0$. Thus Lemma 21 is true. $\square$

Proof of Theorem 4. Choose a collection of n equally spaced points on $\mathbb{T}$ and use Lemma 19 to construct a sequence of normalized circle chain pairs $f_n : \mathbb{D} \to \Omega_n$, $g_n : \mathbb{D} \to \Omega_n^*$. By Lemma 21 there is an R so that the circle chains all remain inside $D(0, R)$, for large enough n (since h is uniformly continuous and harmonic measure $= 1/n$ on Ω, the harmonic measures $\to 0$ in Ω^*). Fix an $\varepsilon > 0$ and let $\{D_j^n\}$ be an enumeration of the at most $(R/\varepsilon)^2$ disks in the nth chain which have radius $\geq \varepsilon$. By passing to a subsequence we may assume the number is the same for every n, say $N \leq (R/\varepsilon)^2$. Let $E_j^n = f_n^{-1}(\partial D_j^n)$, $n = 1, \dots, N$. By passing to another subsequence we may assume that for a fixed $j = 1, \dots, N$, the intervals E_j^n converge to a point x_j. For $x \notin F_\varepsilon = \cup_{j=1}^N \{x_j\}$, x is eventually disjoint from every E_j^n and hence

$$\limsup_{n \to \infty} |f_n(x) - g_n(h(x))| \leq 2\varepsilon,$$

since $f_n(x)$ and $g_n(h(x))$ belong to the boundary of the same disk of radius $\leq \varepsilon$. Taking a sequence $\varepsilon_n \to 0$ and diagonalizing prove the theorem. $\square$

4. Quasisymmetric maps are conformal weldings

In this section we will use Koebe's circle domain theorem to show that if $h : \mathbb{T} \to \mathbb{T}$ is the boundary extension of a K-quasiconformal map of the disk to itself, then h is a conformal welding corresponding to a K-quasicircle, i.e., a curve Γ which is the image of $\mathbb{T}$ under a K-quasiconformal map of the plane. The only fact about quasiconformal maps we shall need is that normalized K-quasiconformal maps form a compact family. (This section is not used later, so can be skipped by readers interested only in Theorems 1-8.)

By a famous result of Beurling and Ahlfors [4], h is the boundary extension of a quasiconformal map if and only if it is quasisymmetric, i.e., there is an $M < \infty$ such that

$$M^{-1} \leq |h(I)|/|h(J)| \leq M,$$

whenever $I, J \subset \mathbb{T}$ are adjacent arcs of equal length. Thus every quasisymmetric map is a conformal welding. This well known fact was proved by Pfluger [36] using the measurable Riemann mapping theorem, and a different proof was given by Lehto and Virtanen [30], [31]. Our proof seems new, is fairly elementary and very geometric, so perhaps it will be of interest.

Given a homeomorphism h and n equidistributed points $\{x_k\}_1^n \subset \mathbb{T}$, let $y_k = h(x_k)$ for $k = 1, \dots n$ and consider the corresponding circle chain $\mathcal{C}_n$ as given by Lemma 19. As before, let Ω_n, Ω_n^* denote the bounded and unbounded complementary domains. By reflecting through each circle we obtain a new chain with $n(n-1)$ circles. Continuing in this way we obtain, in the limit, a Jordan curve Γ_n, with complementary components D_n (bounded) and D_n^* (unbounded). See Figure 4 which shows the original chain and the domain Ω_n on the left, three iterations of the reflections in the center and the corresponding domain D_n on the right.

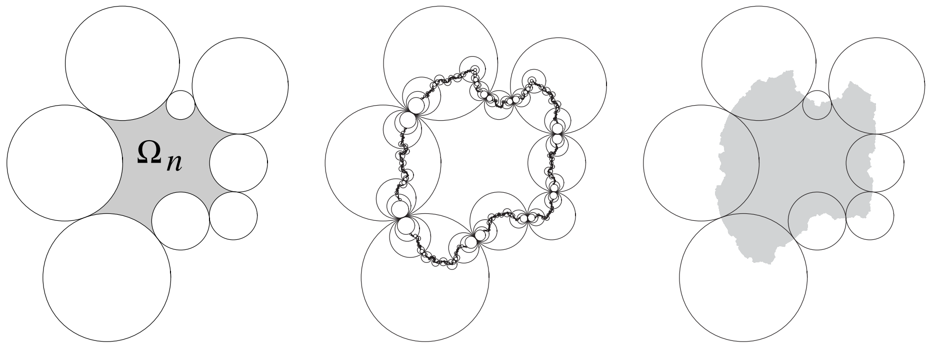

Figure 4: Reflections in a circle chain give a curve

Similarly, given a circle chain $\mathcal{D}_n$ of n circles of equal size, with tangent points along the unit circle, we can reflect through the circles, getting a nested sequence of circle chains which limit on the unit circle, as in Figure 5. We claim that if h is the boundary extension of a K-quasiconformal selfmap of the disk, then there is a K-quasiconformal map of the plane sending the circles in Figure 5 to those in Figure 4. We will prove this by constructing the map separately inside and outside the unit circle.

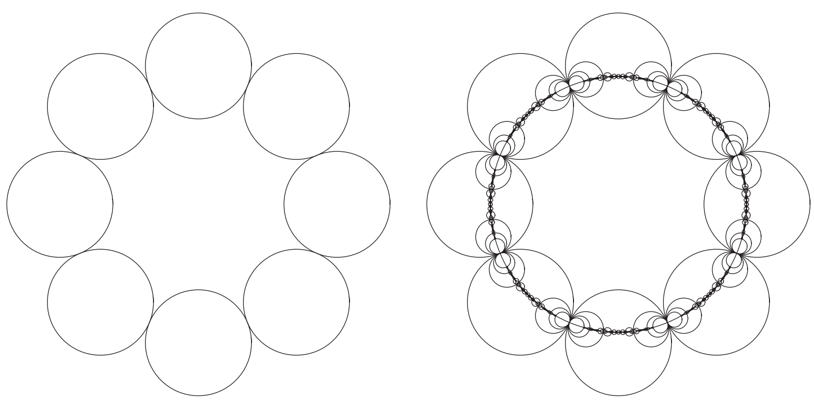

Figure 5: A symmetric circle chain with limit $\mathbb{T}$

Let $W_n = S^2 \setminus \{x_1, \ldots, x_n\}$. We may assume $n \geq 3$, so there is a universal covering map $\Pi : \mathbb{D} \to W_n$. Let U_n be the component of $\Pi^{-1}(\mathbb{D})$ containing the origin, and note that by symmetry U_n may be chosen to be bounded by hyperbolic geodesics with endpoints at the x_k's (the arcs $\mathbb{T} \setminus \cup\{x_k\}$ are hyperbolic geodesics in W_n; this is even clearer if we map $\mathbb{T}$ to $\mathbb{R}$ by a Möbius transformation). Reflecting these arcs across $\mathbb{T}$ gives the circle chain $\mathcal{D}_n$ in Figure 5 with $\{x_k\}_1^n$ as the points of tangency. The conformal map $f_n \circ \Pi : U_n \to \Omega_n$ can be extended by repeated Schwarz reflection to a conformal map $F_n : \mathbb{D} \to D_n$. See Figure 6.

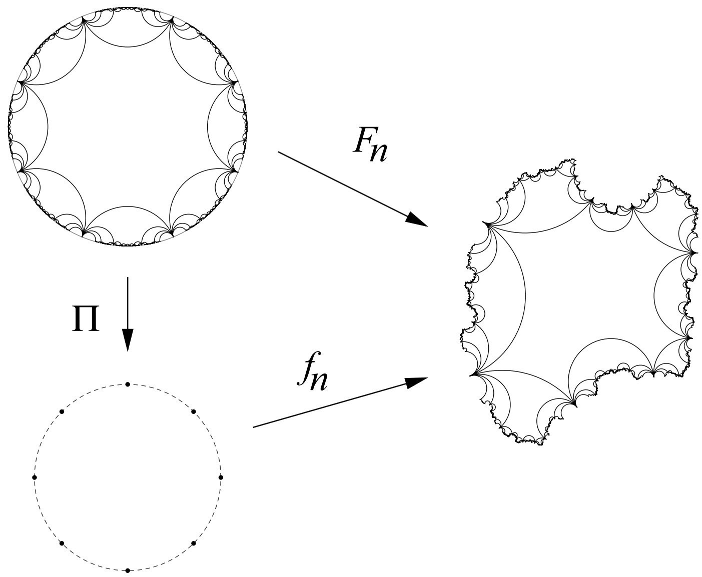

Figure 6: Lifting and extending the Koebe map

Similarly, Koebe's theorem gives a conformal map $g_n : \mathbb{D}^* \to \Omega_n^*$. Let $W_n^* = S^2 \setminus \{y_1, \ldots, y_n\}$ and consider $\Pi : \mathbb{D}^* \to W_n^*$ as the universal cover of W_n^*. As above, we can lift g_n to map of $\Pi^{-1}(\mathbb{D}^*) \to \Omega_n^*$ and use Schwarz reflection to extend it to a map G_n from $\mathbb{D}^* \to D_n^*$. See Figure 7.

By assumption h is the boundary extension of a K-quasiconformal map of the disk to itself. By reflection we can extend this as a K-quasiconformal map H of S^2 to itself. Then H maps W_n to W_n^* and lifts to a K-quasiconformal map of the universal covers. We can represent these by $\mathbb{D}^*$ so that we get a K-quasiconformal map $H_n : \mathbb{D}^* \to \mathbb{D}^*$ which conjugates the covering groups. See Figure 7.

