Renormalization operator for multimodal maps

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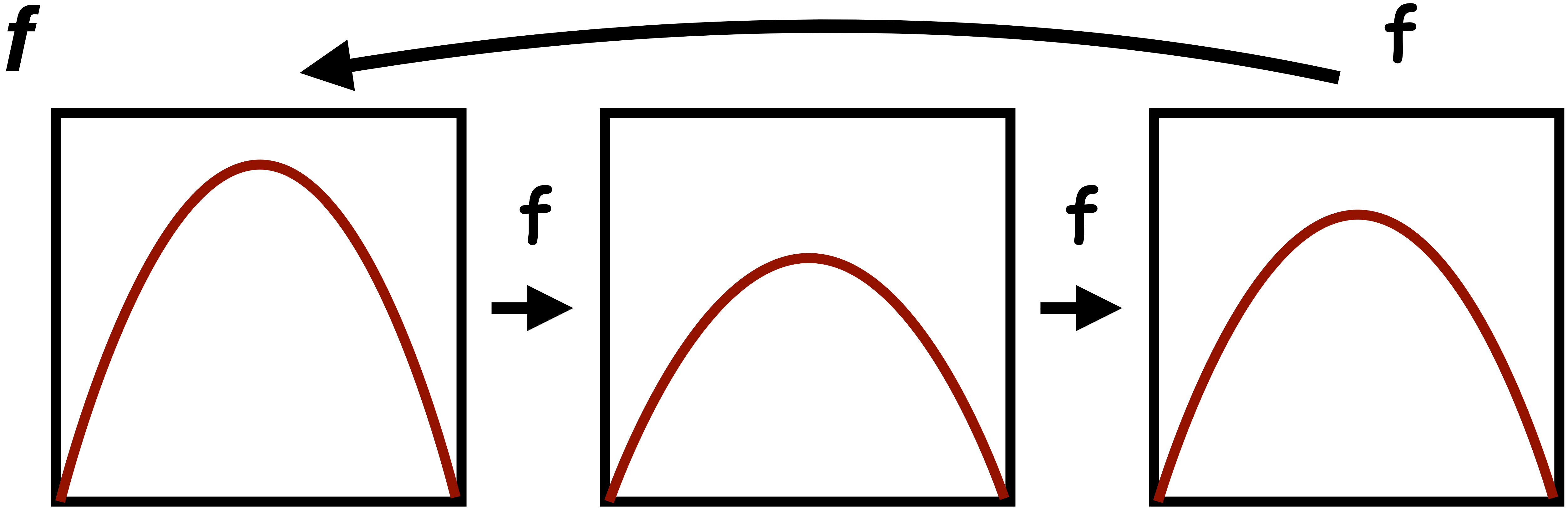
Renormalization for multimodal maps



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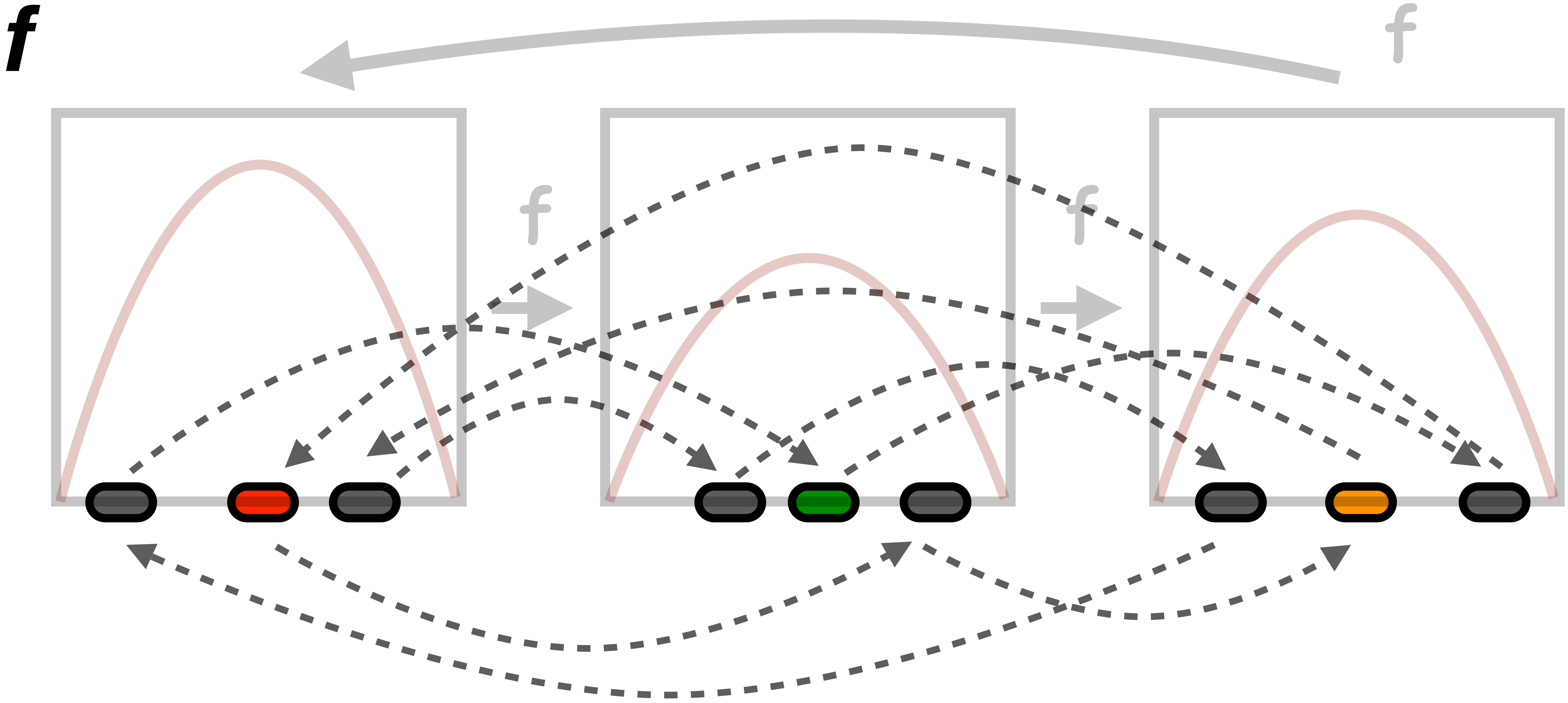

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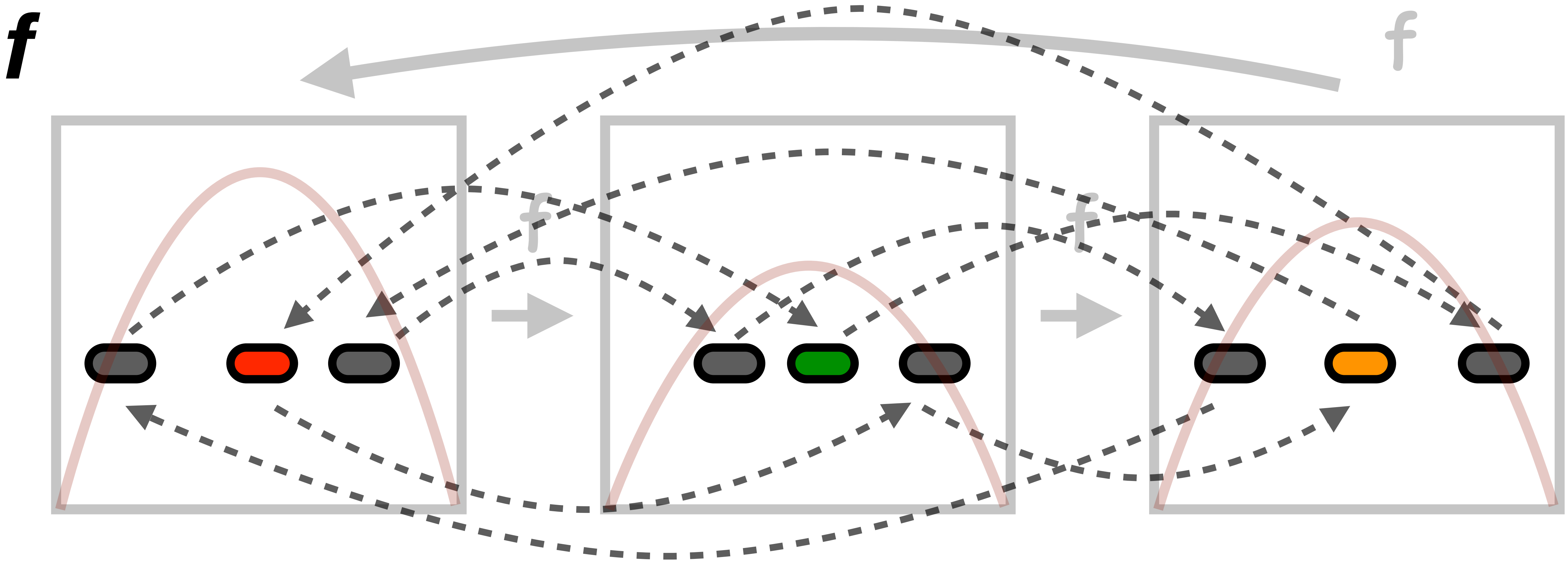

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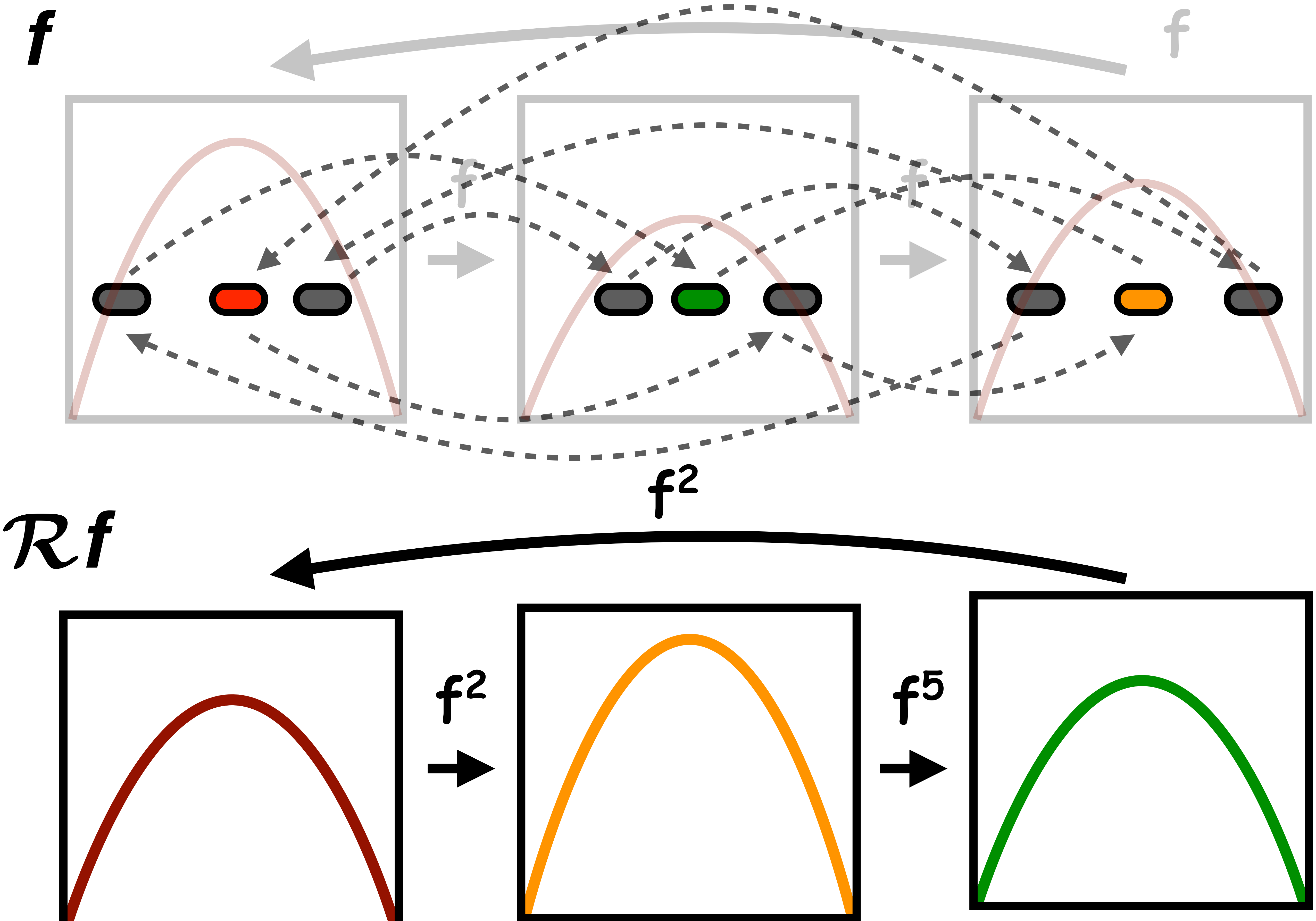
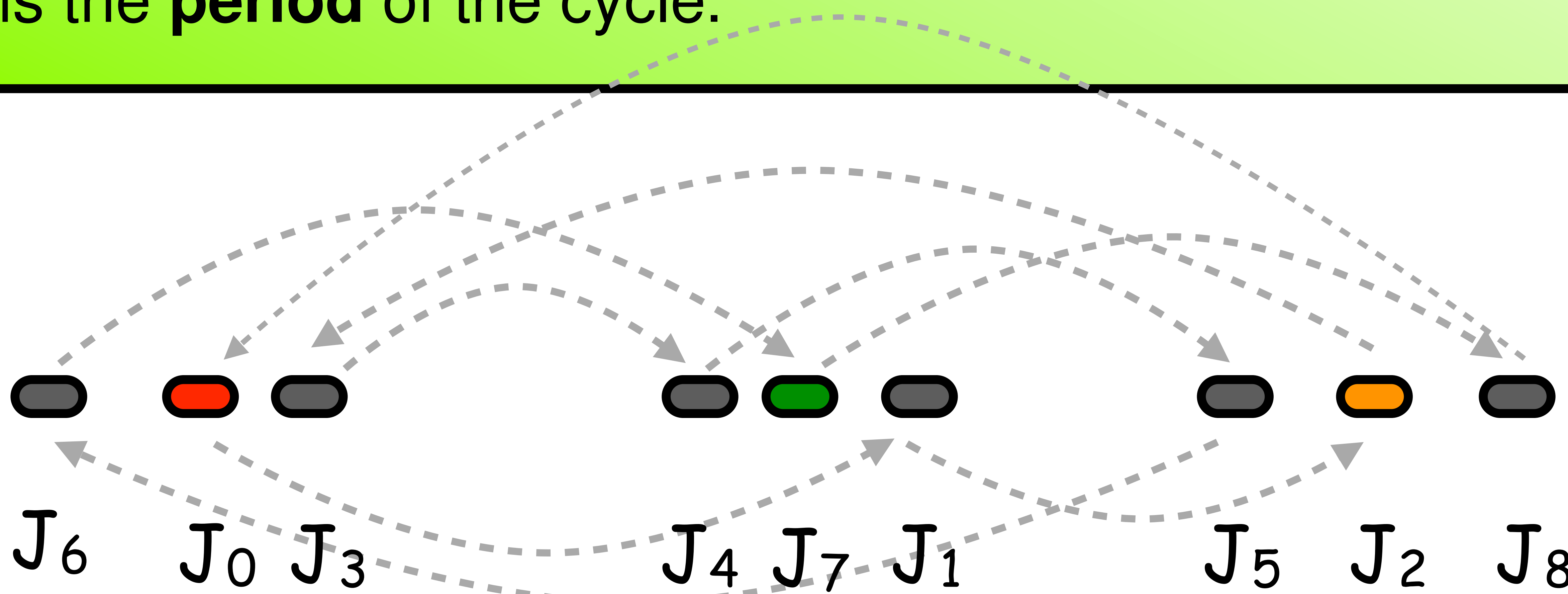


Cycles of intervals

$J_0, \dots J_{p-1}$ is a **cycle of intervals** for f if

- Interiors of J_i are pairwise disjoint.
- $f(J_i) \subset J_{i+1} \mod p$
- $f(\partial J_i) \subset \partial J_{i+1} \mod p$
- All critical points of f belong to $\cup_i J_i$.

p is the **period** of the cycle.