Thus $G_n \circ H_n$ is a K-quasiconformal map of $\mathbb{D}^*$ to D^* whose boundary values agree with F_n on $\mathbb{T}$, and hence these maps together define a K-quasiconformal map of S^2 (easy to check using the analytic definition of quasiconformal in [2]). This map takes $\mathbb{T}$ to Γ_n and the circle chain $\mathcal{D}_n$ to the chain $\mathcal{C}_n$. Taking $n \to \infty$, using the uniform continuity of K-quasiconformal mappings and passing to a subsequence if necessary, we see that our circle chains converge uniformly to a K-quasicircle and that h is the corresponding conformal welding, as desired.

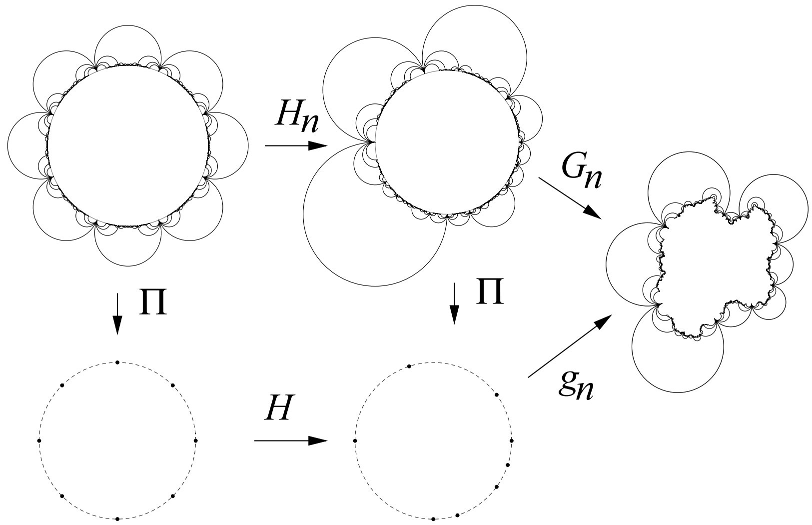

Figure 7: Lifting the maps H and g_n.

5. Passing to the limit: proof of Theorem 2

In this section we will prove Theorem 2. The idea is to take a sequence of map pairs $\{f_n, g_n\}$ as given by Theorem 4 and pass to a subsequence which converges uniformly on compact subsets of $\mathbb{D} \cup \mathbb{D}^*$ to maps f, g. We will then show these maps satisfy Theorem 2. As noted in Remark 5, general results are not enough to give convergence of boundary values off a set of zero log-capacity, so we will need to use special properties of our maps. The proof of Theorem 2 uses two lemmas. The first is a criterion for dividing a set E into subsets of zero capacity.

LEMMA 22. *Suppose $E \subset \mathbb{T}$ is compact, $0 < A < 1$ and $h : \mathbb{T} \to \mathbb{T}$ is a homeomorphism. Suppose that for every $r < 1$ there are Borel sets E_1 and E_2 with $E = E_1 \cup E_2$ and such that the two path families connecting $D(0, r)$ to E_1 and to $h(E_2)$ respectively, both have extremal length $\geq A$. Then there are Borel sets F_1 and F_2 with $E = F_1 \cup F_2$ such that both F_1 and $h(F_2)$ have logarithmic capacity zero.*

Proof. We claim we can choose sets $E_k \subset E$ and numbers $\{r_k\} \nearrow 1$ for $k = 1, 2, \ldots$ such that the extremal length of the path families $\mathcal{C}_k$ and $\mathcal{D}_k$ connecting $D_k = D(0, r_k)$ to E_k and D_k to $h(E \setminus E_k)$ are both greater than $A/2$. We also assume that we have chosen metrics ρ_k and σ_k which are admissible for these path families. i.e.,

$$\int_\gamma \rho_k ds \geq 1, \quad \gamma \in \mathcal{C}_k \qquad \text{and} \qquad \int_\gamma \sigma_k ds \geq 1, \quad \gamma \in \mathcal{D}_k$$

which almost maximize, i.e.,

$$\iint \rho_k^2 dxdy \leq 2/A, \qquad \iint \sigma_k^2 dxdy \leq 2/A.$$

Moreover, we claim we can choose metrics so that

$$\iint (\sum_{j=1}^{k} \rho_j)^2 dxdy < \frac{4k}{A}, \tag{5.1}$$

and the same inequality for the σ's.

For $k = 1$, we take $r_1 = 1/2$ and take E_1 as given by the hypothesis. We can then take ρ_1 and σ_1 by the definition of extremal length and (5.1) is trivial since there is only one term in the sum.

In general, suppose we have satisfied the induction hypothesis up to $n-1$. For any $r_n > r_{n-1}$ we can clearly choose a set E_n and metrics ρ_n and σ_n so that all the conditions are satisfied, except possibly for (5.1). However, since ρ_n is supported in $A_n = \{z : r_n < |z| < 1\}$,

$$\begin{aligned}\iint_{\mathbb{D}} (\sum_{k=1}^{n} \rho_k)^2 dxdy &\leq \iint_{\mathbb{D}} (\rho_n + \sum_{k=1}^{n-1} \rho_k)^2 dxdy \\ &\leq \iint_{\mathbb{D}} \rho_n^2 dxdy + 2\iint_{\mathbb{D}} \rho_n (\sum_{k=1}^{n-1} \rho_k) dxdy + \iint_{\mathbb{D}} (\sum_{k=1}^{n-1} \rho_k)^2 dxdy \\ &< \frac{2}{A} + 2(\iint_{A_n} \rho_n^2 dxdy)^{1/2} (\iint_{A_n} (\sum_{k=1}^{n-1} \rho_k)^2 dxdy)^{1/2} + \frac{4(n-1)}{A}.\end{aligned}$$

In the middle term of the last line we know

$$\iint_{A_n} (\sum_{k=1}^{n-1} \rho_k)^2 dxdy \to 0,$$

as $r_n \nearrow 1$, since we are integrating a fixed L^1 function over sets of smaller and smaller area. Thus for r_n close enough to 1 this term will be strictly less than $1/4$. Using this, the fact $A < 1$, and the induction hypothesis, we see that the sum above is less than

$$\frac{2}{A} + 2(\frac{2}{A})^{1/2}(\frac{1}{4})^{1/2} + \frac{4(n-1)}{A} \leq \frac{2}{A} + \frac{2}{A} + \frac{4(n-1)}{A} = \frac{4n}{A},$$

as desired. Taking r_n even closer to 1, if necessary, gives the same inequality for σ_n. This determines r_n and completes the inductive proof of our claims.

Now fix some integer N and consider the set F_N of points $x \in E$ which are in at least N of the sets $E_1, \ldots E_{2N}$ chosen above. Then points of $E \setminus F_N$ are in at least N of the sets $E \setminus E_n$, $n = 1, \ldots, 2N$. Consider the metric

$$\rho = \frac{1}{N} \sum_{k=1}^{2N} \rho_k.$$

This is clearly admissible for the family of paths connecting $D(0,1/2)$ to F_N and by (5.1),

$$\iint \rho^2 dxdy \leq \frac{1}{N^2}\sum_{k=1}^{2N}\iint \rho_k^2 dxdy \leq \frac{8}{NA}.$$

Thus the extremal length of this path family is $\geq \frac{1}{8}NA$ which is large (since A is fixed and N is as large as we please), similarly for the extremal length associated to $h(E \setminus F_N)$. Lemma 22 now follows from Lemmas 11 and 12. $\square$

The next step is to show that the set where our limit functions f, g fail to equal each other satisfies the hypotheses of the previous lemma.

LEMMA 23. *Given any $\delta > 0$ and $R < \infty$, there is a $c > 0$ so that the following holds. Suppose $f_n : \mathbb{D} \to \Omega_n$ and $g_n : \mathbb{D}^* \to \Omega^*$ is a normalized circle chain pair with $\partial\Omega_n^* \subset D(0,R)$ and assume $f_n \to f$ and and $g_n \to g$ uniformly on compact sets and that the number of disks in the n^{th} chain with diameter $\geq \varepsilon$ is at most $N(\varepsilon)$ (independent of n) for any $\varepsilon > 0$. Suppose $E \subset \mathbb{T}$ is such that f has radial limits on E, g has radial limits on $h(E)$ and for every $x \in E$, $|f(x) - g(x)| \geq \delta > 0$. Then for any $0 < r < 1$ sufficiently close to 1 there is a decomposition $E = F_1 \cup F_2$ such that the two path families connecting $D(0,r)$ to F_1 and to $h(F_2)$ each have extremal length $\geq c$.*

Proof. Suppose $0 < r < 1$. First consider the subsets $E_1, E_2 \subset E$ such that

$$E_1 = \{x \in E : |f(rx) - f(x)| \geq \delta/8\},$$
$$E_2 = \{x \in E : |g(rh(x)) - g(h(x))| \geq \delta/8\}.$$

By Lemma 14, the path families $\mathcal{C}_1$, $\mathcal{C}_2$ which connect these sets to $D(0,r)$ have extremal length bounded below by $\delta^2/C\tilde{a}(r)$, where

$$\tilde{a}(r) = \max(\text{area}(\Omega \setminus f(D(0,r))), \text{area}(\Omega^* \setminus g(D(0,r)))).$$

As $r \to 1$, $\tilde{a}(r) \to 0$, so these extremal lengths are as large as we wish, say ≥ 100.

Now consider $F = E \setminus (E_1 \cup E_2)$. By the uniform convergence on compact sets we can choose an integer n_1 (depending on r) so large that $n \geq n_1$ implies

$$|f_n(rx) - f(rx)| \leq \delta/8,$$
$$|g_n(rh(x)) - g(rh(x))| \leq \delta/8,$$

for all $x \in F$.

By our assumption on the chains, we can choose $n_2 > n_1$ so that $n \geq n_2$ implies $|f_n(x) - g_n(h(x))| \leq \delta/8$ for all $x \in F \setminus E_3$, where E_3 is a finite union of intervals (number depending only on δ) which are as short as we wish if n_2 is chosen large enough. In particular we can arrange for the extremal length of the path family $\mathcal{C}_3$ connecting $D(0,r)$ to E_3 to be as large as we wish, say ≥ 100.

Finally, for $x \in F \setminus E_3$ we must have

$$\begin{aligned}
&|f_n(rx) - f_n(x)| + |g_n(rh(x)) - g_n(h(x))| \\
&\quad \geq |f(x) - g(h(x))| - |f(x) - f(rx)| - |f(rx) - f_n(rx)| \\
&\qquad -|g(h(x)) - g(rh(x))| - |g_n(rh(x)) - g(rh(x))| - |f_n(x) - g_n(h(x))|.
\end{aligned}$$

See Figure 8. Since the first term on the right is $\geq \delta$ and the five other terms

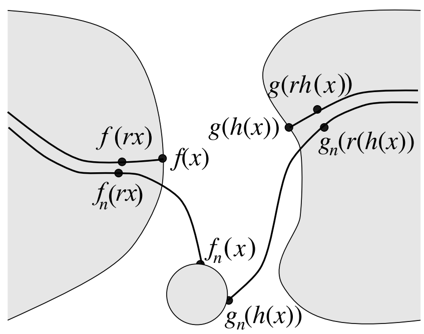

Figure 8: Estimating $|f_n(x) - g_n(x)|$

are all $\leq \delta/8$, we deduce that for every $x \in F \setminus E_3$ either

$$|f_n(rx) - f_n(x)| \geq \delta/8,$$

or

$$|g_n(rh(x)) - g_n(h(x))| \geq \delta/8.$$

Let E_4 and E_5 be the subsets of E where each of these inequalities occurs respectively and note that by Lemma 14 the corresponding path families $\mathcal{C}_4$, $\mathcal{C}_5$ connecting E_4 and $h(E_5)$ to $D(0,r)$ have extremal length $\geq \delta^2/C\tilde{a}_n(r)$, where

$$\tilde{a}_n(r) = \max(\text{area}(\Omega_n \setminus f_n(D(0,r))), \text{area}(\Omega_n^* \setminus g_n(D(0,r)))).$$

Unfortunately, we don't know that $\tilde{a}_n(r) \to 0$ uniformly as $r \nearrow 1$, but at least $\tilde{a}_n(r)$ is uniformly bounded above for large n (since Ω_n^* converges to Ω^*, it contains a uniform neighborhood of ∞ for large n). Thus these extremal lengths can be bounded below uniformly for large n (depending on r).

Thus we can write $E = F_1 \cup F_2 = (E_1 \cup E_3 \cup E_4) \cup (E_2 \cup E_5)$ where the path family for each set has the right estimate. By Lemma 16 we are done. $\square$

Proof of Theorem 2. Suppose $\{f_n\}$ and $\{g_n\}$ are the normalized circle pairs given by Theorem 4 and replace them (if necessary) by a subsequence converging uniformly on compact sets to conformal maps f and g. By a result of Beurling (also a consequence of Lemma 18) f and g each have radial boundary values except on a set of zero logarithmic capacity. Suppose E is a set where all the radial limits exist but $f \neq g \circ h$ on E. Let $E_\delta \subset E$ be the subset such that $|f(x) - g(h(x))| \geq \delta$. By Lemmas 22 and 23, we see that we can always find $F \subset E_\delta$ so that F and $h(E_\delta \setminus F)$ have zero logarithmic capacity. Since a countable union of zero capacity sets has zero capacity, taking $\delta = \delta_n \to 0$ shows that E itself has the desired decomposition. This completes the proof. □

6. Extending partial weldings: Proof of Theorem 8

Suppose $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ are maps onto disjoint domains, each with a closed Jordan curve as boundary (and assume $\infty \in \Omega^*$). Assume that $X = \partial\Omega \cap \partial\Omega^* \neq \emptyset$ and let $E = f^{-1}(X)$. Then on E we can define $h = g^{-1} \circ f$. We wish to prove that h can be extended to a conformal welding on all of $\mathbb{T}$.

We claim it suffices to find Jordan domains Ω_1 and Ω_1^*, with common boundary curve Γ, which contain Ω and Ω^* respectively and quasiconformal maps $\psi : \Omega \to \Omega_1$ and $\psi^* : \Omega^* \to \Omega_1^*$ which are the identity on X. By solving the Beltrami equation in the usual way we can find a quasiconformal map Ψ of the whole plane so that $\Psi \circ \psi$ and $\Psi \circ \psi^*$ are conformal. Then

$$F_1 = (\Psi \circ \psi \circ f), \qquad F_2 = (\Psi \circ \psi^* \circ g)$$

are conformal maps of $\mathbb{D}$ and $\mathbb{D}^*$ onto two sides of the Jordan curve $\Psi(\Gamma)$ and $H = (F_2)^{-1} \circ F_1$ is a conformal welding which restricted to E gives

$$H = (F_2)^{-1} \circ F_1 = (g)^{-1} \circ f = h,$$

as desired.

The first step is to note that each component of $\partial\Omega \setminus X$ and $\partial\Omega^* \setminus X$, may be assumed to be an analytic arc. This is because we can replace Ω by $f(W_E)$ and Ω^* by $g(W_{h(E)})$, where W_E denotes the sawtooth domain corresponding to E (see Section 2). Since W_E is a 2-quasiconformal image of $\mathbb{D}$, $f(W_E)$ is a 2-quasiconformal image of Ω, similarly for Ω^*. Thus if we can do the extension assuming analytic arcs, we can do it in general with just a larger constant; so we assume this is the case.

Now suppose $\mathcal{U}$ is the collection of components of $S^2 \setminus \overline{\Omega \cup \Omega^*}$. Then each $U \in \mathcal{U}$ is a Jordan domain and its boundary is the union of two analytic arcs with common endpoints $a, b \in X$. Map U conformally to the horizontal strip $S = \{(x+iy : -1 < y < 1, -\infty < x < \infty\}$ with the points a, b mapping to $\pm\infty$. Because U is bounded by analytic arcs, the inverse mapping can be extended analytically to some domain V of the form $V = \{x+iy : -\infty < x < \infty, |y| < 1+\eta(x)\}$, for some positive, continuous function η on $\mathbb{R}$. Let φ denote this conformal map on V. If we choose η small enough then we may assume the sets of the form $\varphi(V) \cap \Omega$ are all disjoint (e.g. lie in different components of $\mathbb{D} \setminus W_E$) as we range over different components in $\mathcal{U}$, similarly for $\varphi(V) \cap \Omega^*$.

Now divide $\mathbb{R}$ into disjoint (except for endpoints) intervals $\{I_n\}_{-\infty}^{\infty}$, such that

$$|I_n| \leq \frac{1}{10} \inf_{x \in 2I_n} \eta(x).$$

Form an oscillating curve γ in S by taking the union of horizontal and vertical line segments (see Figure 9)

$$\begin{aligned} V_n &= \{x+iy : -1 \leq y \leq 1, \quad x \text{ an endpoint of some } I_n\}, \\ H_n &= \{x+iy : y = -1, \quad x \in I_n, n \text{ even }\}, \\ H_n &= \{x+iy : y = 1, \qquad x \in I_n, n \text{ odd }\}. \end{aligned}$$

For each n let c_n denote the center of H_n and consider the rectangle

$$R_n = \{x+iy : -1 \leq y \leq 1, x \in I_n\},$$

and the semicircles

$$S_n = \{x+iy : |z - c_n| \leq |I_n|/2, y > 1\},$$

if n is even and

$$S_n = \{x+iy : |z - c_n| \leq |I_n|/2, y < -1\},$$

if n is odd. See Figure 9.

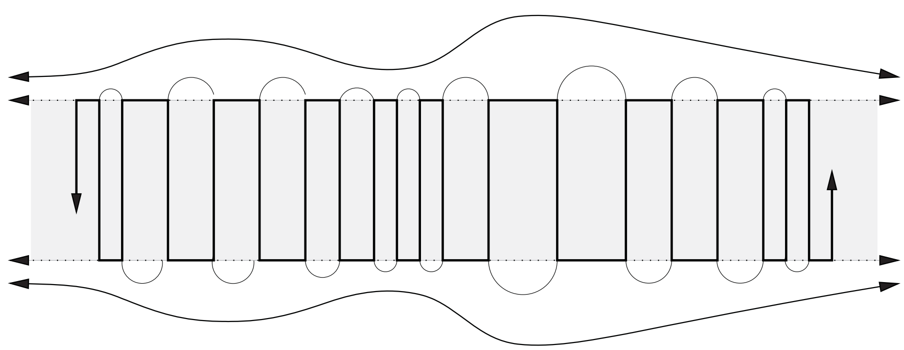

Figure 9: An oscillating curve which “fills” the strip

There is a simple lemma that states there is a uniform C such that $R_n \cup S_n$ is a C-quasiconformal image of S_n by a map which equals the identity on the circular arc in ∂S_n.

Given the component U we can define new domains $\Omega_1 = \Omega \cup \cup_{n \text{ even}} \varphi(R_n)$, and $\Omega_1^* = \Omega^* \cup \cup_{n \text{ odd}} \varphi(R_n)$. Do this for every component in $\mathcal{U}$. Then clearly the resulting domains are the two complementary components of a Jordan curve and are C-quasiconformal images of Ω and Ω^* by maps that are the identity on E. This completes the proof of Theorem 8.