Infinitely renormalizable maps

f is **infinitely renormalizable** if there exists a sequence of cycles

$$J_0^n, \ldots, J_{p_n}^n$$

with $p_n < p_{n+1}$ and

$$\bigcup_i J_i^{n+1} \subset \bigcup_i J_i^n$$

f has B-bounded combinatorics if moreover

$$\sup_n \frac{p_{n+1}}{p_n} \leq B$$

Main Theorem

Let f_λ be a finite-dimensional smooth family of real analytic multimodal maps and let Λ_B be the subset of parameters λ such that f_λ is infinitely renormalizable with B-bounded combinatorics.

For a generic finite-dimensional family f_t the set Λ_B has zero Lebesgue measure.

The meaning of generic

For a generic finite-dimensional family f_t the set Λ_B has zero Lebesgue measure.

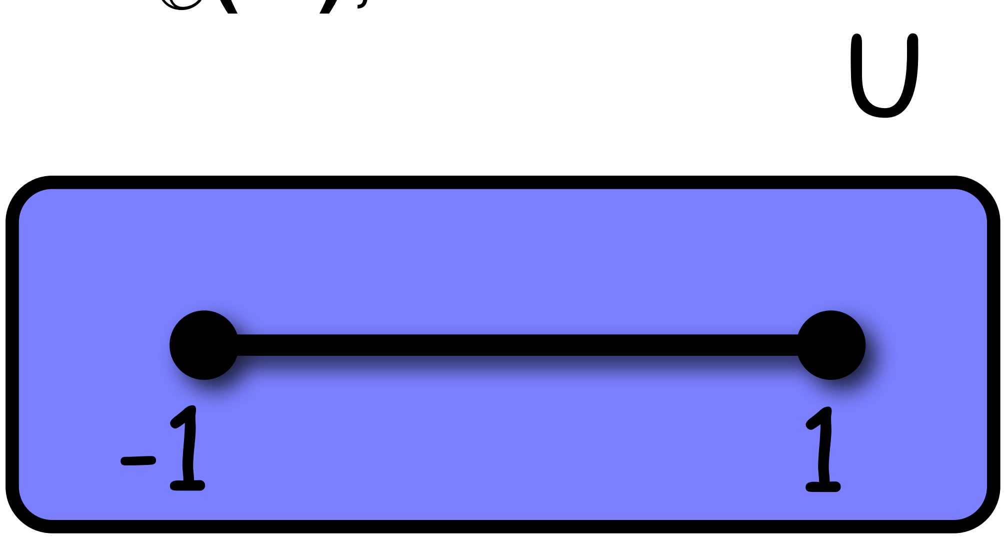
$f \in B_{\mathbb{R}}(U)$ *iff*

- f is continuous in $\overline{U}$,
- f is complex analytic in U and
- $f(\overline{z}) = \overline{f(z)}$.

We mean generic C^k families $t \in [0,1]^n \longrightarrow B_{\mathbb{R}}(U)$, $k > 1$.

and also generic C^ω families $t \in \overline{\mathbb{D}}^n \longrightarrow B_{\mathbb{C}}(U)$,

real in real parameters



Facts on the renormalization operator

Unimodal (Douady&Hubbard, Sullivan, McMullen, Lyubich)
and multimodal (Hu, S. (2001,2005), + stuff in progress)

(Complex bounds) If f is infinitely renormalizable then $\{R^n f\}_n$ is precompact.

(Universality) The Omega-limit set Ω of R is a compact set. The dynamics of R on Ω is conjugate with a full shift with finitely many symbols.

There exists $\lambda \in (0,1)$ s.t. if f is infinitely renormalizable then there exists $f_\star \in \Omega$ such that

$$|R^n f - R^n f_\star| \leq C_f \lambda^n.$$

Steps of the proof

1. Complexification of R (Complex bounds).

2. The Omega limit set Ω of R is hyperbolic.

3. If a family f_t is transversal to the stable lamination $W^s(\Omega)$ then Λ has zero Lebesgue measure.

   (easy) adaptation of results by Bowen and Ruelle (1975) for the finite-dimensional case.

4. The result for generic families follows from step 3 using...Fubini's Theorem!!

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Quasiconformal vector fields

The vector field $\alpha : \mathbb{C} \to \mathbb{C}$ is **quasiconformal** if it has distributional derivatives in L^2_{loc} and

$$|\overline{\partial}\alpha|_\infty < \infty$$

$$\alpha(x + iy) = u(x, y) + i \cdot v(x, y)$$

$$\overline{\partial}\alpha = \frac{u_x - v_y}{2} + i \cdot \frac{v_x + u_y}{2}$$

Horizontal directions

(Lyubich, 1999)

$\boldsymbol{f}: \boldsymbol{U} \longrightarrow \boldsymbol{V}$ polynomial-like map. $\boldsymbol{v}: \boldsymbol{U} \longrightarrow \boldsymbol{V}$ is **horizontal** if there exists a quasiconformal vector field α, defined in a neighborhood of $\boldsymbol{K(f)}$ such that

$$\boldsymbol{v(x) = \alpha \circ f(x) - Df(x) \cdot \alpha(x)}$$

Moreover $\overline{\partial}\alpha = 0$ on the filled-in Julia set $\boldsymbol{K(f)}$.

$$\boldsymbol{E^h_f := \{v \; : \; v \text{ is horizontal for } f\}}$$

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Moreover $\overline{\partial}\alpha = \mathbf{0}$ on the filled-in Julia set $\boldsymbol{K(f)}$.

automatic in our setting (no invariant line fields on J(f)), so don't pay too much attention to this...

$$\boldsymbol{E_f^h := \{v \;:\; v \text{ is horizontal for } f\}}$$

Facts on horizontal directions

Unimodal(Lyubich, 1999) and multimodal(S., in progress)

(Continuity) The codimension of E_f^h is finite and it depends only on the number of unimodal components. Moreover $f \to E_f^h$ is continuous.

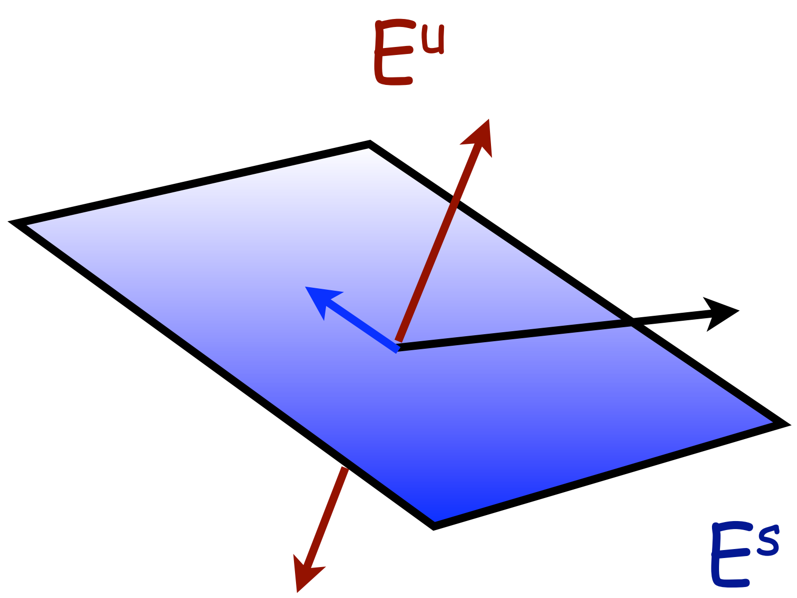
(Invariant vector bundle) if $v \in E^h$ then $DR_f \cdot v \in E^h_{\mathcal{R}f}$.

(Contraction) $|DR_f^n \cdot v| \leq C\lambda^n$, $\lambda < 1$.

Detecting hyperbolicity

Autonomous case

T is a hyperbolic linear map



Detecting hyperbolicity

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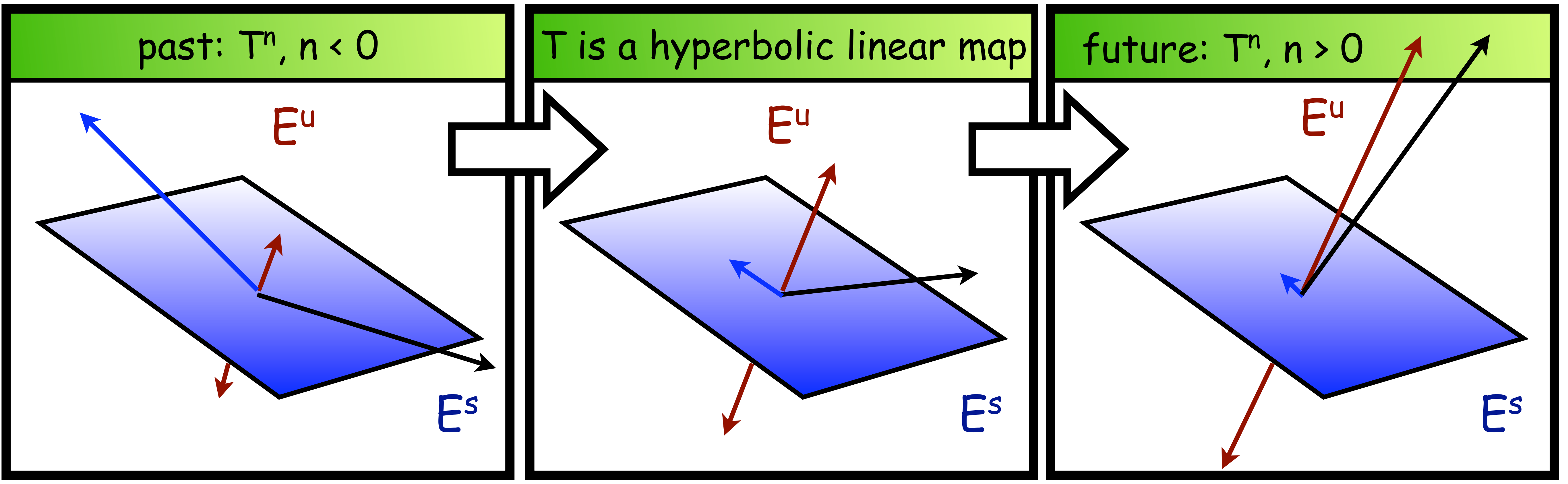

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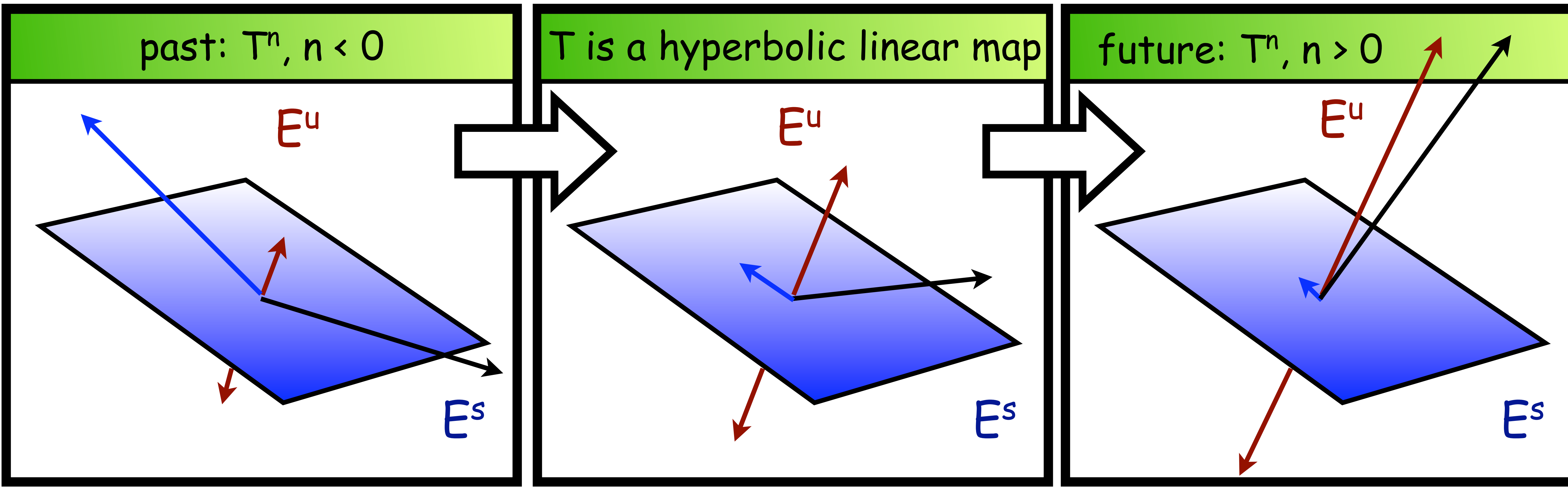

T is hyperbolic if and only if

$$B = \{v \in \mathbb{R}^n : \sup_{i \in \mathbb{Z}} |T^i v| < \infty\} = \{0\}$$

Detecting hyperbolicity

Non-autonomous case (Sacker & Sell, 1974)

X compact metric space.

$f: X \longrightarrow X$ homeomorphism such that the minimal sets are dense in X.

$A: X \longrightarrow GL(n, \mathbb{R})$ continuous.

Let $T: X \times \mathbb{R}^n \longrightarrow X \times \mathbb{R}^n$ be the linear cocycle defined by

$$T(x, v) = (f(x), A(x) \cdot v)$$

Detecting hyperbolicity

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T is a hyperbolic cocycle if and only if $B = X \times \{0\}$.

PS: Same result for vector bundles with same assumption on the base X

Back to renormalization

Considering the finite-dimensional vector bundle defined by

$$f \in \Omega \longrightarrow \mathbb{B}/E_f^h$$

and the cocycle

$$\tilde{D}_f[v] = [D\mathcal{R}_f \cdot v]$$

and using Sacker & Sell Theorem we can get:

If

$$B_f^+ = \{(f, v) \in \Omega \times \mathbb{B} \; s.t. \; \sup_{i \geq 0} |D\mathcal{R}_f^i \cdot v| < \infty\} \subset E_f^h$$

for every $f \in \Omega$ then the renormalization operator is hyperbolic on Ω with $E_f^s = E_f^h$.