7. Characterization of flexible curves: proof of Theorem 3

We start with a few comments on the definition of flexible curves. Suppose γ and γ' are closed Jordan curves, $\varepsilon > 0$, F is a homeomorphism of the sphere which is conformal off γ, z_1, z_2 are points in each of the two complementary components of γ and w_1, w_2 are points in each of the two complementary components of γ'. Then three possible definitions of γ being flexible are

(1) For any $\varepsilon > 0$ there is an F so that $F(\gamma)$ approximates γ' to within ε in the Hausdorff metric.

(2) Assume (1) holds and in addition, there are fixed points z_1, z_2 so that for any w_1, w_2, we can choose F so that $F(z_1) = w_1$ and $F(z_2) = w_2$.

(3) Assume (1) holds and in addition, for any points z_1, z_2 and any w_1, w_2, we can choose F so that $F(z_1) = w_1$ and $F(z_2) = w_2$.

The third definition is the one we gave in the introduction. However, the proof of Lemma 24 below shows that the second definition implies h is log-singular, while Theorem 25 below shows that log-singularity implies the third definition. Thus these two are equivalent. The first definition above was used in [7] (although the construction of examples given there yields the stronger definitions), but is not equivalent. One can show that if h is a conformal welding which is log-singular on some nondegenerate interval, then the corresponding curve satisfies the first definition above (one can map the corresponding arc of γ to approximate most of γ' and map the rest of γ into a small ball near γ'). I thank Steffan Rohde and Don Marshall for pointing out an error in an earlier version of this paper concerning the definition of flexible curves and for suggesting the correct alternative.

Now we proceed with the proof of Theorem 3. We start with the "easy" direction:

LEMMA 24. *Suppose h is a conformal welding homeomorphism associated to a flexible curve. Then there is a set $E \subset \mathbb{T}$ such that both E and $h(\mathbb{T} \setminus E)$ have zero logarithmic capacity.*

Proof. Suppose n is large and let $\Gamma_n = \partial W$ where $W = [-1, n^2] \times [-1, 1]$. Since h corresponds to a flexible curve, there is a curve corresponding to h which lies within Hausdorff distance 1/4 of Γ. Let $f : \mathbb{D} \to \Omega$ be a conformal map onto the bounded complementary component of Γ with $f(0) = 0$. Let

$E = f^{-1}(\{x + iy \in \Gamma : x < n\})$. Then both E and $h(\mathbb{T} \setminus E)$ have small logarithmic capacity by Lemma 17, depending only on n. Taking $n \to \infty$ and applying Lemma 11, we are done. $\square$

Next we prove the opposite direction. Although stated slightly differently, the following implies the rest of Theorem 3.

THEOREM 25. *Suppose h is an orientation-preserving homeomorphism and that there is a Borel set E such that both E and $h(\mathbb{T} \setminus E)$ have zero logarithmic capacity. Suppose there are two conformal maps $F : \mathbb{D} \to \Omega$ and $G : \mathbb{D}^* \to \Omega^*$ onto disjoint domains such that $\infty \in G(\mathbb{D}^*)$. Then for any $r < 1$ and any $\eta > 0$, there are conformal maps f and g of $\mathbb{D}$ and $\mathbb{D}^*$ onto the two complementary components of a Jordan curve Γ such that $h = g^{-1} \circ f$ on $\mathbb{T}$, $|f(z) - F(z)| \leq \eta$ for all $|z| \leq r$ and $|g(z) - G(z)| \leq \eta$ for all $|z| \geq 1/r$.*

Before giving the proof, we first describe in more detail how this implies Theorem 3. Suppose h is log singular. We have to show h is the conformal welding of some curve γ and that given a closed Jordan curve γ' there is a homeomorphism of the sphere conformal off γ which maps γ to within ε of γ' and which also maps any two prescribed points (one on either side of γ) to any given points on either side of γ'.

Theorem 25 applied to any two suitable maps F and G (say the identity maps) gives that h is a conformal welding corresponding to some curve γ. Suppose f_1 and g_1 are conformal maps onto the two sides of γ which give $h = g_1^{-1} \circ f_1$. Suppose z_1 and z_2 are the given points on either side of γ, and w_1, w_2 are the two points on either side of γ'. Fix some $\varepsilon > 0$.

We will apply Theorem 25 when F and G are conformal maps of $\mathbb{D}$ and $\mathbb{D}^*$ onto the two sides of γ' which map $x_1 = f_1^{-1}(z_1)$ to w_1 and $x_2 = g_1^{-1}(z_2)$ to w_2. Since F and G map onto Jordan domains we can choose $r < 1$ close enough to 1 so that $F(D(0,1)) \setminus F(D(0,r))$ is contained in an $\varepsilon/4$ neighborhood of γ', similarly for G. Also assume r is so close to 1 that $|x_1| < r$ and $|x_2| > 1/r$.

Then from Theorem 25 we get maps f and g which map onto two sides of a Jordan curve γ'' and so that f and g approximate F and G to within ε/C for $|z| < r$ and $|z| > 1/r$ respectively, where we choose

$$C = \max(4, 8(\operatorname{diam}(\gamma') + 1)|w_1 - w_2|^{-1}).$$

Since f and g approximate F and G to within $\varepsilon/4$ for $|z| < r$ and $|z| > 1/r$ respectively, γ'' must lie in an $\varepsilon/2$ neighborhood of γ' (otherwise either f maps a point of $|z| = r$ outside this neighborhood or g maps a point of $|z| = 1/r$ outside it; in either case this contradicts our assumptions).

Moreover, f and g map the special points x_1 and x_2 to within $\varepsilon|w_1 - w_2|/(8 + 8\operatorname{diam}(\gamma')$ of the desired image points w_1 and w_2. Thus there is a Euclidean similarity which maps $f(x_1)$ and $g(x_2)$ to w_1 and w_2 respectively, while moving points on the curve γ'' by less than $\varepsilon/2$. Thus by composing f and g with this similarity, we may assume $f \circ f_1^{-1}$ and $g \circ g_1^{-1}$ define conformal maps from the two sides of γ to the two sides of a curve γ''' and they extend to a homeomorphism of the sphere since the maps agree on γ (since both γ and γ''' correspond to h). Furthermore, γ''' approximates γ' to within ε and the given points z_1, z_2 map to the given points w_1, w_2. Thus γ is a flexible curve. This proves the remaining half of Theorem 3.

Proof of Theorem 25. It actually suffices to construct f and g which are quasiconformal with constant arbitrarily close to 1; we can then solve the Beltrami equation to obtain conformal maps which give the same welding and which are uniformly close to the quasiconformal maps (depending only on the size of the QC constant).

We will build f and g by an inductive construction: at the nth step we will have K_n-quasiconformal maps $f_n : \mathbb{D} \to \Omega_n$ and $g : \mathbb{D} \to \Omega_n^*$ onto smooth Jordan domains with disjoint closures. Given any $\varepsilon_0 > 0$ we can choose the quasi-constants to satisfy $K_0 = 1$ and $|K_{n+1} - K_n| \leq \varepsilon_0 2^{-n}$, which means that the limiting maps will be $(1 + C\varepsilon_0)$-quasiconformal.

The topological annulus $A_n = S^2 \setminus (\Omega_n \cup \Omega_n^*)$ is foliated by a family hyperbolic geodesics $\mathcal{C}_n$ so that for each $x \in \mathbb{T}$ there is exactly one element $\gamma_n(x) \in \mathcal{C}_n$ which connects the points $f_n(x) \in \partial\Omega_n$ and $g_n(h(x)) \in \Omega_n^*$. Moreover, we will show the length ℓ of γ_n will satisfy

$$\ell(\gamma_n(x)) \leq \frac{2}{3}\ell(\gamma_{n-1}(x)).$$

Since obviously

$$|f_n(x) - g_n(h(x))| \leq \ell(\gamma_n(x)),$$

this implies

$$|f_n(x) - g_n(h(x))| \leq (\frac{2}{3})^n L, \tag{7.1}$$

where L is the maximum length of any curve in $\mathcal{C}_0$. We will also show that for all x

$$|f_n(x) - f_{n-1}(x)| \leq (\frac{2}{3})^n L, \tag{7.2}$$

$$|g_n(x) - g_{n-1}(x)| \leq (\frac{2}{3})^n L. \tag{7.3}$$

Moreover, for any sequence $\delta_0 > \delta_1 > \delta_2 > \cdots \to 0$, we can choose our initial maps so that

$$|F(x) - f_0(x)| \leq \delta_0, \quad |z| \leq 1 - \delta_0, \tag{7.4}$$

$$|G(x) - g_0(x)| \leq \delta_0, \quad |z| \geq 1 + \delta_0. \tag{7.5}$$

(7.6) $$|f_{n-1}(x) - f_n(x)| \le \delta_n, \quad |z| \le 1 - \delta_n, n = 1, 2, \dots,$$

(7.7) $$|g_{n-1}(x) - g_n(x)| \le \delta_n, \quad |z| \ge 1 + \delta_n, n = 1, 2, \dots .$$

First suppose we can construct sequences with these estimates. Then by the Weierstrass M-test, (7.2) and (7.3) imply f_n and g_n converge on $\overline{\mathbb{D}}$ and $\overline{\mathbb{D}^*}$ respectively to continuous functions f and g which map $\mathbb{D}$ and $\mathbb{D}^*$ onto disjoint domains with locally connected boundaries. Equation (7.1) implies $f(x) = g \circ h(x)$ for all $x \in \mathbb{T}$, which implies f and g are both one-to-one on $\mathbb{T}$ and hence these are maps onto two sides of a Jordan curve. Finally, equations (7.4) to (7.7) imply that f and g can be taken to approximate F and G as closely as we want on compact sets, as desired. Thus it suffices to build the sequences as described.

To begin the induction, suppose $t < 1$ and let $f_0(z) = F(tz)$ and $g_0(z) = G(z/t)$. Choose t so close to 1 that equations (7.4) and (7.5) are satisfied. We let $\Omega_0 = f_0(\mathbb{D})$, $\Omega_0^* = g_0(\mathbb{D}^*)$ and $A_0 = S^2 \setminus (\Omega_0 \cup \Omega_0^*)$. Let $\mathcal{C}_0$ be the collection of hyperbolic geodesics connecting points $f_0(x) \in \partial\Omega_0$ to $g_0(h(x)) \in \partial\Omega_0^*$ in A_0. Since A_0 has analytic boundary, these curves have length bounded by some $L < \infty$. This completes the initial step of the induction.

Now suppose we have verified the induction hypotheses up to step n. We describe the construction of f_{n+1}. We will not give the details for g_{n+1}, but the construction is similar, with only typographical changes (i.e., replace f_n by g_n, Ω by Ω^*, E by $F = h(\mathbb{T} \setminus E)$, $\dots$).

We are going to build f_{n+1} as a composition of two maps, Φ and Ψ. The first map Φ will be a conformal map from $\mathbb{D}$ to a simply connected domain W which roughly looks like a disk with large radius R with finitely many radial slits, many of which connect the boundary of the disk to the circle of radius 1. See Figure 10. Moreover, we may choose Φ so that on compact subsets of the disk it is as close to the identity as we wish and so that Φ maps a given set E of zero capacity close to the circle of radius R. The second map, Ψ, is a quasiconformal extension (with constant close to 1) of f_n from $\mathbb{D}$ to a domain V containing W (in fact, V will equal W with some of its radial slits added back in). The image $D = \Psi(V)$ is a domain containing $\Omega_n = f_n(\mathbb{D})$ and contained in a larger smooth domain $\tilde{\Omega}_n$ and consists of $\tilde{\Omega}_n$ minus a finite number of smooth slits (which are all subarcs of arcs from our collection γ_n). Then $f_{n+1} = \Psi \circ \Phi$ is quasiconformal with a small constant, is close to f_n on compact subsets of the disk, and is closer to $g \circ h$ on the set E. See Figure 10.

Construction of Φ. The mapping Φ will be of the form $\exp(U + i\tilde{U})$ where $U(z) = G(z) + G(1/\bar{z})$ and G is the potential of a measure supported on some set $E \subset \mathbb{T}$. The map is similar to the one in Proposition 9.15 of Pommerenke's book [39], where the measure is taken to be equilibrium measure for E. However, for technical reasons it will be easier for us to take a slightly different measure here, namely a sum of equilibrium measures for pieces of E.

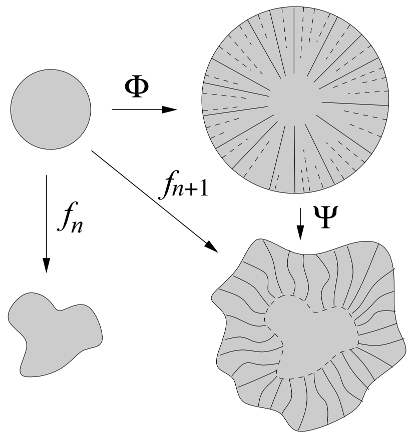

Figure 10: The maps Φ and Ψ

Suppose N is a large integer (to be chosen later) and divide $\mathbb{T}$ into N equilength intervals $\{I_k\}$. Given any small $\varepsilon_1 > 0$ we can, by hypothesis, choose a set $E \subset \mathbb{T}$ which is a finite union of intervals, so that $\text{cap}(E_k) = \text{cap}(I_k \cap E) < \varepsilon_1$ and $\text{cap}(F_k) = \text{cap}(h(I_k \setminus E)) < \varepsilon_1$. By taking ε_1 small enough we can make the Robin constant of E_k as large as we wish, say $\geq A$ (A will be chosen later with $A \gg N$). By enlarging E_k in each I_k, if necessary, we can make the Robin constant of E_k equal to A. Similarly we may assume the Robin constant of F_k is A.

Put the equilibrium measure μ_k of mass $1/N$ on each E_k and let $\mu = \sum_k \mu_k$. Now consider the potential

$$(7.8)\ G(z) = \int \log \frac{1}{|x-z|} d\mu(x) = \sum_k G_k(x) = \sum_k \int \log \frac{1}{|x-z|} d\mu_k(x).$$

By standard potential theory $NG_k(z) - A$ is the Green's function for $\Omega_k = S^2 \setminus E_k$ with pole at ∞ (and hence is zero on E_k). Thus $G_k(x) = A/N$ on E_k.

Lemma 26. *G is continuous on $\mathbb{R}^2$ and harmonic on $\mathbb{R}^2 \setminus E$ and*

(1) *$G(z) \to \log |z|^{-1}$ for $|z| > 1$ and $G(z) \to 0$ for $|z| < 1$ as $N \to \infty$.*

(2) *For any $\delta > 0$ and any interval I, $|\{x \in I : |G(x)| > \delta\}|/|I| \to 0$ as $N \to \infty$.*

The rate of convergence is independent of A, if A is large enough.

Proof. That G is continuous and harmonic off E are standard results. Condition (1) holds since μ clearly converges weakly to normalized Lebesgue measure on $\mathbb{T}$ and $\int \log |z-x|^{-1} dx = 0$ if $|z| \leq 1$ and $= \log |z|^{-1}$ if $|z| > 1$.

To prove (2), fix $x \in \mathbb{T}$ and assume the N intervals in the definition of μ are relabeled so that $x \in I_0$, so that I_1, I_2 are adjacent to I_0 and so that $\mathrm{dist}(I_k, x) \simeq k/N$ for $k = 3, \dots, N-1$. Let $I = I_0 \cup I_1 \cup I_2$. Then

$$\begin{aligned} G(x) &= \int_I \log \frac{1}{|z-x|} d\mu(z) + \sum_{k>2} \int_{I_k} \log \frac{1}{|z-x|} d\mu(z) \\ &= H_1(x) + H_2(x). \end{aligned}$$

Note that $H_1 > 0$ on I if $N \geq 12$. If we integrate H_1 over I and apply Fubini's theorem, then

$$\begin{aligned} \int_I H_1 dx &\leq \int_I \log \frac{1}{|z-x|} d\mu(z) |dx| \\ &\leq \mu(I) \max_{z \in I} \int_I \log \frac{1}{|z-x|} |dx| \\ &\leq C \frac{\log N}{N^2}. \end{aligned}$$

Thus by Tchebyshev's inequality

$$|\{x \in I : H_1(x) \geq \frac{\lambda}{2} \frac{\log N}{N}\}| \leq C \frac{|I|}{\lambda} = C \frac{1}{N\lambda}.$$

To estimate H_2, note that for $k > 2$, the variation of $\log |z-x|^{-1}$ over I_k is at most C/k. Thus

$$\begin{aligned} |H_2(x)| &\leq |\sum_{k>2} \int_{I_k} \log \frac{1}{|z-x|} (d\mu(z) - \frac{|dz|}{2\pi})| + |\sum_{k>2} \int_{I_k} \log \frac{1}{|z-x|} \frac{|dz|}{2\pi}| \\ &\leq \sum_{k>2} 2 \frac{k}{N} + \int_I \log \frac{1}{|z-x|} \frac{|dz|}{2\pi} \leq C \frac{\log N}{N}, \end{aligned}$$

where we have also used the fact that $\int_{\mathbb{T}} \log |z-x|^{-1} |dz| = 0$ for $x \in \mathbb{T}$. Thus for $\lambda = 2\delta N / \log N$,

$$|\{x \in I_0 : G(x) \geq \delta\}| \leq \frac{C|I| \log N}{\delta N},$$

which implies the second conclusion. □

Also note that these estimates also prove that for $x \in E$,

$$G(x) \geq \frac{A}{N} - O(\frac{\log N}{N}). \tag{7.9}$$

Now symmetrize G by setting $U(z) = G(z) + G(z/|z|^2)$. Then U is harmonic on Ω, equal to $2G(x)$ on E and has negative logarithmic poles at 0 and ∞. Since U is symmetric with respect to $\mathbb{T}$ it has normal derivative zero on $\mathbb{T} \setminus E$.

Lemma 27. *We can choose a (multi-valued) harmonic conjugate $\tilde{U}$ of U on $\mathbb{D}$ so that $\exp(i\tilde{U}(x)) \to x$ uniformly as $N \to \infty$ (with an estimate independent of A).*

Proof. Since U is symmetric with respect to $\mathbb{T}$, it has normal derivative zero on $\mathbb{T} \setminus E$, and hence $\tilde{U}$ is constant on each component of $\mathbb{T} \setminus E$. On each component I of E, we have

$$\int_I \frac{d\tilde{U}}{d\theta} d\theta = \int_I \frac{dU}{dn} d\theta = \frac{1}{2} \int_I \Delta U d\theta = \int_I \Delta G d\theta = 2\pi\mu(I).$$

Because of our choice of μ, we see that we can choose $\tilde{U}$ so that $\exp(i\tilde{U}(x)) = x$ at N equidistributed points around the circle. Since $\tilde{U}$ is monotonic, this implies the desired result. $\square$

Note that $\varphi(z) = \exp(U(z) + i\tilde{U}(z))$, is a conformal map of the disk onto a region W which is a Jordan region $\tilde{W}$ with a finite number of radial slits removed. By (7.9) $\tilde{W}$ contains the disk $D(0, \exp((2A/N) - 1))$ if N is large enough and W contains the disk $D(0, 1 - C \log N/N)$ for some $C < \infty$. On the other hand, every interval of length $1/N$ must contain a point x where $U(x) < C \log N/N$ and hence every such interval contains a point where $|\varphi(x)| \leq 1 + C(\log N)/N$. (If we had taken μ to be the equilibrium measure for all of E then the domain W would be a disk with radial slits. On the other hand, Lemmas 26 and 27 would have been harder to prove. This is the technical reason mentioned above for choosing μ as we did.)