Key Lemma

If $f \in \Omega$ and

$$|D\mathcal{R}^i_f \cdot v| \leq C$$

for every $i \geq 0$ then there exists a quasiconformal vector field α defined in a neighborhood of $K(f) = J(f)$ such that

$$v(x) = \alpha \circ f(x) - Df(x) \cdot \alpha(x).$$

Infinitesimal pullback argument

(Avila, Lyubich and de Melo, 2003)

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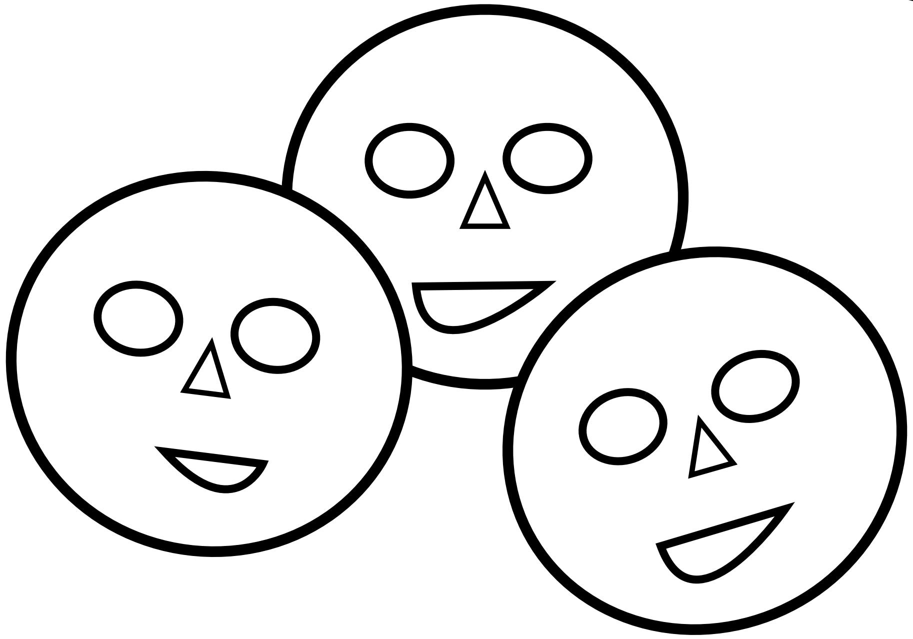
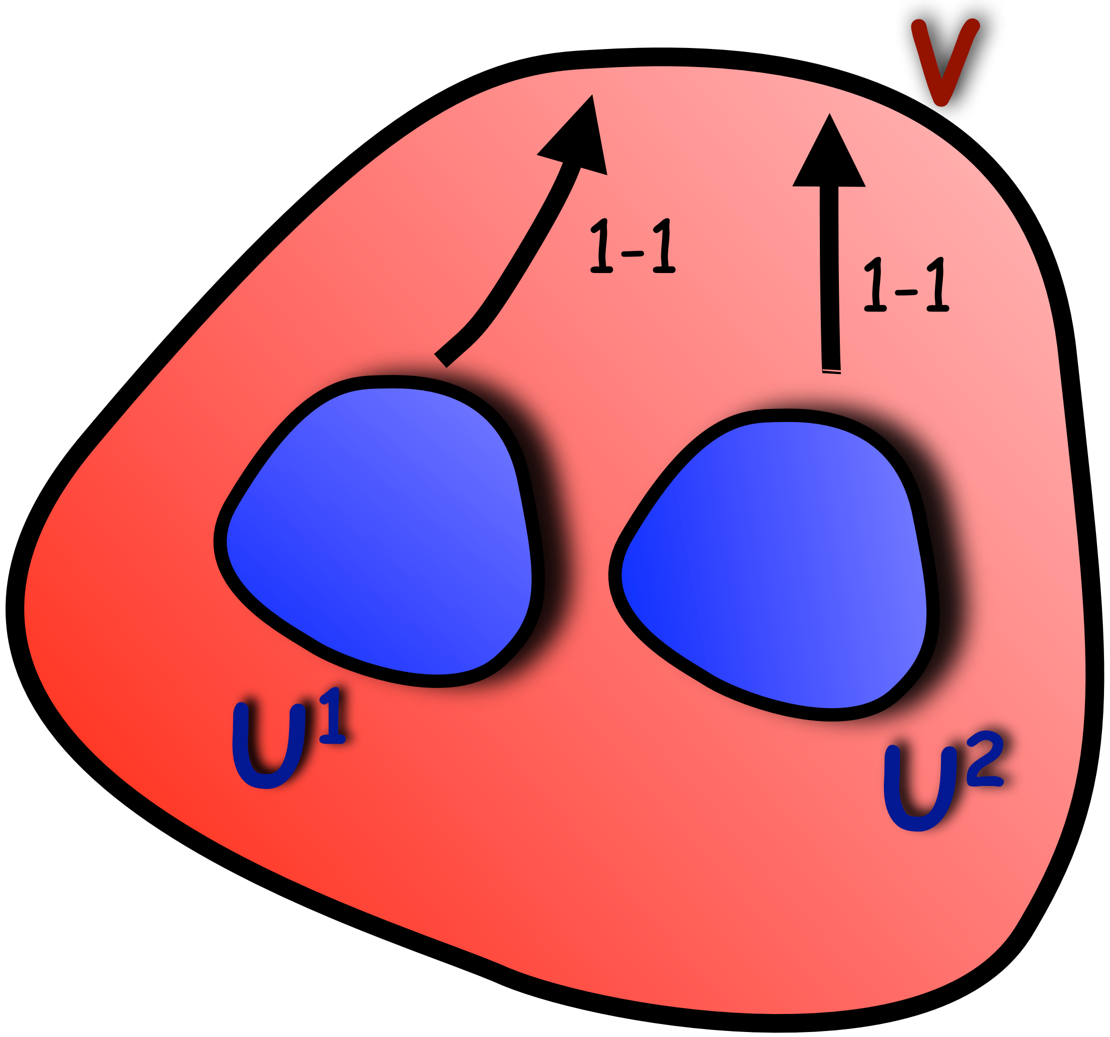


To find a quasiconformal vector field solution to the t.c.e. we just need to find a quasiconformal vector field which is the solution on the boundary of the domain and the postcritical set.

Easy case: Conformal iterated function systems (no critical points)

$f : U^1 \cup U^2 \longrightarrow V$

$f : U^i \longrightarrow V$ conformal and onto, $i = 1, 2$.



Problem: Given

$$v : U^1 \cup U^2 \longrightarrow \mathbb{C},$$

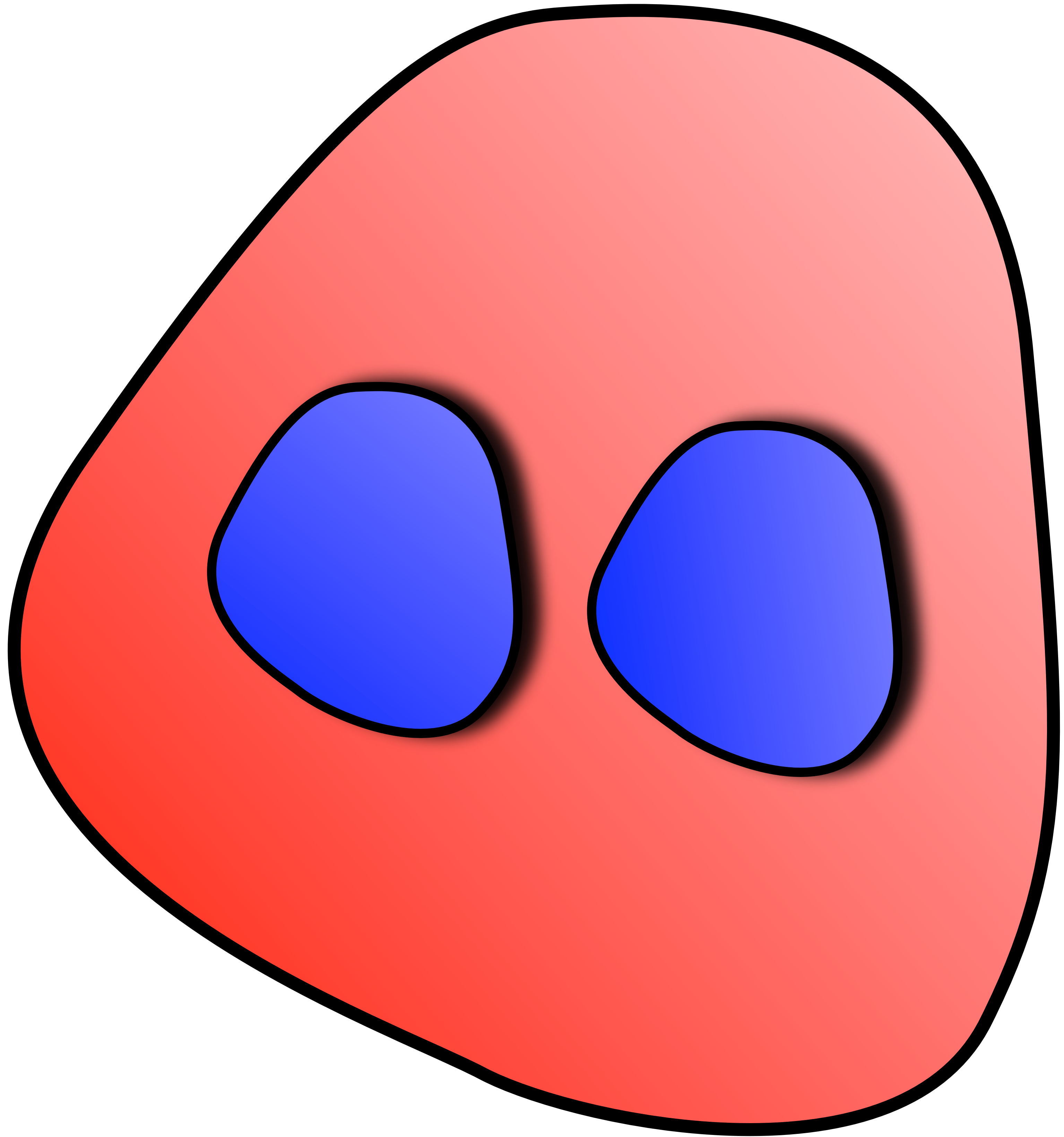
find a quasiconformal vector field

$$\alpha : V \longrightarrow \mathbb{C}$$

such that

$$v(x) = \alpha(f(x)) - Df(x) \cdot \alpha(x)$$

Easy case of Infinitesimal pullback argument : Conformal iterated function systems



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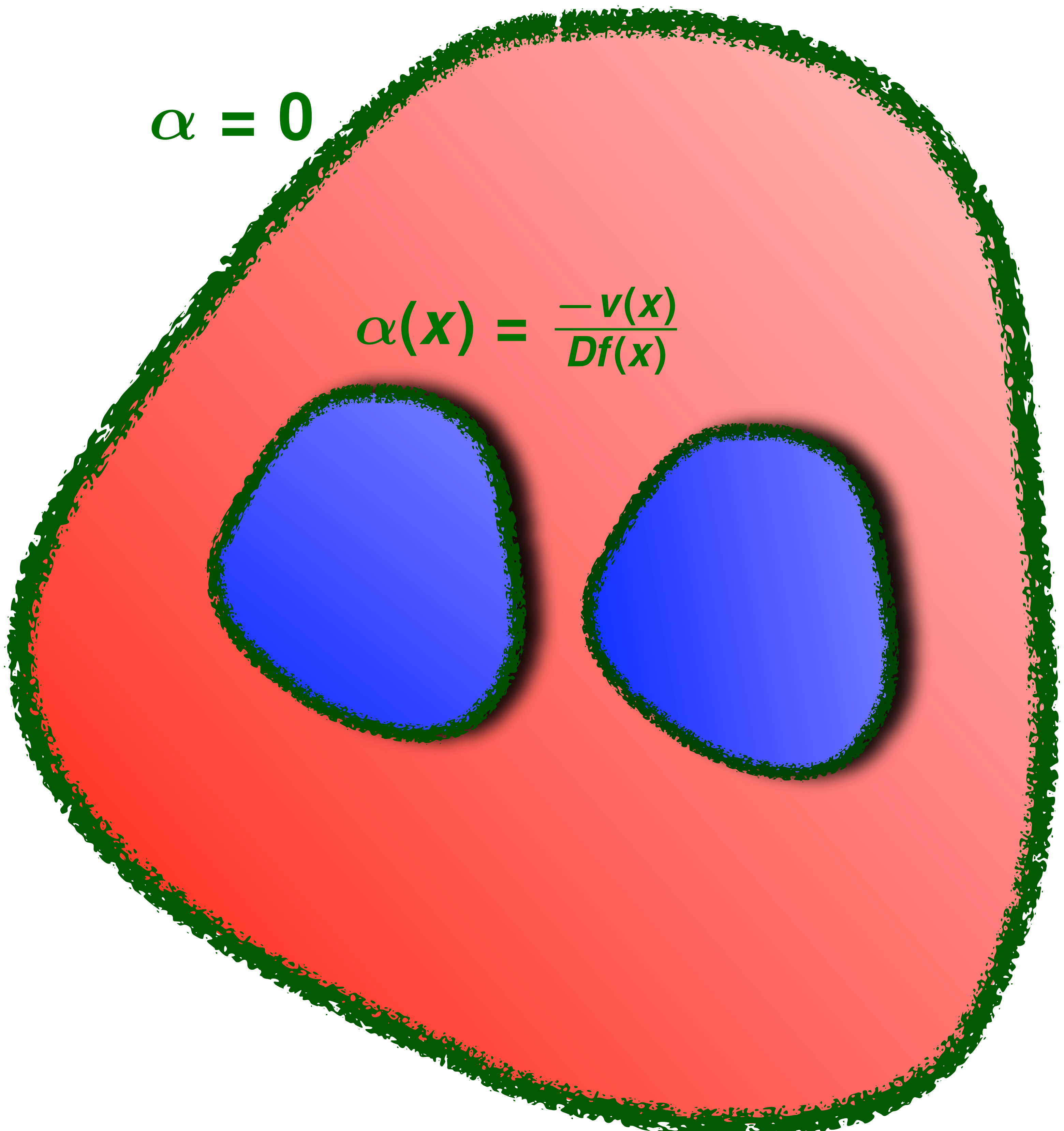

α is defined.

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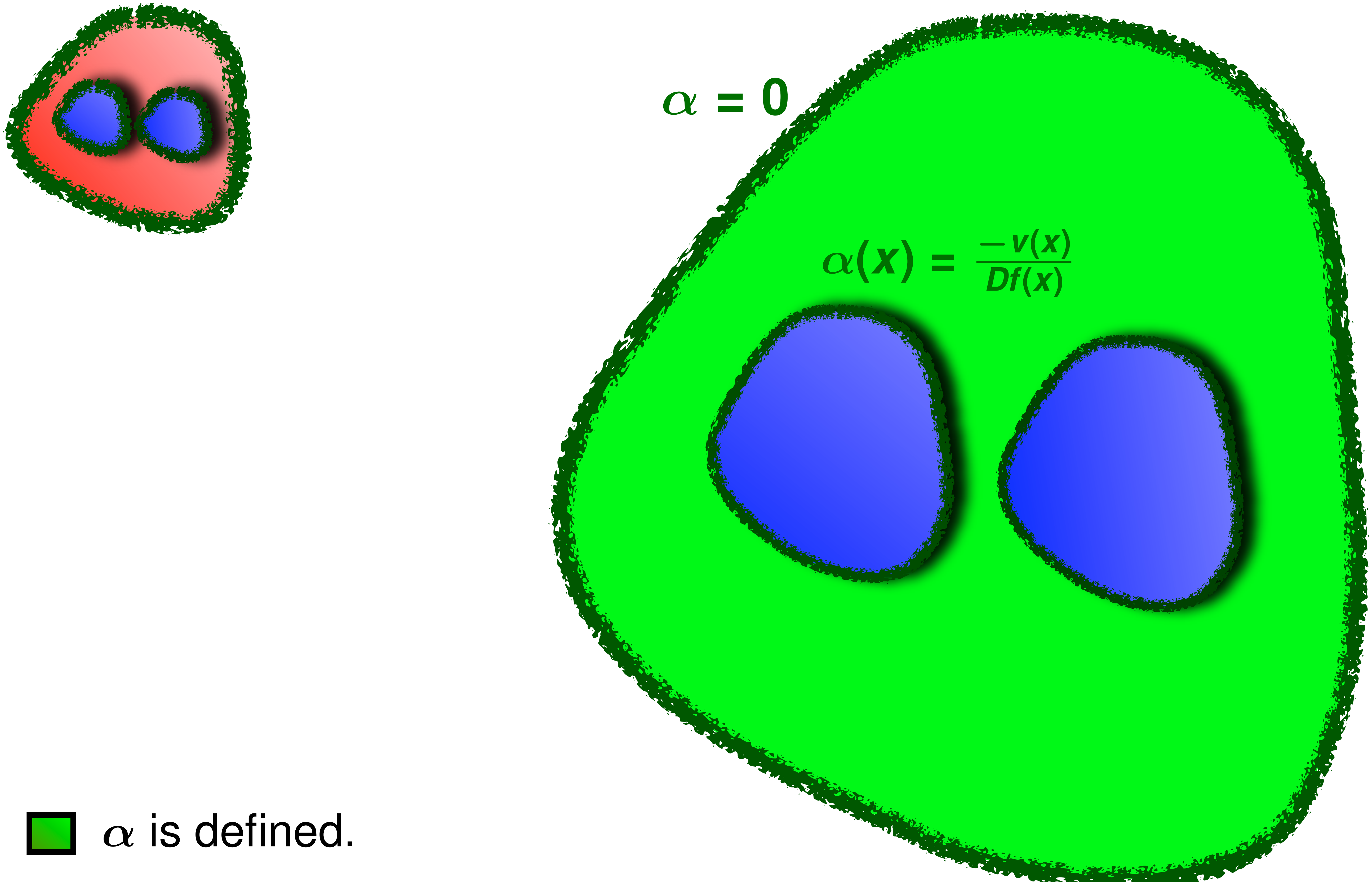

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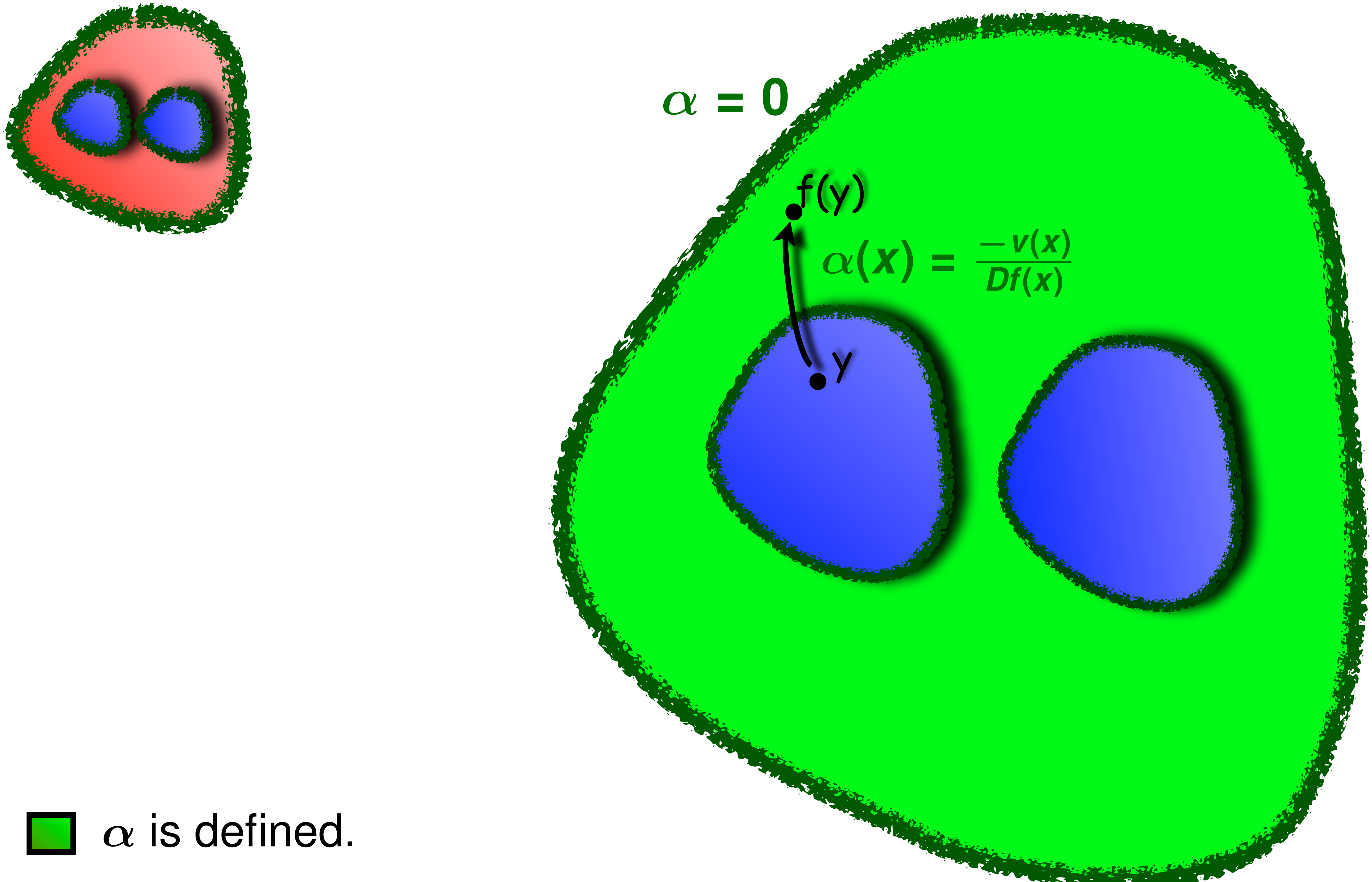

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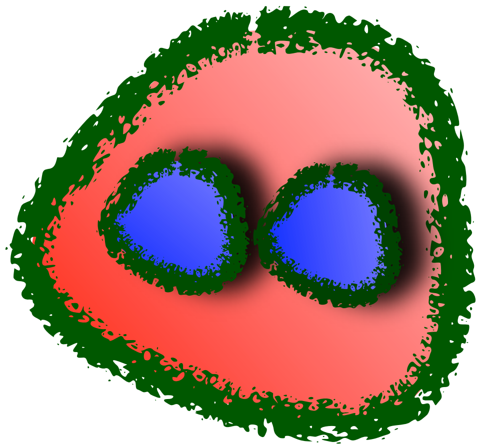
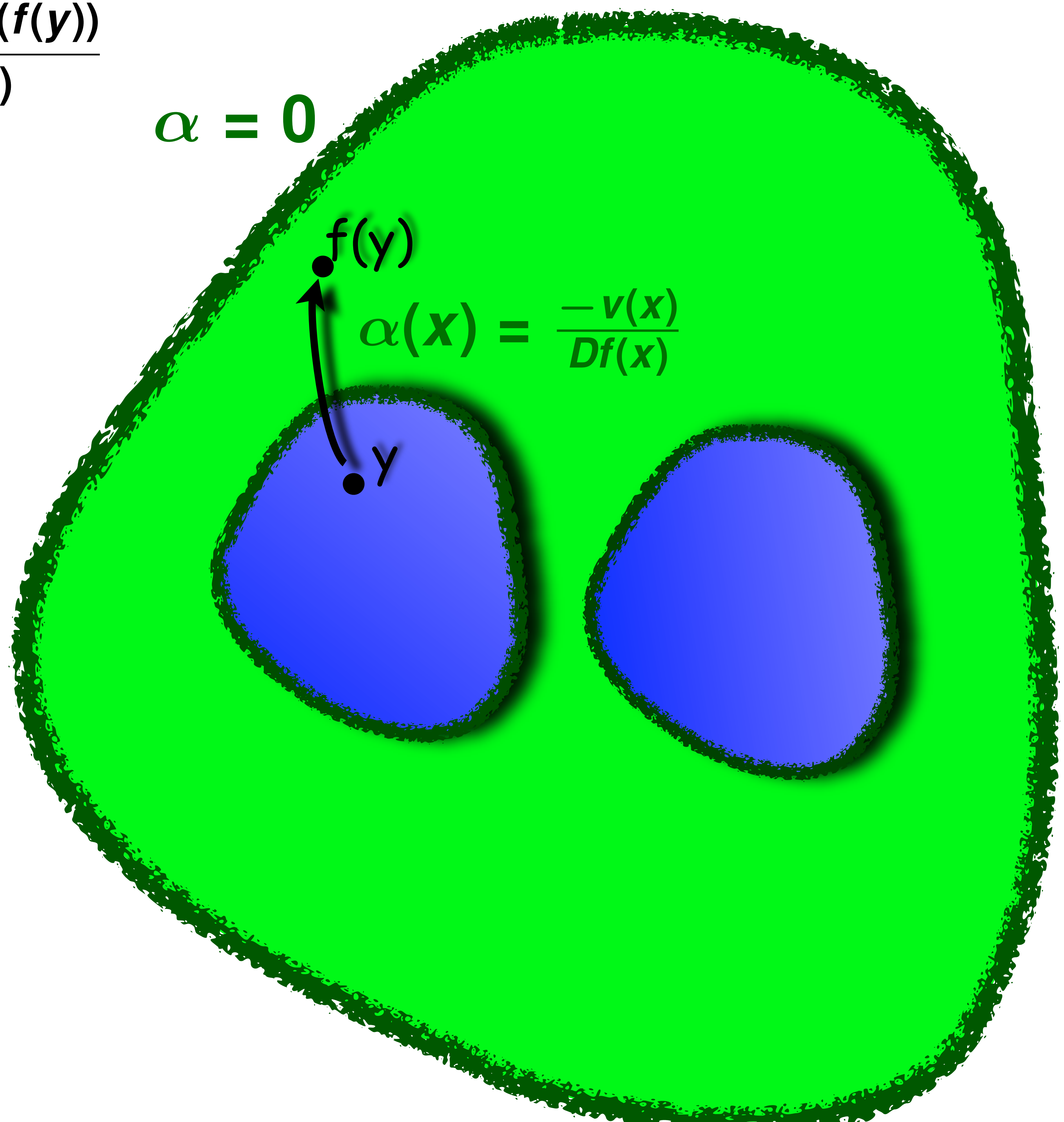


$$\alpha(y) := \frac{v(y) - \alpha(f(y))}{Df(y)}$$



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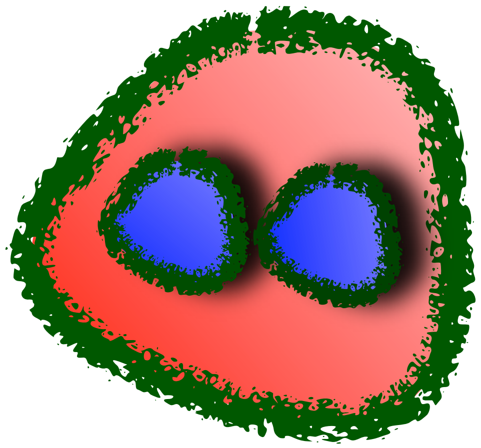

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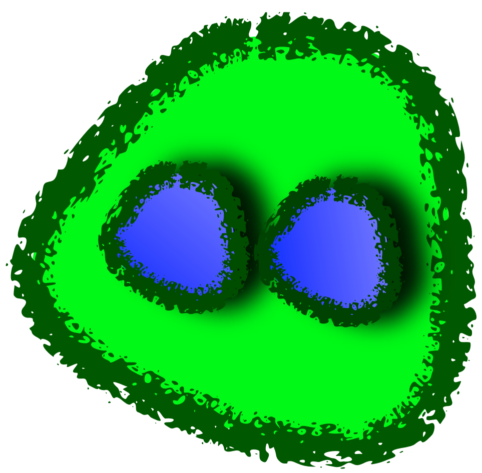

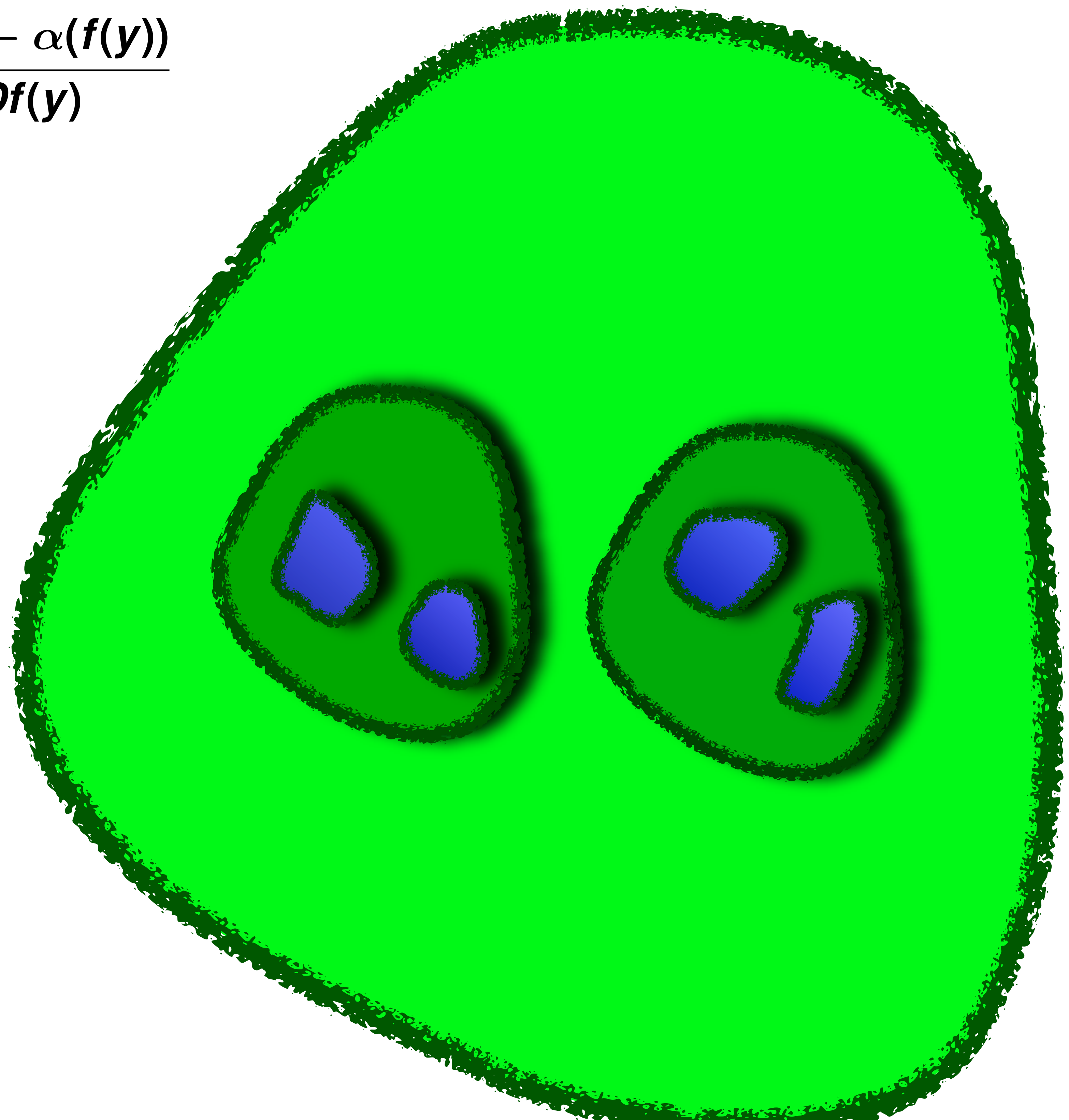

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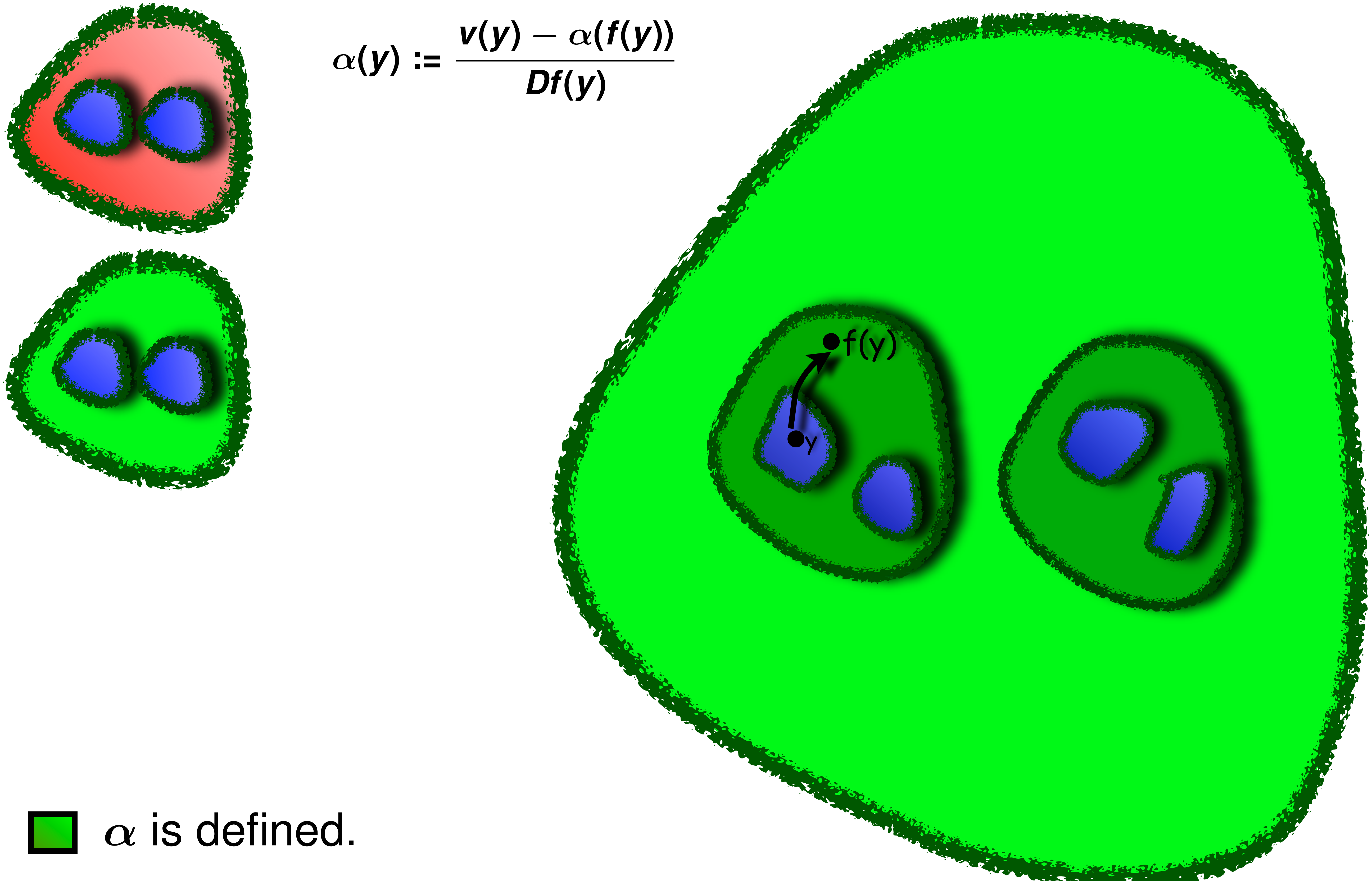