Now suppose M is a large integer and $\delta > 0$ is a small real number. If A and N are large enough we can choose a collection of M critical points $\{x_k\}$ of G on $\mathbb{T}$ so that

$$|x_k - \exp(2\pi i/M)| \leq \delta,$$

and such that

$$1 - \delta \leq \varphi(x_k) \leq 1 + \delta,$$

for all $k = 1, \dots, M$. Finally let $\Phi = \varphi/(1 + \delta)$.

Construction of Ψ. Let W_k be the interior of the closure of the part of $W = \Phi(\mathbb{D})$ separated from 0 by the arc of the circle $\mathbb{T}$ between the angles corresponding to x_k and x_{k+1}. Then W_k can be conformally mapped to the 1 by R_k rectangle $\{x + iy : 0 < y < 1, 0 < x < R_k\}$ with x_k and x_{k+1} going to the vertices 0 and i respectively and the radial sides of W_k corresponding to x_k and x_{k+1} going to the horizontal sides of the rectangle. Moreover, given any large enough R, by (7.9) we can take $R_k \geq R$ for all k if δ is small enough and A, N are large enough.

Next recall that the annulus $A_n = S^2 \setminus (\Omega_n \cup \Omega_n^*)$ is foliated by a family of hyperbolic geodesics $\mathcal{C}_n$. For each curve $\gamma \in \mathcal{C}_n$ let x denote the midpoint (with respect to arclength measure). Since $\mathcal{C}_n$ is a smooth foliation, the set of such points is a smooth curve Γ_n which separates the two boundary components of A_n. Given a pair of points $x_1, x_2 \in \mathbb{T}$, we form a topological quadrilateral $Q(x_1, x_2)$ whose four sides are:

(1) the arc on $\partial\Omega_n$ from $f_n(x_1)$ to $f_n(x_2)$,

(2) the subarc of $\gamma_n(x_2)$ joining $f_n(x_2)$ to a point $y_2 \in \Gamma_n$,

(4) the arc of Γ_n from y_2 to y_1 (the intersection of Γ_n and $\gamma(x_1)$),

(3) the arc of $\gamma_n(x_1)$ joining y_1 to $f_n(x_1)$.

See Figure 11.

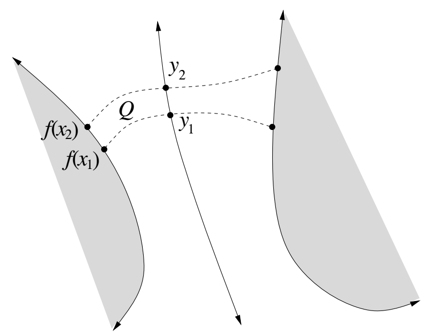

Figure 11: Defining the quadrilateral $Q(x_1, x_2)$

We will assume x_1 and x_2 are so close together that several conditions hold. First, near $\partial\Omega_n$, the quadrilateral is a $1 + 2^{-n}\varepsilon_0$ quasiconformal image of a rectangle (this uses the fact that our arcs are geodesics and hence are smooth and perpendicular to the boundary). Second, given a fixed η, we can assume x_1 and x_2 are so close together that the side of $Q(x_1, x_2)$ along Γ_n has length $\leq \eta$. Moreover, $Q(x_1, x_2)$ can be conformally mapped to a rectangle $\{x + iy : 0 < x < T, 0 < y < 1\}$ with corners going to corners. The value $T = T(x_1, x_2)$ depends continuously on the choice of x_1, x_2 and tends to ∞ as $|x_1 - x_2| \to 0$. We can define a similar function $T^*(x_1, x_2)$ but for quadrilaterials with vertices $g(x_1), g(x_2) \in \partial\Omega^*$ and $y_1, y_2 \in \Gamma_n$. For each $\delta > 0$ we let

$$T(\delta) = \max\{T(x_1, x_2) + T^*(h(x_1), h(x_2)) : |x_1 - x_2| \geq \delta\}. \tag{7.10}$$

Make A compared to N that $R_k > T(1/2M)$ for every k, where T is as given in (7.10).

Since $R_k \geq T_k = T(x_k, x_{k+1})$, we can map our $1 \times R_k$ rectangle into our $1 \times T_k$ rectangle with the "left" side mapping bijectively to the left side and the "right" side mapping into (but not necessarily onto) the right side. Now map it to the quadrilateral Q_k by a conformal map. This gives us a conformal map of our kth "radial sector" D_k to our kth quadrilateral Q_k. By making the map quasiconformal with constant $1 + 2^{-n}\varepsilon_0$ we can glue it to the map $f : \mathbb{D} \to \Omega_n$ so that every point of E is mapped into the curve Γ_n; indeed, it is mapped to within η of the point where $\gamma(x)$ crosses Γ_n. This defines the map Ψ.

Now, repeating the whole argument for Ω^* and we get a $(1+2^{-n}\varepsilon_0)$-quasiconformal map g_{n+1} on the unit disk which maps each point of $h(\mathbb{T} \setminus E)$ to within η of the point where $\gamma(x)$ crosses Γ_n. Thus for every $x \in \mathbb{T}$, $f_{n+1}(x)$ and $g_{n+1}(x)$ can now be joined by a curve of length $\leq \ell(\gamma_n(x))/2 + C\varepsilon$ in $S^2 \setminus (f_{n+1}(\mathbb{D}) \cup g_{n+1}(\mathbb{D}))$. This is still true if we replace f_{n+1} and g_{n+1} by $f_{n+1}(tz)$ and $g_{n+1}(z/t)$ with $t < 1$ close enough to 1, and then our maps are onto smooth, disjoint Jordan domains. We let $\Omega_{n+1} = f_{n+1}(t\mathbb{D})$ and $\Omega^*_{n+1} = g_{n+1}(\mathbb{D}^*/t)$. Finally, join $f_{n+1}(tx)$ to $g_{n+1}(h(x)/t)$ by the hyperbolic geodesic in $A_{n+1} = S^2 \setminus (\Omega_{n+1} \cup \Omega^*_{n+1})$. To see that this geodesic also has length $\leq \ell(\gamma_n(x))/2 + C\varepsilon$, we apply the following lemma.

Lemma 28. *Suppose Γ is a smooth closed Jordan curve and Ω, Ω^* are simply connected domains obtained from the complementary components of Γ by removing a finite number of smooth, disjoint arcs $\{\gamma_k\}$ each disjoint from Γ except for one endpoint on Γ. Let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be conformal maps and let $A_t = S^2 \setminus (f(t\mathbb{D}) \cup g(\frac{1}{t}\mathbb{D}^*)$). Then for any $\varepsilon > 0$ there is a $t < 1$ so that the hyperbolic geodesic in A_t connecting the boundary points $f(tx)$ and $g(y/t)$ has length at most $L(x, y)+\varepsilon$, where $L(x, y)$ is the length of the shortest path connecting $f(x)$ and $g(y)$ in $X = \partial\Omega \cup \partial\Omega^*$.*

Proof. If t is close enough to 1, then locally A_t looks like a strip, except at the finitely many tips of the arcs γ_k and at the finitely many points where they intersect Γ. See Figure 12. Suppose there are M such tips and intersection points and cover each by an $\varepsilon/(2M)$ disk. If γ is a geodesic in A_t, then inside these disks we can estimate its length by the Gehring-Hayman inequality (see proof of Lemma 14) to be less than $C\varepsilon/M$. Away from these points, A_t locally looks like a strip and normal families imply the Euclidean length of γ is close to the corresponding length on X. □

Also note that the argument above shows that as $t \to 1$, the geodesic from $f(x)$ to $g(h(x))$ in A_t converges (in the Hausdorff metric) to the path in X between these points. This observation will be used in the proof of the next lemma.

This completes the proof of the induction step and hence completes the proof of Theorem 25 (and hence Theorem 3). □

In the next section we will need a slightly stronger version of Theorem 25, which follows from the proof we have already given. It says that if F and G

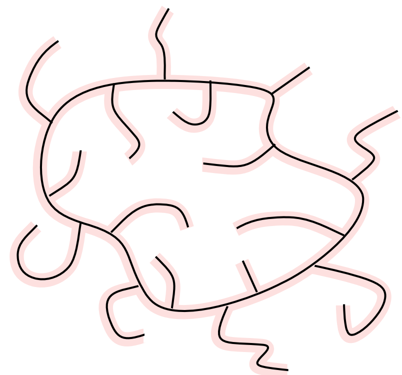

Figure 12: Γ and A_t in Lemma 28

map onto smooth Jordan domains then the maps f and g are in some sense uniform approximations to F and G, if we quotient out by the arcs $\gamma(x)$.