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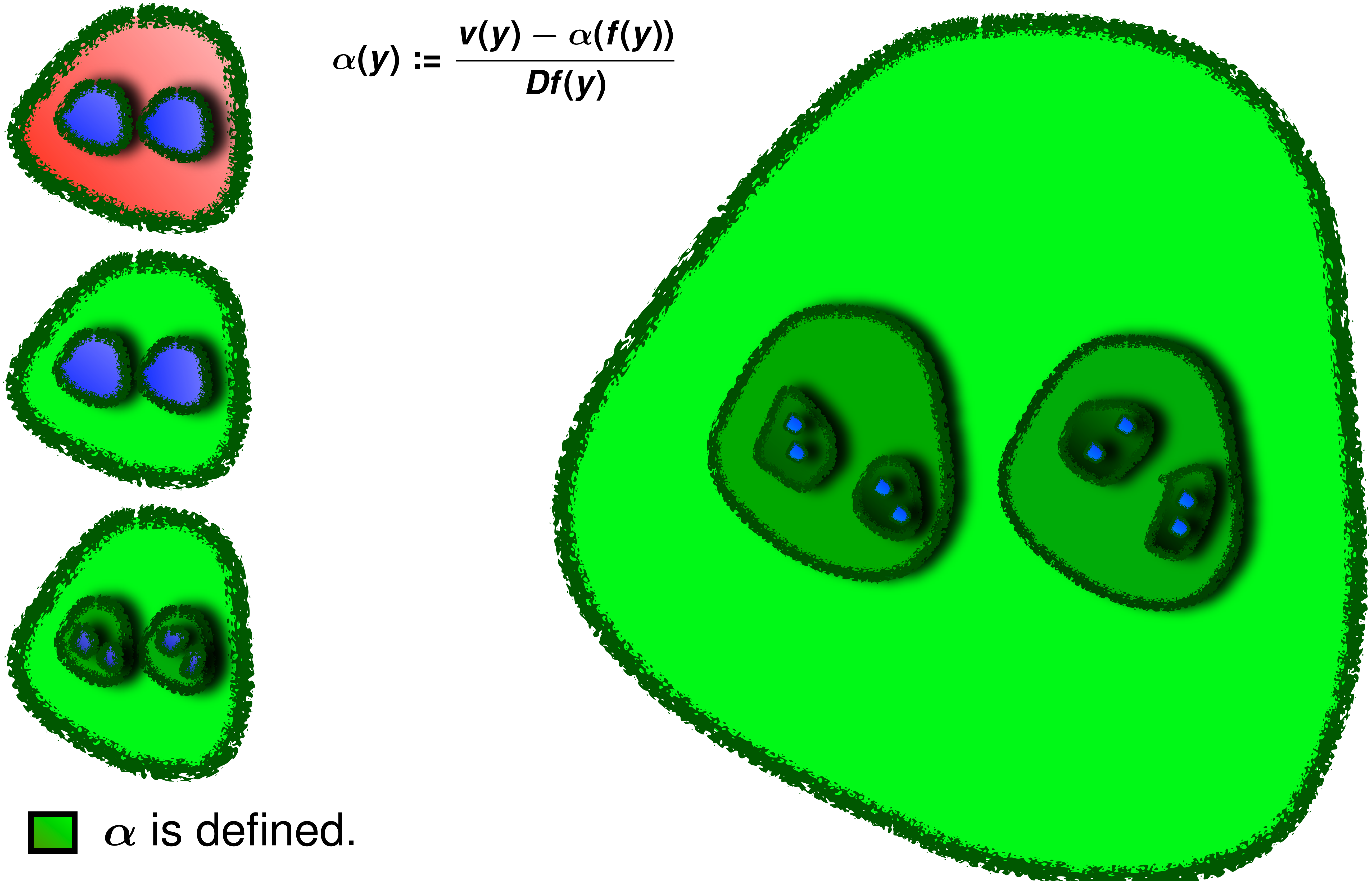

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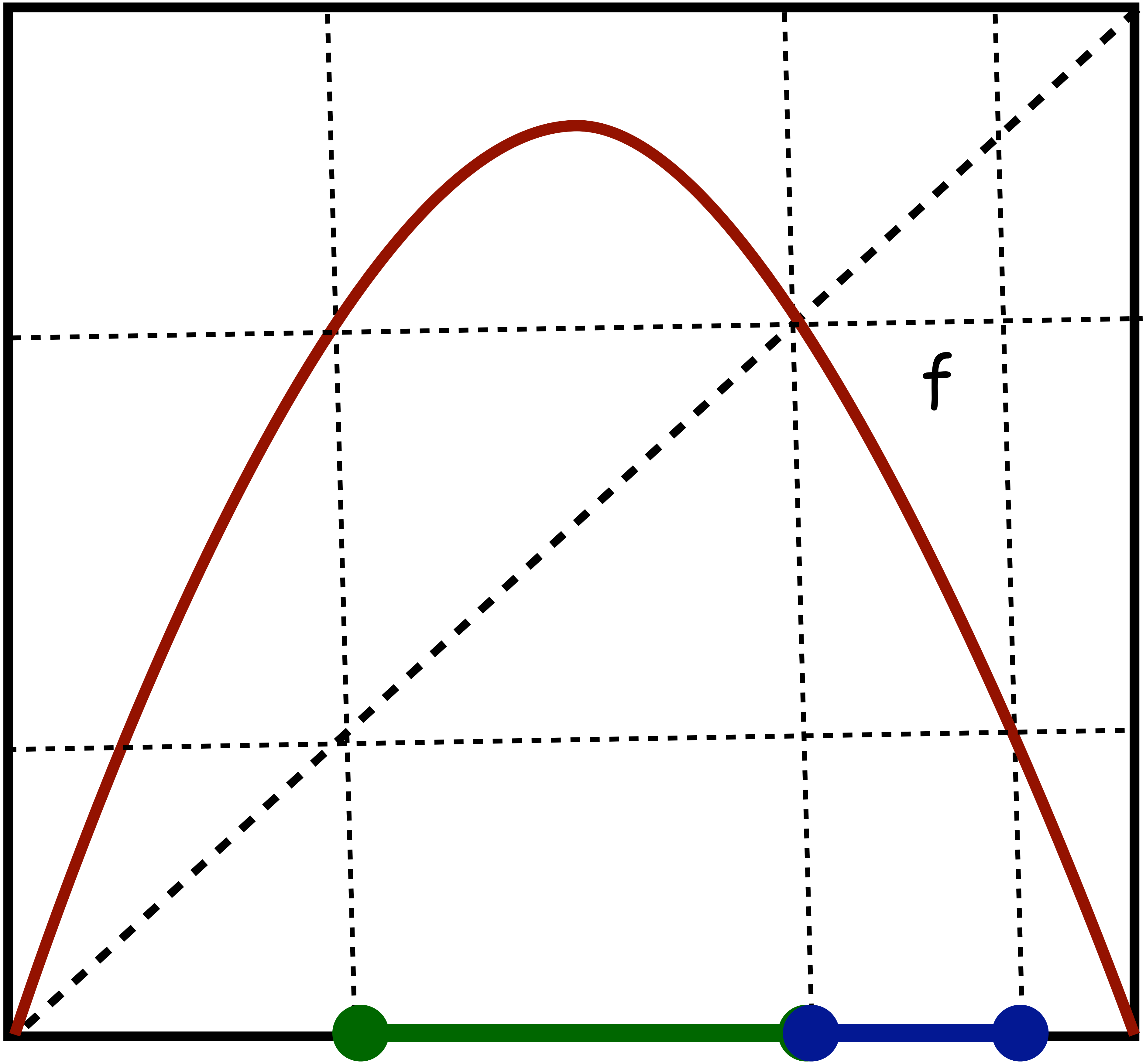
Period-doubling case: induced map



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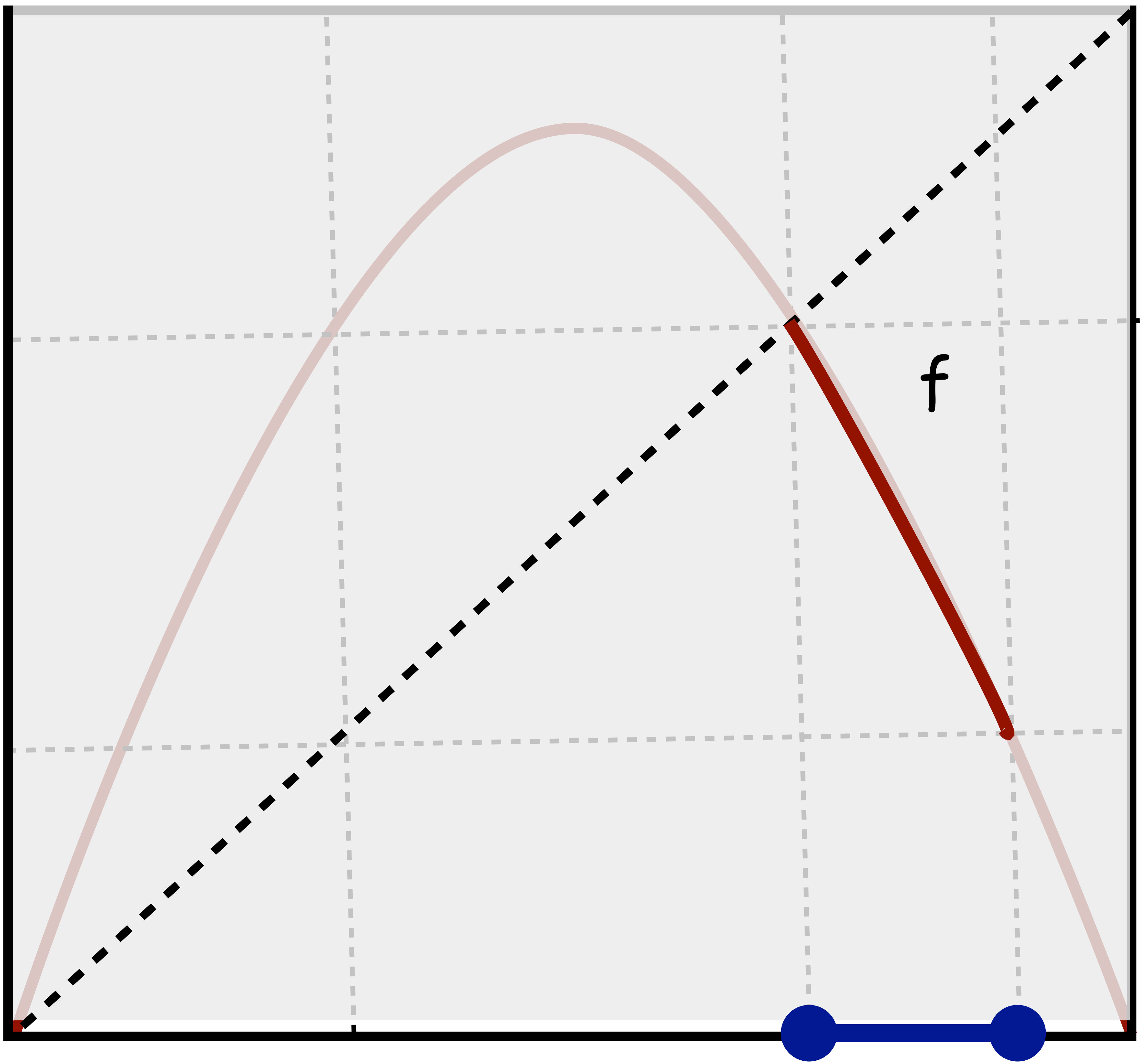

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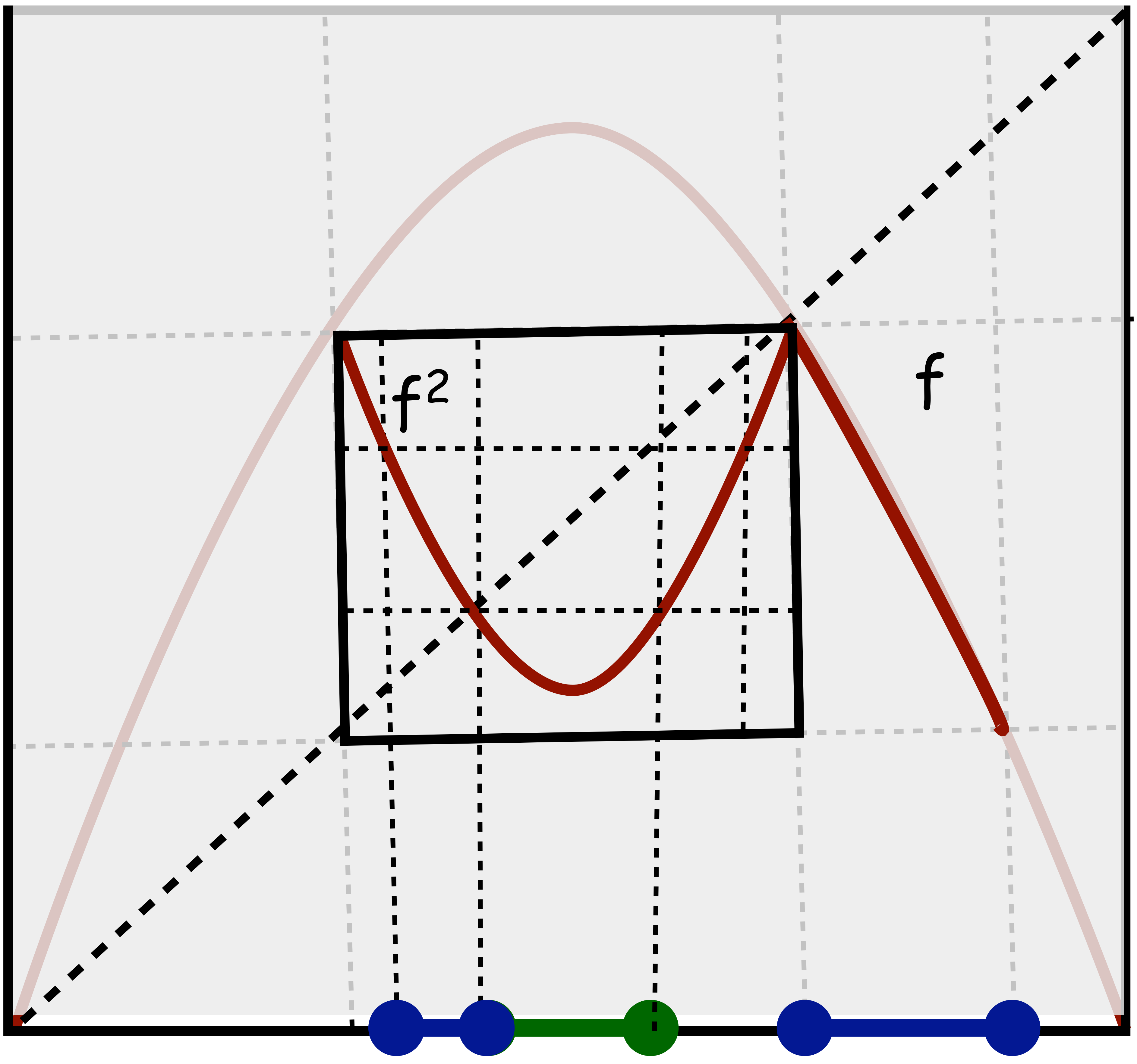