LEMMA 29. *Suppose that F and G in Theorem 25 map onto smooth domains Ω, Ω^* with disjoint closures and that $\gamma(x)$ is the hyperbolic geodesic in $S^2 \setminus \overline{\Omega \cup \Omega^*}$ which connects $F(x)$ to $G(h(x))$. Then for any $\eta > 0$, choose the map f in Theorem 25 so that for $z \in \overline{\mathbb{D}}$, and $x = z/|z| \in \mathbb{T}$,*

$$\mathrm{dist}(f(z), F(z) \cup \gamma(x)) \leq \eta. \tag{7.11}$$

Also, choose g so that for $z \in \overline{\mathbb{D}^*}$, and $x = z/|z| \in \mathbb{T}$

$$\mathrm{dist}(g(z)), G(z) \cup \gamma(h^{-1}(x)) \leq \eta.$$

Proof. Given the function F, we will show that the first approximation $f_1 = \Psi \circ \Phi$, constructed in the proof of Theorem 25, satisfies the desired estimates. We will then iterate the argument to obtained the desired result. The argument for g is similar, so we will not give it here.

By compactness, given $\eta > 0$ we can find $0 < \rho < \eta$ so that if $\gamma(y)$ contains a point within ρ of $\gamma(x)$ then $\gamma(y)$ is itself contained in an $(\eta/4)$-neighborhood of $\gamma(x)$.

Given the map F and $\rho > 0$, we can choose $s > 0$ so small that $|z - w| < s$ implies $|F(z) - F(w)| < \rho/4$ for all $z, w \in \overline{\mathbb{D}}$. By taking s smaller, if necessary, we can also assume that $x, y \in \mathbb{T}$, $|x - y| \leq s$ implies $\gamma(y)$ lies in a ρ-neighborhood of $\gamma(x)$. Thus if the points $\{x_k\}$ in the definition of Φ are chosen to be less than $s/4$ apart, then any of our regions W_k which intersect the interval of length $s/2$ centered at x must map under Ψ into a ρ-neighborhood of $\gamma(x)$.

By Lemmas 26 and 27, given any $s > 0$, we can choose Φ as close to the identity on $D(0, 1 - s)$ as we wish. In particular, we may assume that

$|z| \leq 1 - s/2$ implies $|\Phi(z) - z| \leq s/4$. Thus $|f_1(z) - z| \leq \rho/4 < \eta$ for such points.

This choice also implies that $|\Phi(z)| > 1 - s$ for $|z| > 1 - s/2$. By Lemma 27, we can choose Φ so that $\arg(z)$ and $\arg(\Phi(z))$ are as close as we wish. Thus if $|z| > 1 - s/2$, then z is either mapped to within s of itself, or is mapped into one of the domains W_k whose boundary comes within $s/4$ of x. In either case, $\Psi \circ \Phi$ maps z to within ρ of $\gamma(x)$. Thus we can take the function $f_1 = \Psi \circ \Phi$ in the first step of our iteration to satisfy (7.11) with any $\rho > 0$ we wish. Thus, by our choice of ρ, $f_1(x)$ and $g_1(x)$ can be joined by a path in $S^2 \setminus (\Omega \cup \Omega^*)$ consisting of arcs of our foliation and an arc of Γ_1 which stay within an $(\eta/4)$-neighborhood of $\gamma(x)$.

As noted following the previous lemma, this implies that the geodesic $\gamma_1(x)$ between $f(tx)$ and $g(h(x)/t)$ in A_t also lies in an $\eta/2$-neighborhood of $\gamma(x)$ if t is close enough to 1.

At the next step, we approximate f_1 by a function f_2 so that for every z either $|f_1(z) - f_2(z)| \leq \eta/4$ or $f_2(z)$ lies within an $\eta/4$-neighborhood of geodesic $\gamma_2(x)$, $x = z/|z|$. Thus $f_2(z)$ is either within $\eta/2 + \eta/4$ of $F(z)$ or is inside an $\eta/2 + \eta/4$ neighborhood of the curve $\gamma(x)$. Continuing by induction, we see that the nth step approximation f_n can be taken to satisfy (7.11) with constant $(\eta/2)(1 + \cdots + 2^{-n}) < \eta$. Thus the limiting function satisfies (7.11) with constant η, as desired. □

8. Every homeomorphism is almost a welding: Proof of Theorem 1

Suppose h is an orientation-preserving homeomorphism of the circle and $\varepsilon > 0$. In this section we will prove that there is a set E with $|E| + |h(E)| < \varepsilon$ and a conformal welding homeomorphism H so that $h(x) = H(x)$ for all $x \in \mathbb{T} \setminus E$.

If h satisfies the hypothesis of Theorem 3, then h is a conformal welding and there is nothing to do. So assume otherwise and apply Theorem 2 to get conformal maps $F : \mathbb{D} \to \Omega$ and $G : \mathbb{D}^* \to \Omega^*$ such that $F = G \circ h$ on $\mathbb{T} \setminus (\tilde{E}_1 \cup \tilde{E}_2)$ where $\operatorname{cap}(\tilde{E}_1) = \operatorname{cap}(h(\tilde{E}_2)) = 0$.

Choose a set E_1 of zero logarithmic capacity so that

$$|h(E_1)| = \sup\{|h(E)| : \operatorname{cap}(E) = 0\}.$$

We can do this since we can take E to a countable union of zero capacity sets such that $|h(E)|$ approaches the supremum. Since $\tilde{E}_1$ has zero capacity we may add it to E_1 and hence assume E_1 contains $\tilde{E}_1$. Similarly, we can choose a set E_2, containing $\tilde{E}_2$, so that $\operatorname{cap}(h(E_2)) = 0$ and

$$|E_2| = \sup\{|E| : \operatorname{cap}(h(E)) = 0\}.$$

Next we need a simple lemma.

Lemma 30. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is a homeomorphism and that $K \subset \mathbb{T}$ has the property that $E \subset K$, $\text{cap}(E) = 0$ implies $|h(E)| = 0$. Then for any $\varepsilon > 0$ there is a $\delta > 0$ so that $E \subset K$ and $\text{cap}(E) < \delta$ implies $|h(E)| < \varepsilon$.*

Proof. If this did not hold, then one could takes sets E_n such that $\text{cap}(E_n) \leq 2^{-n}$, but so that $|h(E_n)| \geq \varepsilon$. But then $F = \cap_n \cup_{k>n} E_k$ would be a set of zero capacity, such that $|h(F)| \geq \varepsilon$. This is a contradiction, proving the lemma. □

Fix an $\varepsilon > 0$. Let F, G and E_1, E_2 be as above. Apply the previous lemma to the set $\mathbb{T} \setminus E_1$ and the map h to choose a δ so small that $E \cap E_1 = \emptyset$, $\text{cap}(E) < \delta$ implies $|h(E)| < \varepsilon/32$. Applying the lemma to the map h^{-1} on the set $\mathbb{T} \setminus h(E_2)$, we may also assume that $E \cap h(E_2) = \emptyset$, $\text{cap}(E) < \delta$ implies $|h^{-1}(E)| < \varepsilon/32$. By Lemma 18, we can choose open sets U_1 and U_2 with capacity $\leq \delta$ so that F is continuous on $W_{\mathbb{T}\setminus U_1}$ and G is continuous on $W^*_{\mathbb{T}\setminus U_2}$. By Lemma 30 and choosing δ small enough we may assume

$$|U_1| + |U_2| + |h(U_1 \setminus E_1)| + |h^{-1}(U_2 \setminus E_2)| \leq \varepsilon/32.$$

Now choose a compact set $K \subset \mathbb{T} \setminus (E_1 \cup E_2 \cup U_1 \cup h^{-1}(U_2))$, so that

$$|K| \geq 1 - |E_2| - \varepsilon/16,$$
$$|h(K)| \geq 1 - |h(E_1)| - \varepsilon/16.$$

By definition F extends continuously to W_K, G extends continuously to $W^*_{h(K)}$ and they satisfy $F = G \circ h$ on K. On W_K and $W^*_{h(K)}$ we will leave F and G alone. On the complementary components we will replace them by approximations which satisfy the welding relation on a large subset of $E_1 \cup E_2$.

Choose a compact set $F_1 \subset E_1$ so that $|h(F_1)| \geq |h(E_1)| - \varepsilon/32$, and a compact set $F_2 \subset E_2$ so that $|F_2| \geq |E_2| - \varepsilon/32$. Let $F_3 = F_1 \cup F_2$. Thus

$$|K \cup F_3| \geq 1 - \varepsilon/4, \qquad |h(K \cup F_3)| \geq 1 - \varepsilon/4. \tag{8.1}$$

Let $\varphi_1 : \mathbb{D} \to W_K$ and $\varphi_2 : \mathbb{D}^* \to W^*_{h(K)}$ be the 2-quasiconformal maps described in Section 2. Assume we have replaced F and G by the maps $F \circ \varphi_1$ and G by $G \circ \varphi_2$. Then F and G map onto disjoint Jordan domains Ω and Ω^*, such that $K = F^{-1}(\partial\Omega \cap \partial\Omega^*)$ is nonempty. Suppose U is a component of $S^2 \setminus \overline{\Omega \cup \Omega^*}$. Then, as in Section 6, U is bounded by two analytic arcs with common endpoints a, b. Let $I = F^{-1}(\partial U)$ and choose two disjoint, closed, subintervals $I_1, I_2 \subset I$, each containing one endpoint of I so that

$$|I_1| + |I_2| + |h(I_1)| + |h(I_2)| \leq |I|\varepsilon/4.$$

Let $J = \overline{I \setminus (I_1 \cup I_2)}$. Let $C_I \subset \mathbb{D}$ be the region bounded by I and the circular arc with the same endpoints as I which makes angle $\pi/4$ with I. Let C_J be the corresponding region for J. Let

$$S = (C_I \setminus C_J) \cap D(t),$$

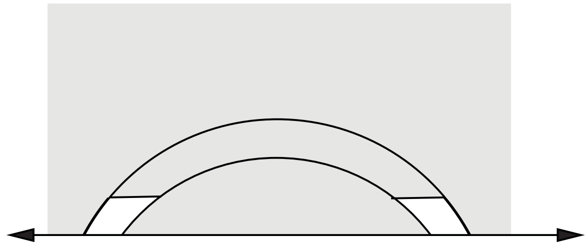

Figure 13: The regions above I

where $D(t) = \{z : 1 - |z| \geq t\}$, and $t = \min(|I_1|, |I_2|)$. See Figure 13.