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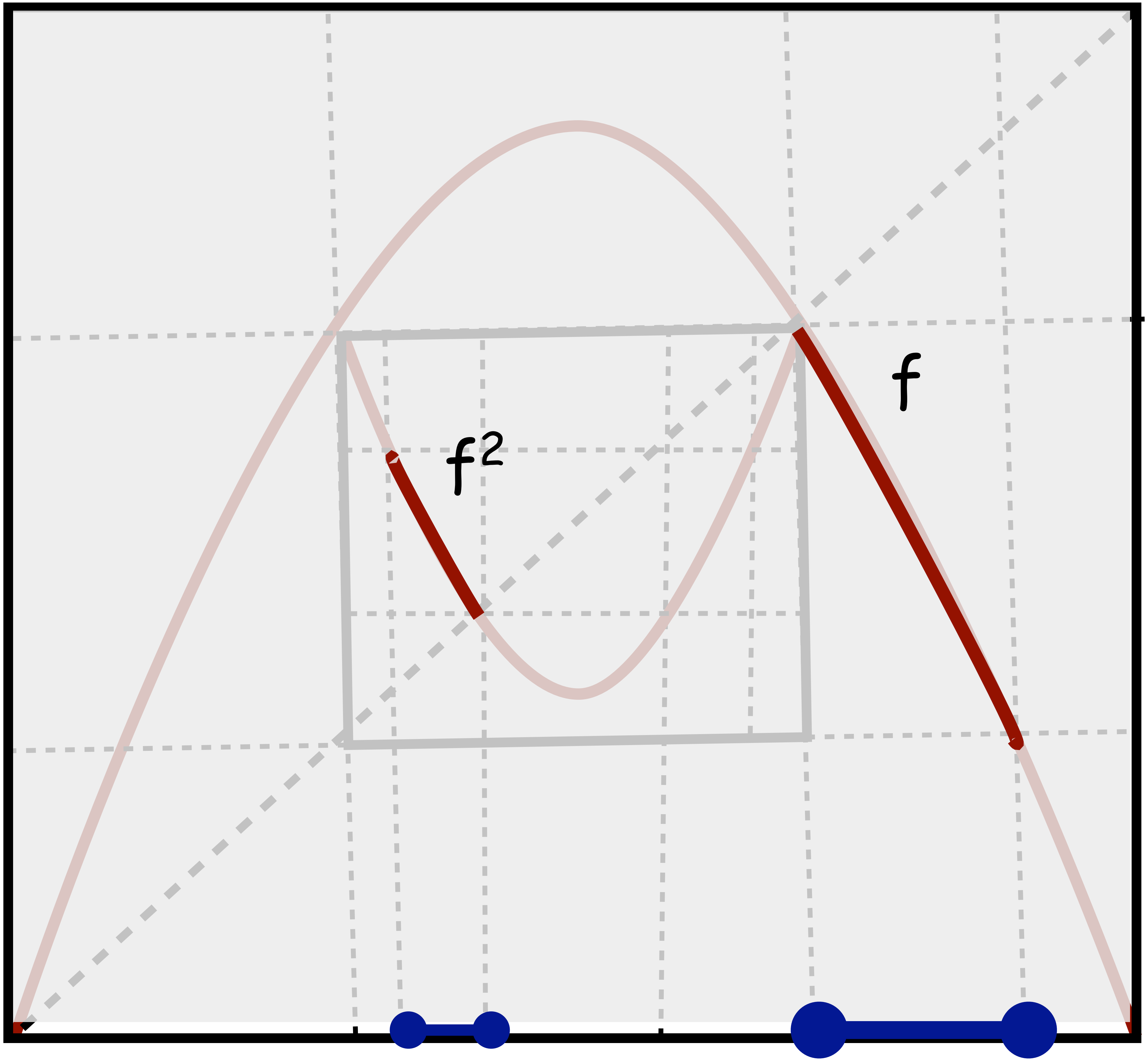

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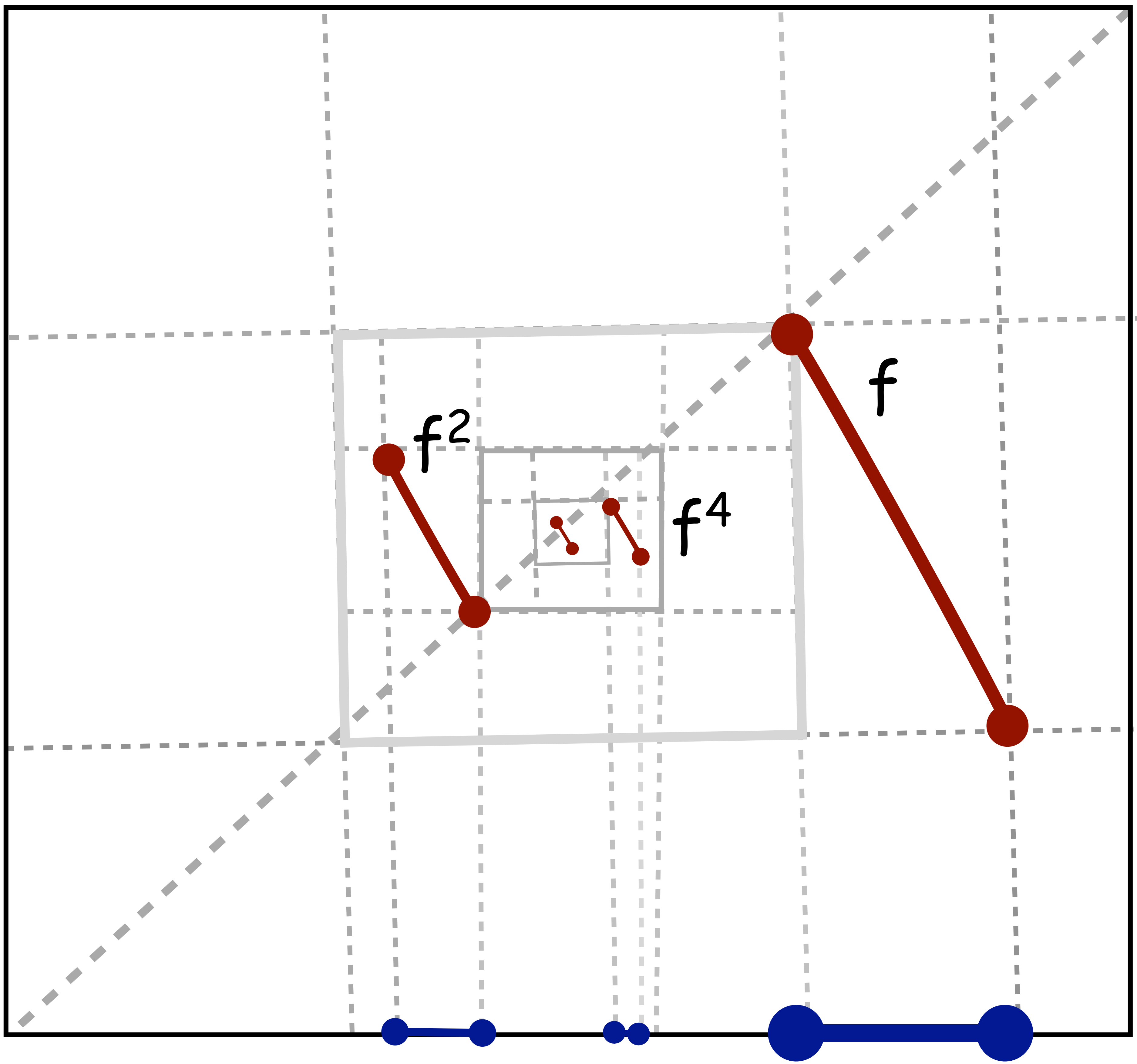
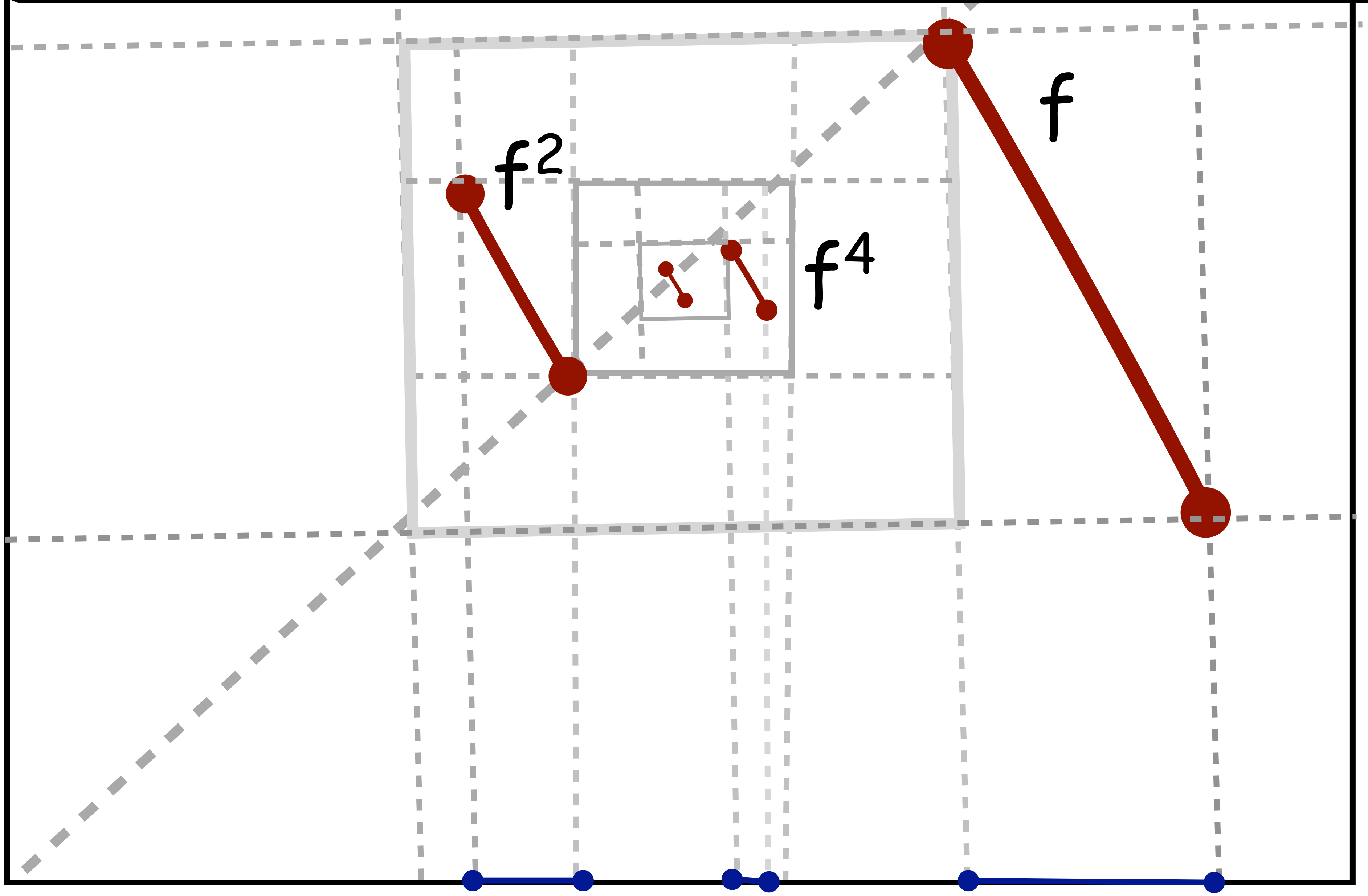


Period-doubling case: Complex induced map



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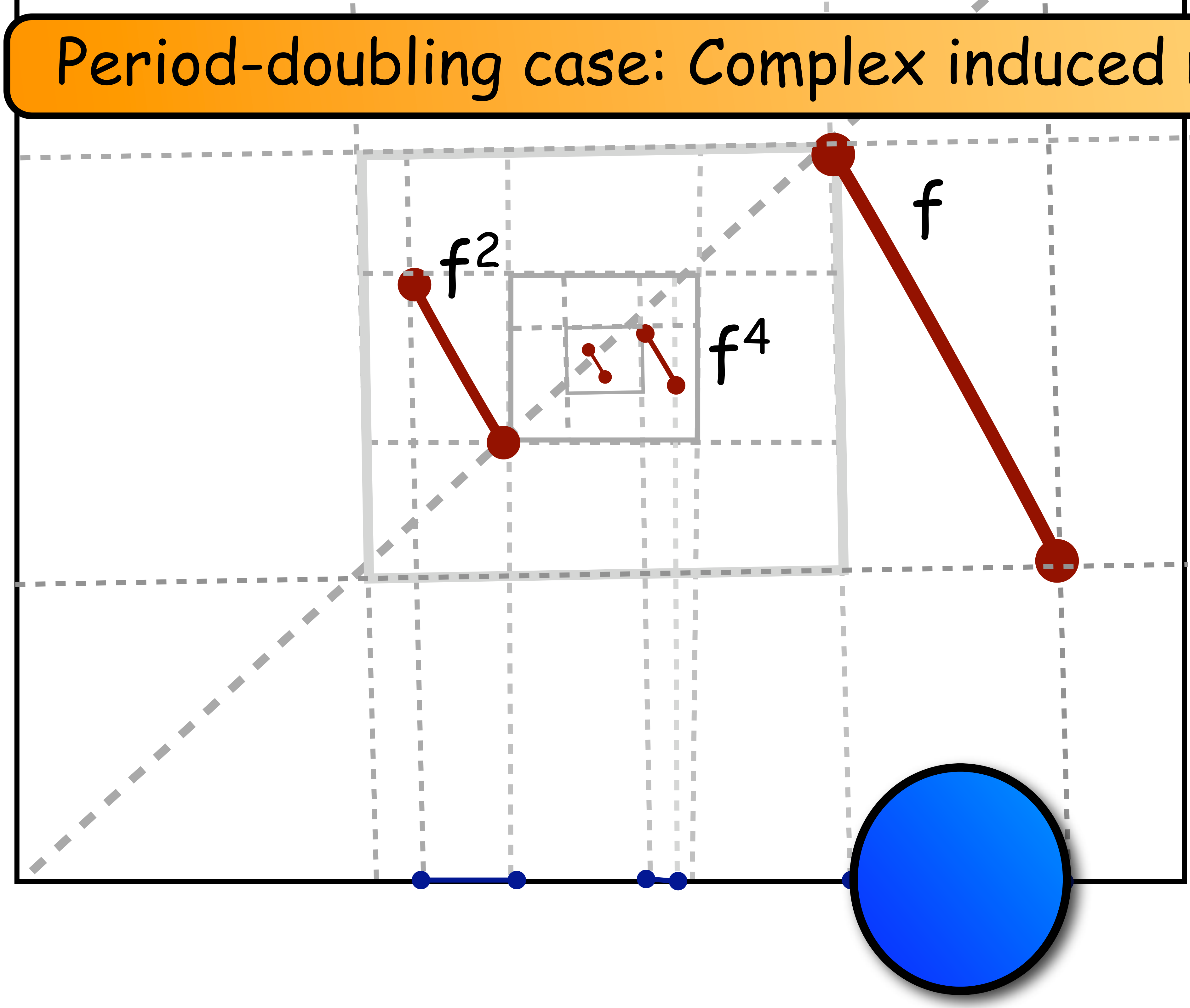