We claim that we can find a log-singular homeomorphism $\tilde{h}$ of $\mathbb{T}$ such that $\tilde{h} = h$ on F_3. To prove this, note that h is log-singular on F_3 and then apply Remark 9 to each of the complementary intervals of F_3, and make it agree with h at the endpoints. Let f and g be approximations to F and G which satisfy $f = g \circ \tilde{h}$. If we take f close enough to F on the compact set S, then we can find a 2-quasiconformal map of S which agrees with F on $\partial S \cap \partial C_I$ and which agrees with f on $\partial S \cap \partial C_J$.

Next we claim that we can choose f so that $\overline{f(C_J)} \subset V = \operatorname{int}(F(\overline{C_I}) \cup U)$. To prove this, note that

$$Y = F(\overline{C_I}) \cup \cup_{x \in J} \gamma(x),$$

is a compact subset of V and hence is at a positive distance η from ∂V. By Lemma 29 we can take $f(\overline{C_J})$ to lie within $\eta/2$ of the set Y. Hence, $f(\overline{C_J})$ is a subset of V as desired. The same argument shows that we can choose g so that $g(\overline{C^*(h(J))})$ is in $G(\overline{C^*_{h(I)}} \cup U$. Moreover, if

$$S^* = (C^*_{h(I)} \setminus C^*_{h(J)}) \cap D^*(t),$$

where $D(t) = \{z : |z| - 1 \geq t\}$, and $t = \min(|h(I_1)|, |h(I_2)|)$, then we can choose a 2-quasiconformal map of S^* which agrees with G on $\partial S \cap \partial C^*_{h(I)}$ and agrees with g on $\partial S \cap \partial C^*_{h(J)}$.

We now carry out the construction above for every component of $S^2 \setminus \overline{\Omega \cup \Omega^*}$. Let $\{U_n\}$ be an enumeration of these components, $\{I_n\}$ the corresponding intervals on $\mathbb{T}$ and $\{J_n\}$ the corresponding subintervals. Let W be the interior of $W_k \cup \cup_n (\overline{S_n} \cup C_{J_n})$. Clearly W can be mapped quasiconformally (with a uniformly bounded constant) to the disk by a map which is the identity on $\partial W \cap \mathbb{T}$. Similarly for the corresponding region $W^* \subset \mathbb{D}^*$. Moreover,

$$(8.2) \qquad |\mathbb{T} \setminus \partial W| = \sum_n |I_n \setminus J_n| \leq \varepsilon/4, \qquad |h(\mathbb{T} \setminus \partial W)| \leq \varepsilon/4.$$

The construction above gives 2-quasiconformal maps f on W and g on W^* onto disjoint Jordan domains so that $g^{-1} \circ f = h$ on $K_1 = K \cup (F_3 \cap \partial W)$. By (8.1) and (8.2), K_1 has Lebesgue measure $\geq 1 - \varepsilon$, as does its h image. By composing with the inverses of the uniformly quasiconformal maps described in the previous paragraph, we get uniformly quasiconformal maps f on $\mathbb{D}$ and g on $\mathbb{D}^*$ onto disjoint Jordan domains so that $f = g \circ h$ holds on K_1. By solving a Beltrami equation in the usual way we may assume f and g are conformal. Now apply Theorem 8 and we have completed the proof of Theorem 1.

9. A generalized Koebe circle conjecture

Based on Theorem 5 it seems reasonable to state

CONJECTURE 1. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is any orientation-preserving homeomorphism. Then h is a generalized conformal welding on $\mathbb{T} \setminus E$ where E is a countable set.*

Given sequences $\{f_n\}$ and $\{g_n\}$ from Theorem 5, we want to pass to convergent subsequences and show that the limiting functions f and g satisfy $f = g \circ h$ except on a countable set. One problem is that conformal maps in general can fail to have radial boundary values on a set of zero logarithmic capacity; thus we must show that we can choose our sequences so that the limiting maps have radial limits except on at most a countable set. Furthermore, even if the limiting maps do have this property, we then have to improve the argument in Section 5 to show $f = g \circ h$ except on a countable set, rather than a set of zero capacity.

Each of these steps will require some special choices of the sequences. Given a log-singular h we can choose corresponding maps $\{f_n\}$, $\{g_n\}$ which converge uniformly on compacta to any conformal maps we want (as long as the images are disjoint). In particular, the limits could fail to have radial limits on an uncountable set. Similarly, we can choose the sequences to converge to the identity, and then the desired equality fails on an uncountable set (even though the limit maps have radial limits everywhere, indeed are continuous). Of course, we already know such an h is a conformal welding everywhere (Theorem 3), but these examples indicate some care needs to be taken in the general case.

Another way to attack Conjecture 1 is to replace our use of Koebe's theorem by a much stronger result. Koebe's conjecture (also known as the Kreisnormierungsproblem) states that any planar domain is conformally equivalent to a circle domain, i.e., a domain whose complementary components are all either points or disks. Also, recall from Remark 3 the theorem of R. L. Moore which states that if we quotient $\mathbb{R}^2$ by a upper semi-continuous decomposition whose elements do not separate the plane, the resulting space is $\mathbb{R}^2$ again. The following can be considered a generalization of both Koebe's conjecture and Moore's theorem (see Remark 3).

CONJECTURE 2 (Generalized Koebe conjecture). *Every Moore decomposition of $\mathbb{R}^2$ is conformally equivalent to a Koebe decomposition.*

This contains Koebe's Kreisnormierungsproblem as a special case, because if Ω is connected then the decomposition of $E = S^2 \setminus \Omega$ into its connected components is an upper semi-continuous decomposition (Theorem 24 of Moore's paper [33]).

To see how Conjecture 2 implies Conjecture 1, let $A = \{z : 1 \leq |z| \leq 2\}$. Make a closed, upper semi-continuous decomposition of A whose elements are curves connecting x and $2h(x)$. Extend this to a decomposition of S^2 by taking singletons off A. Applying the generalized Koebe conjecture, we see that all but countably many of the curves in our decomposition are collapsed to points (since there can only be countably many disjoint disks in the plane). Thus, f restricted to the two complementary components of A gives the desired maps. This is just the idea discussed in the introduction made more precise.

Conformal welding may also offer an approach to the usual version of the Koebe conjecture. First suppose h is a generalized conformal welding on a dense subset E of $\mathbb{T}$ and that $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ are maps that realize this welding. For any $x \in \mathbb{T}$ choose a sequence of intervals $I_n = (a_n, b_n)$ containing x and such that the endpoints are in the set E. Let γ_I be the hyperbolic geodesic with the same endpoints as I and let W_n be the Jordan domain bounded by $f(\gamma_{I_n}) \cup g(\gamma_{h(I_n)})$. Finally, define $I(x) = \cap_n W_n$. It is easy to see this a compact, connected set which does not separate the plane and does not depend on the particular choice of sequences. Moreover, for $x \neq y$, $I(x) \cap I(y) = \emptyset$. In fact, these sets form an upper semi-continuous decomposition $\mathcal{C}$ of $K = S^2 \setminus (\Omega \cup \Omega^*)$. Moreover, by modifying the proof of Moore's triod theorem [34], [37], one should be able to show that $I(x)$ is a singleton for all but countably many points of E. In particular, if h is a generalized conformal welding on the circle minus a countable set, then $I(x)$ is a point except for countably many x's.

Now suppose Ω is any domain containing infinity and let E denote its complement. Then by removing a countable number of smooth arcs $\{\gamma_n\}$ from Ω we can divide it into two subdomains Ω_1 and Ω_2 so that the Riemann mappings f_1, f_2 onto these domains define a map $h = f_2^{-1} \circ f_1$ almost everywhere on $\mathbb{T}$ which can be extended to a homeomorphism (also denoted by h). Then if Conjecture 1 is true, we can write $h = g^{-1} \circ f$ except on a countable set. Then $g \circ f_2^{-1}$ and $f \circ f_1^{-1}$ are conformal maps on Ω_1 and Ω_2 which agree along the smooth arcs $\{\gamma_n\}$ and hence extend to a conformal map on all of Ω. Moreover, the connected components of E must map into elements of our decomposition $\mathcal{C}$ from above and only countably many of these are not singletons. Thus Conjecture 1 should imply

Conjecture 3. *Any planar domain is conformally equivalent to one whose complement has only countably many components which are not points.*

Conjecture 3 is an obvious consequence of Koebe's conjecture, and if true would reduce Koebe's conjecture to

Conjecture 4. *Suppose $\Omega = S^2 \setminus E$ is a domain and at most countably many components of E are not points. Then Ω is conformally equivalent to a circle domain.*

There is some hope that this special case can be proved since it looks similar to the theorem of He and Schramm [23], that the desired conclusion holds if E has at most countably many components (including points). In a later paper [24] they also proved this is true if the accumulation set of the nontrivial components is at most countable.

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