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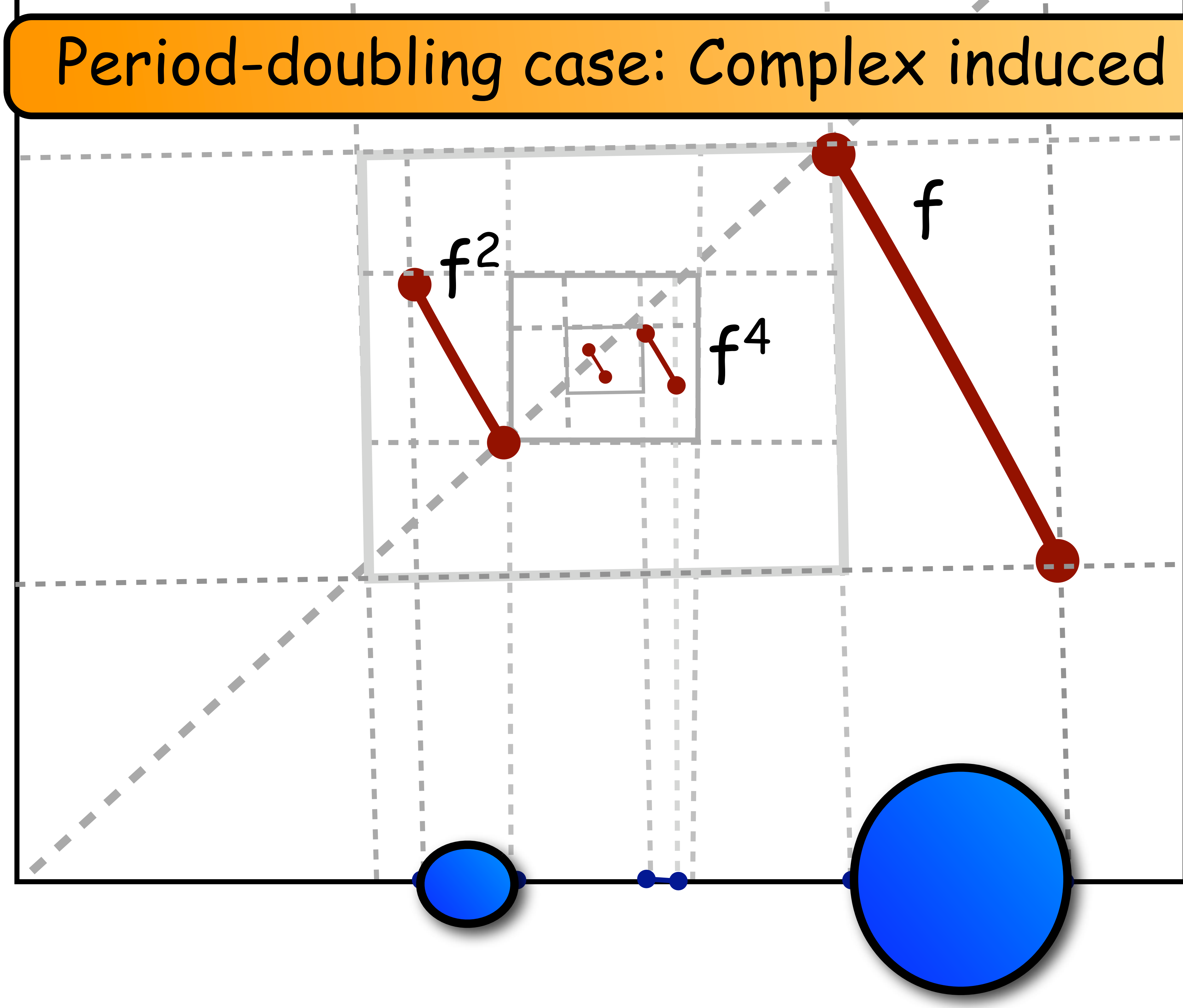

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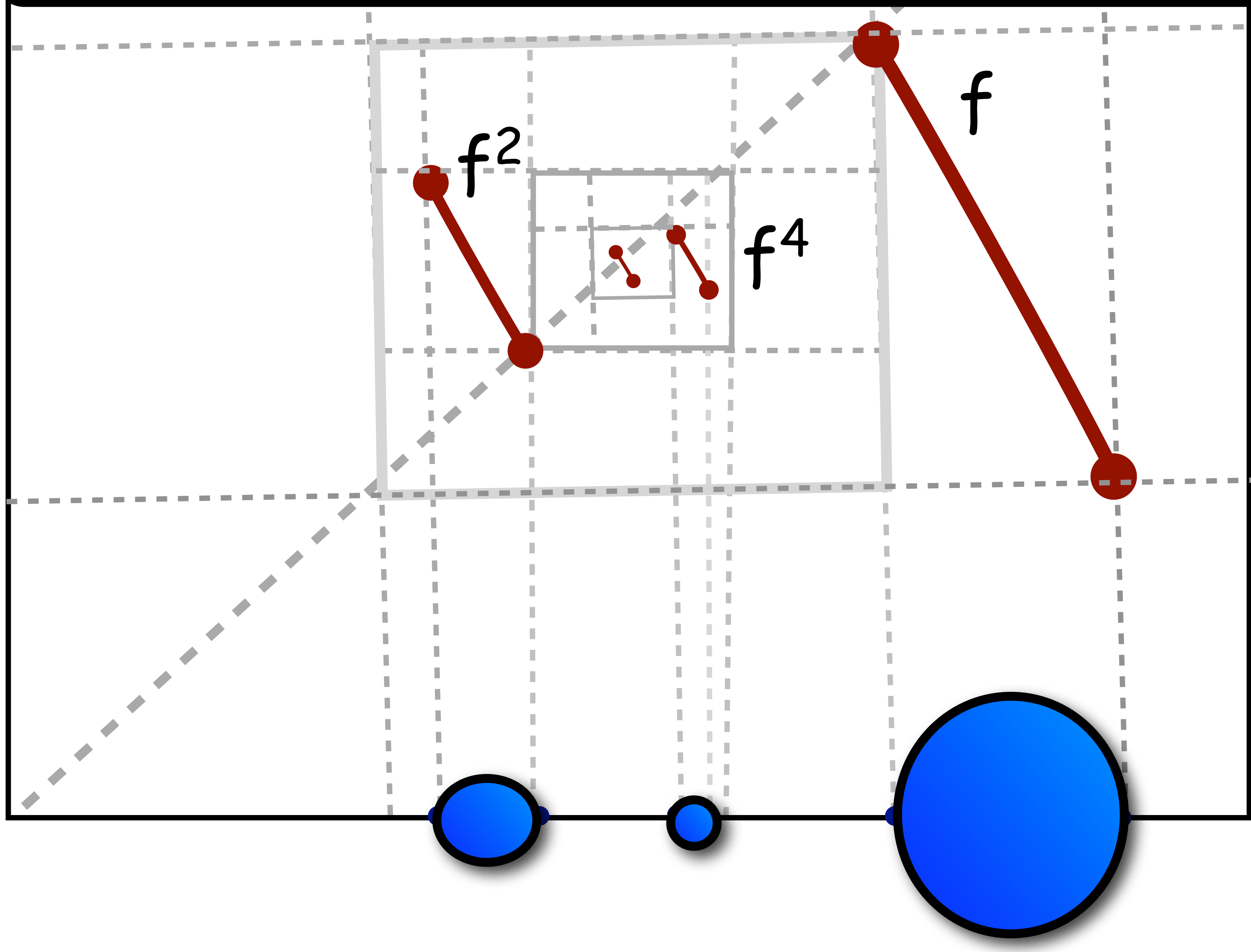

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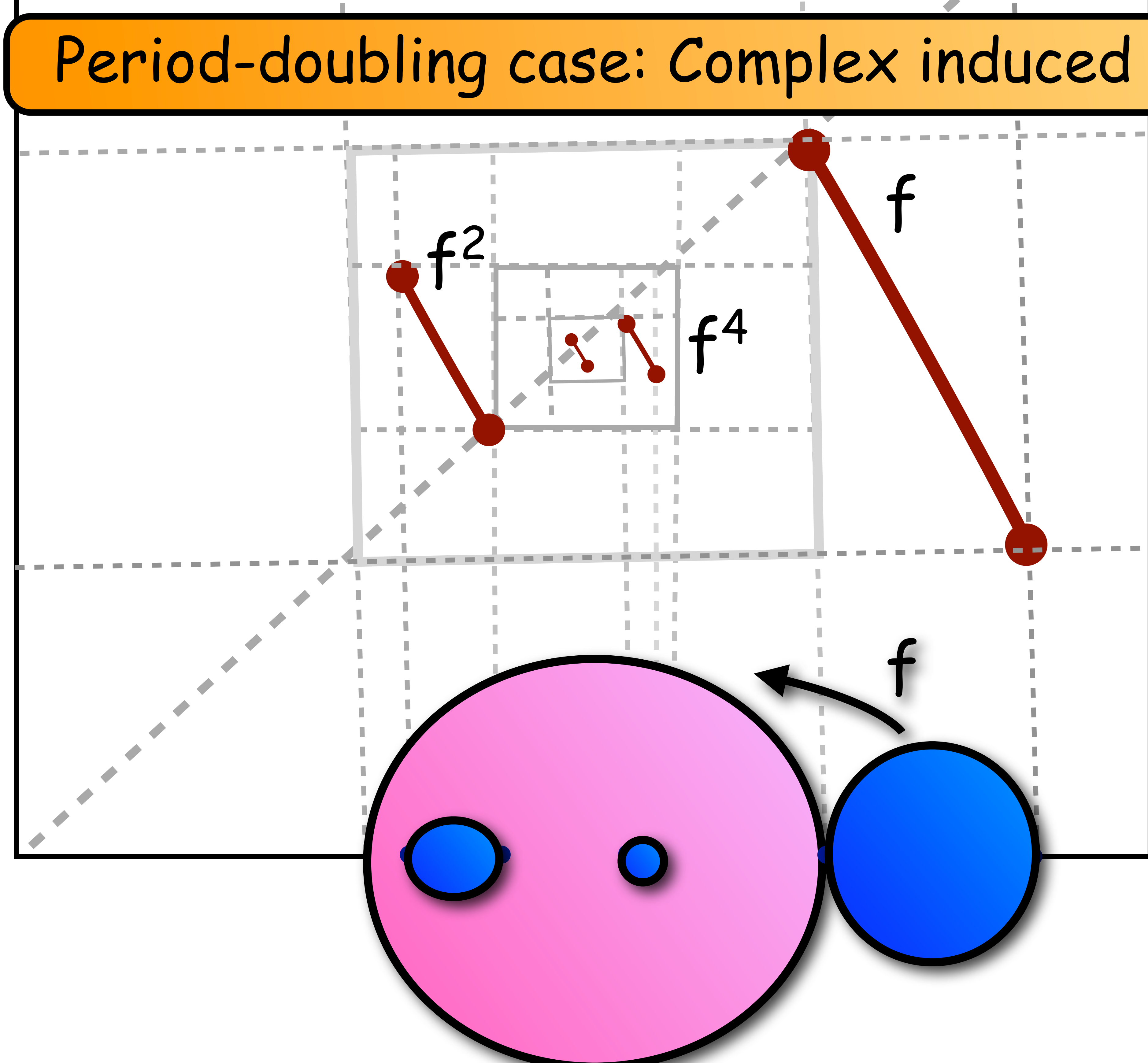

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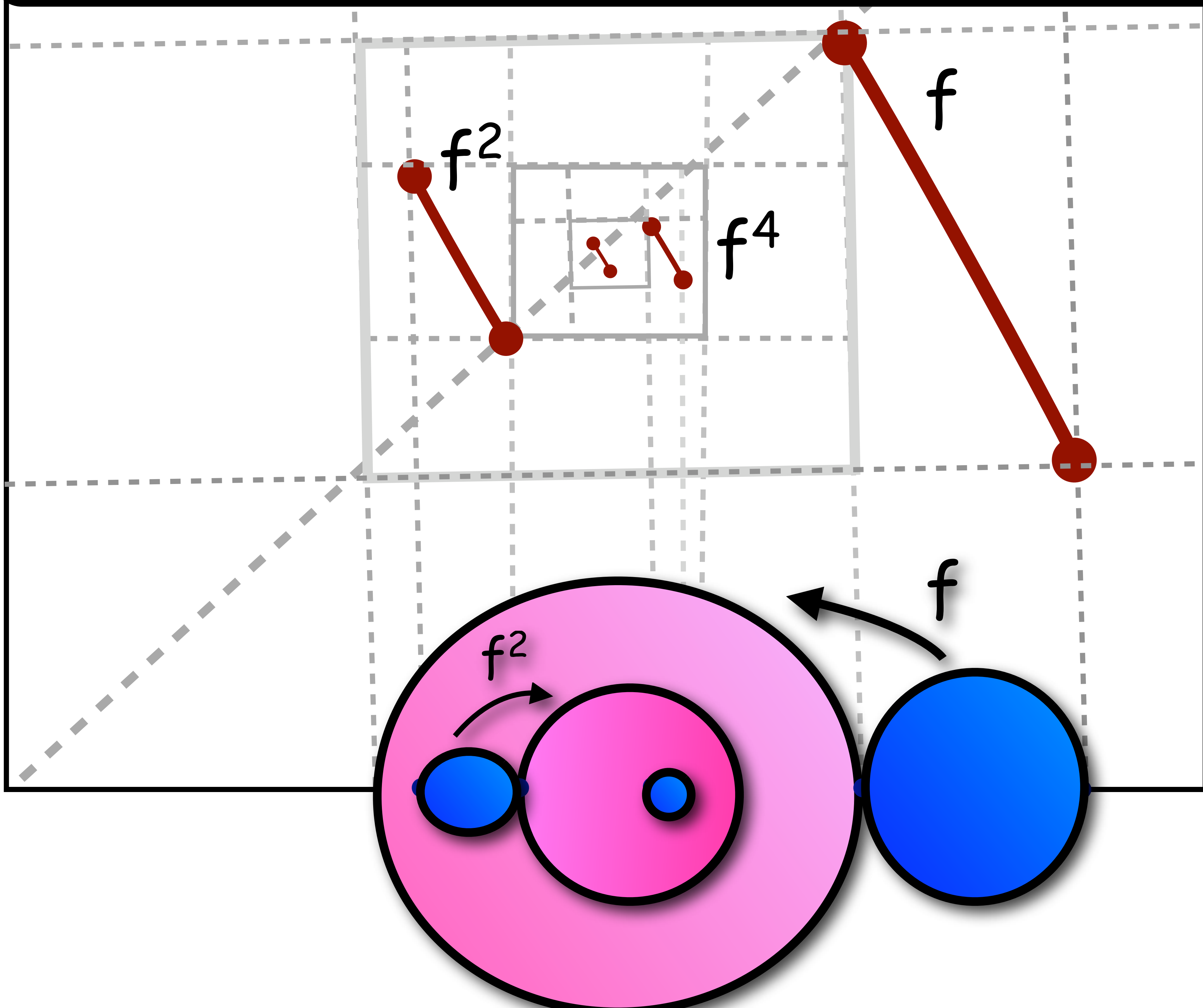

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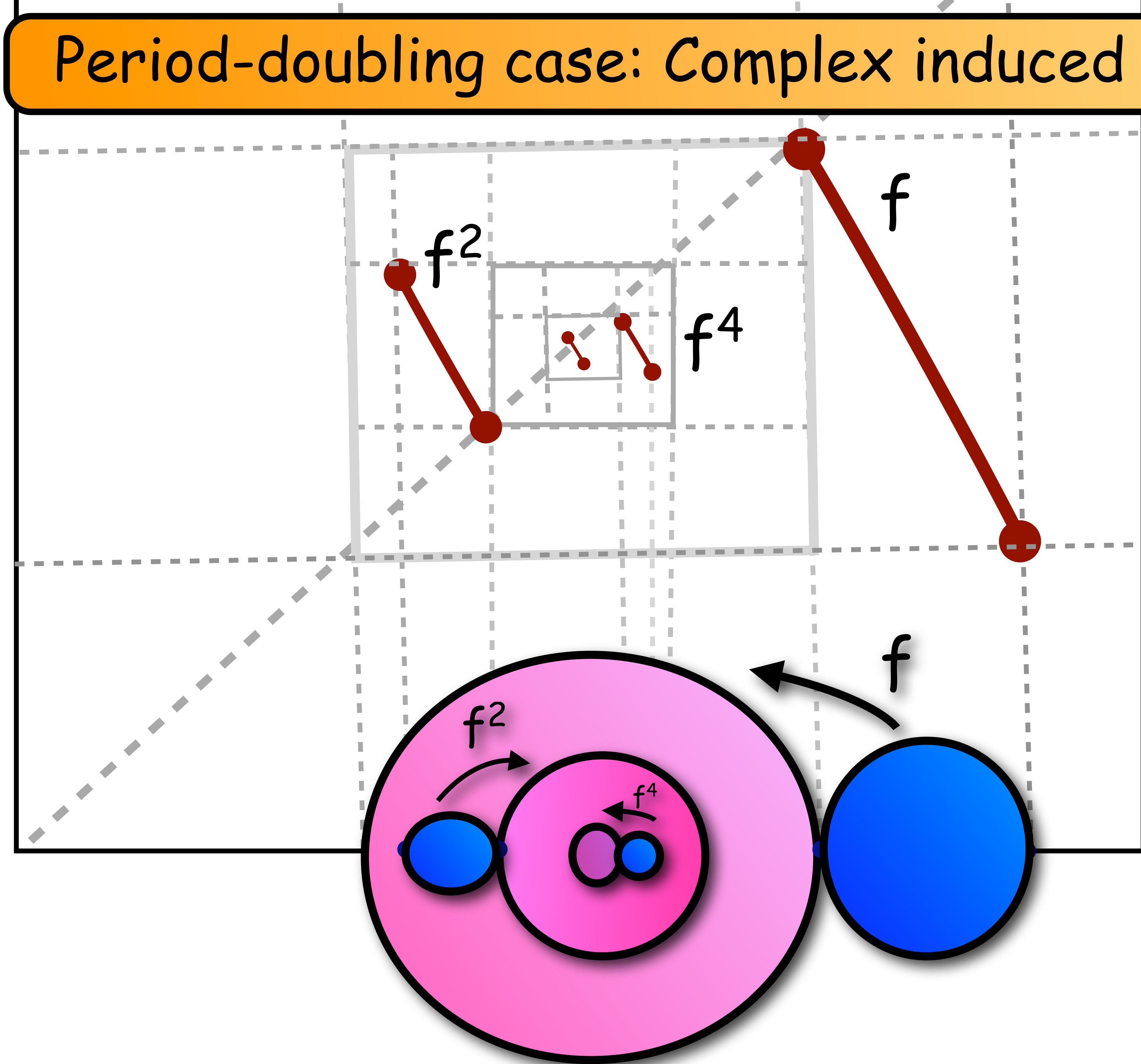
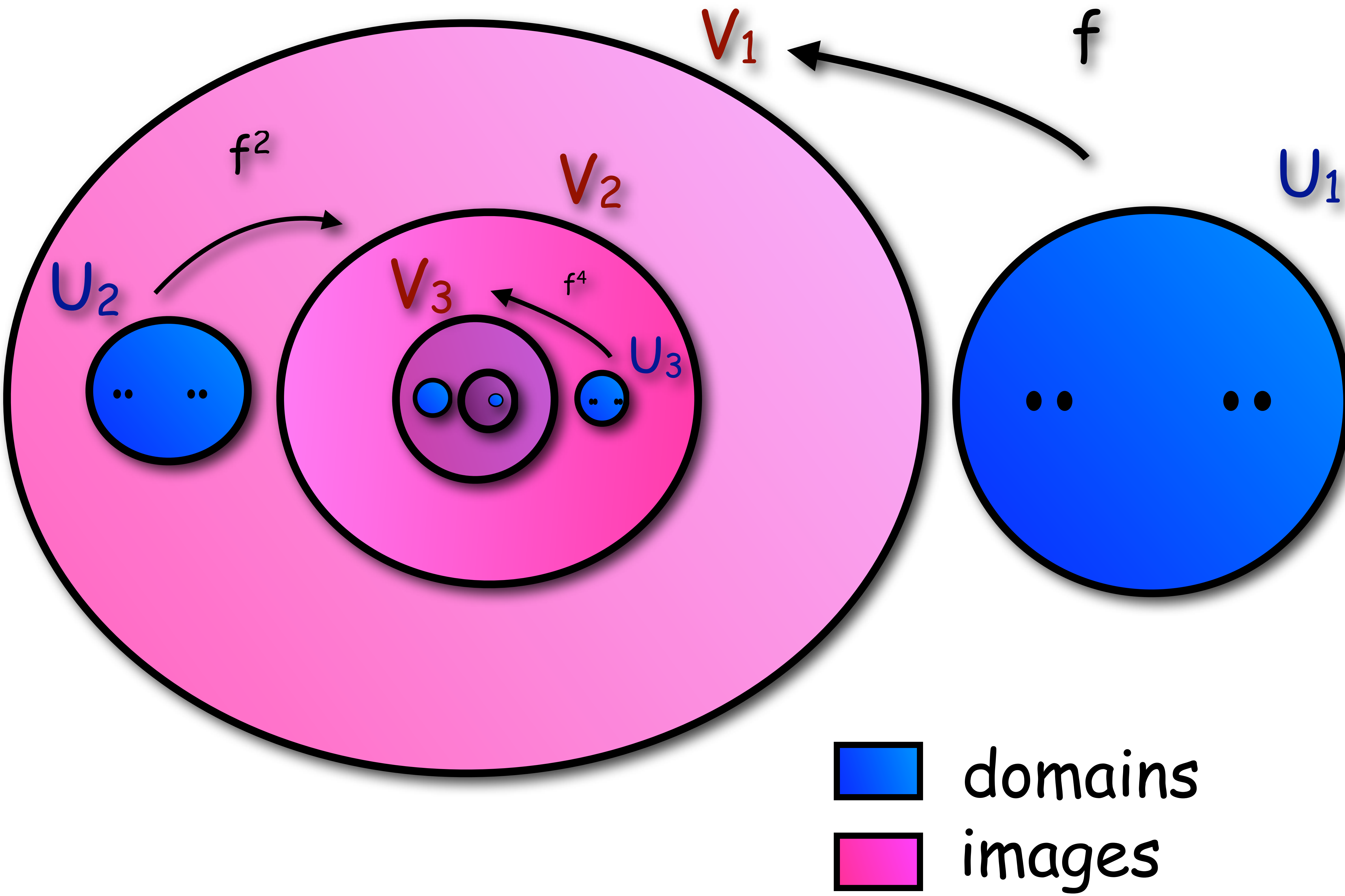
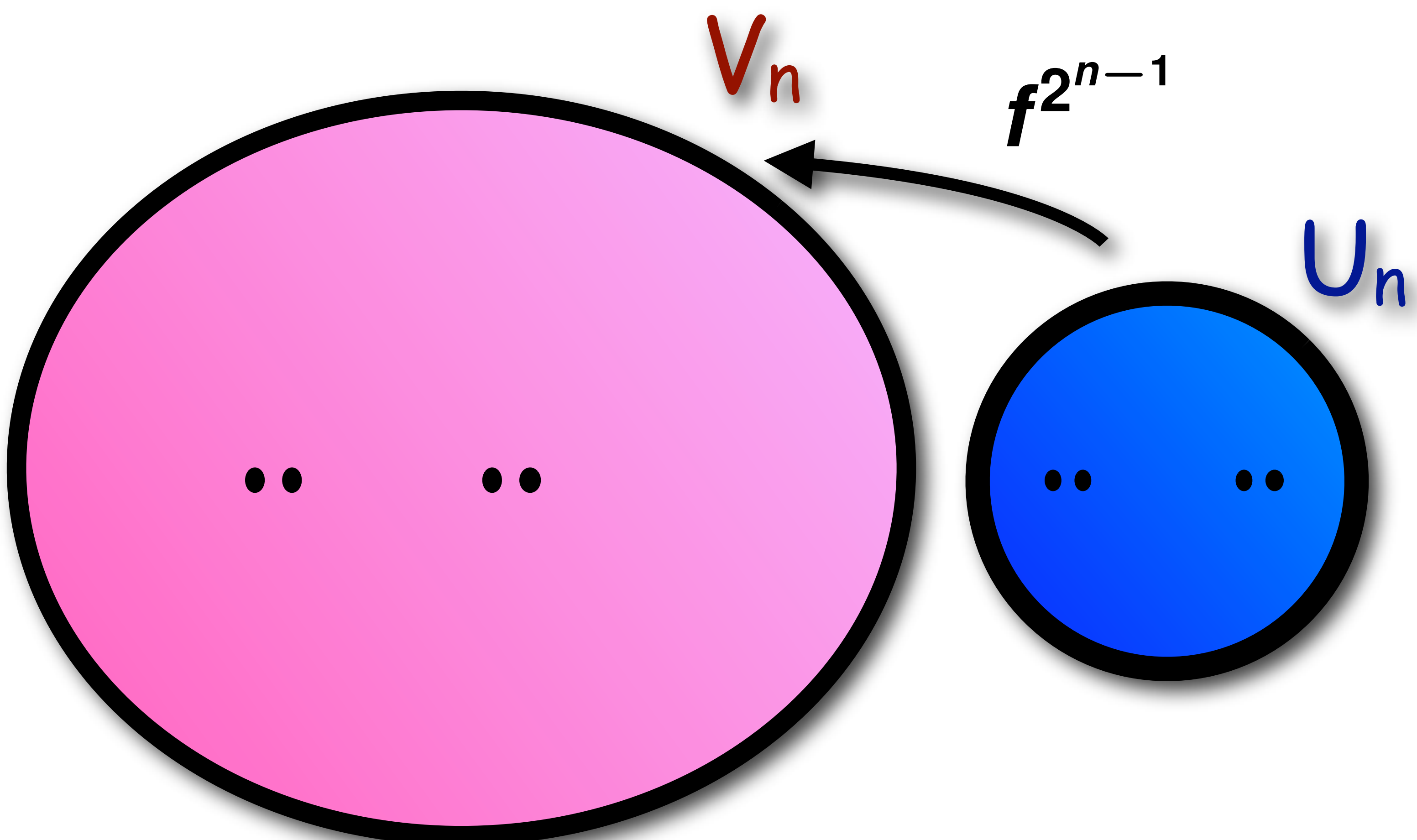


Period-doubling case: Complex induced map

(reducing the domain a little bit)



Induced Problem

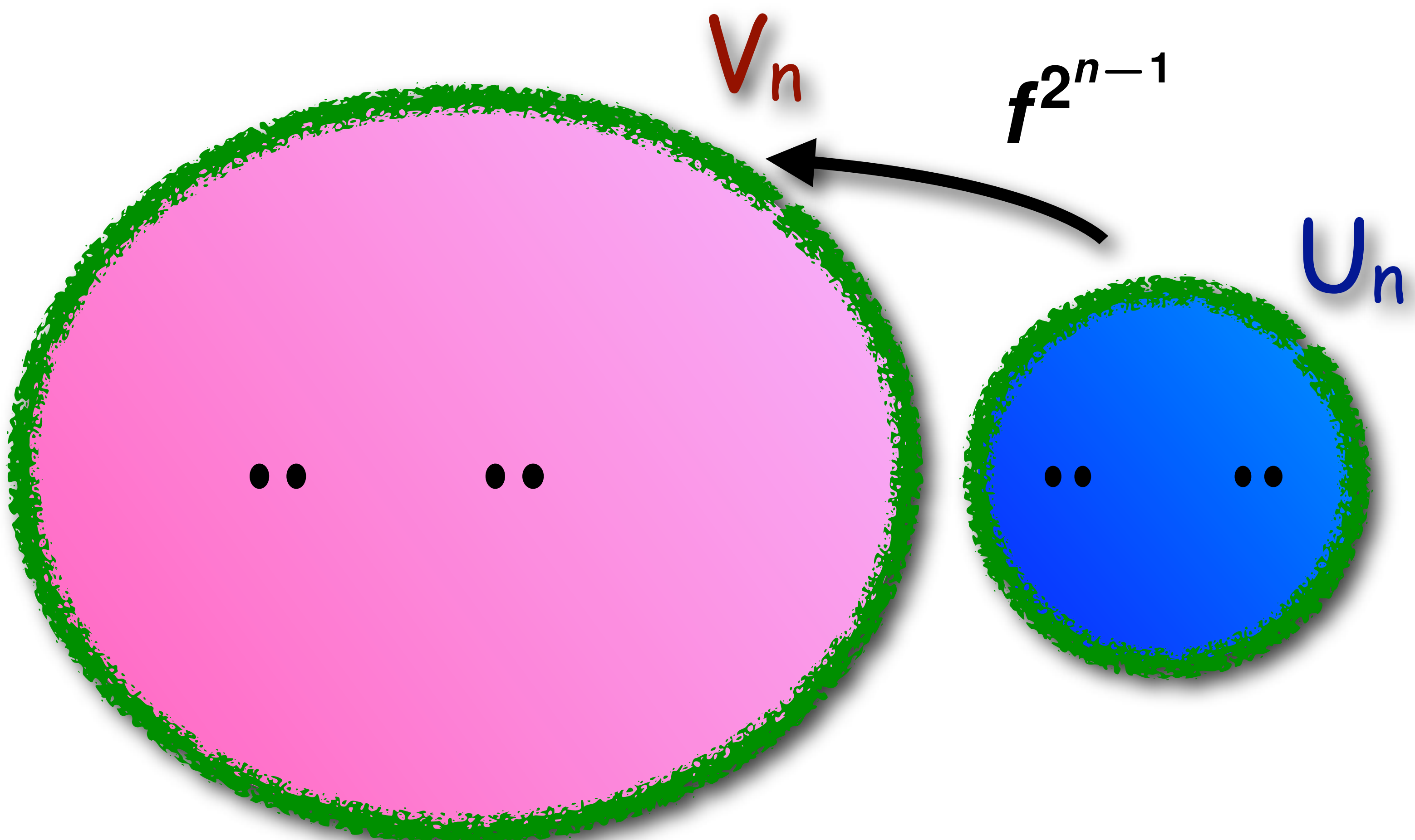


Finding a quasiconformal vector field α such that

$$\partial_t (f + tv)^{2^n-1}\big|_{t=0}(x) = \alpha(f^{2^n-1}(x)) - Df^{2^n-1}(x) \cdot \alpha(x)$$

for every $x \in \partial U_n$ and for all n.

Induced Problem



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More information on $\partial_t(f + tv)^{2^n}|_{t=0}(y)$

$$\partial_t(f + tv)^{2^n}|_{t=0}(y) = p_{n,0} \cdot (D\mathcal{R}_t^n \cdot v)(\frac{y}{p_{n,0}})$$

$$+\partial_x f^{2^n}(y) \cdot \beta_n(y) - \beta_n(f^{2^n}(y))$$

More information on $\partial_t (f + tv)^{2^n}|_{t=0}(y)$

$$\partial_t (f + tv)^{2^n}|_{t=0}(y) = \underbrace{p_{n,0} \cdot (D\mathcal{R}_t^n \cdot v)(\frac{y}{p_{n,0}})}_{w_1}$$

nice!! since $|D\mathcal{R}^n \cdot v| \leq C$ for every n!!

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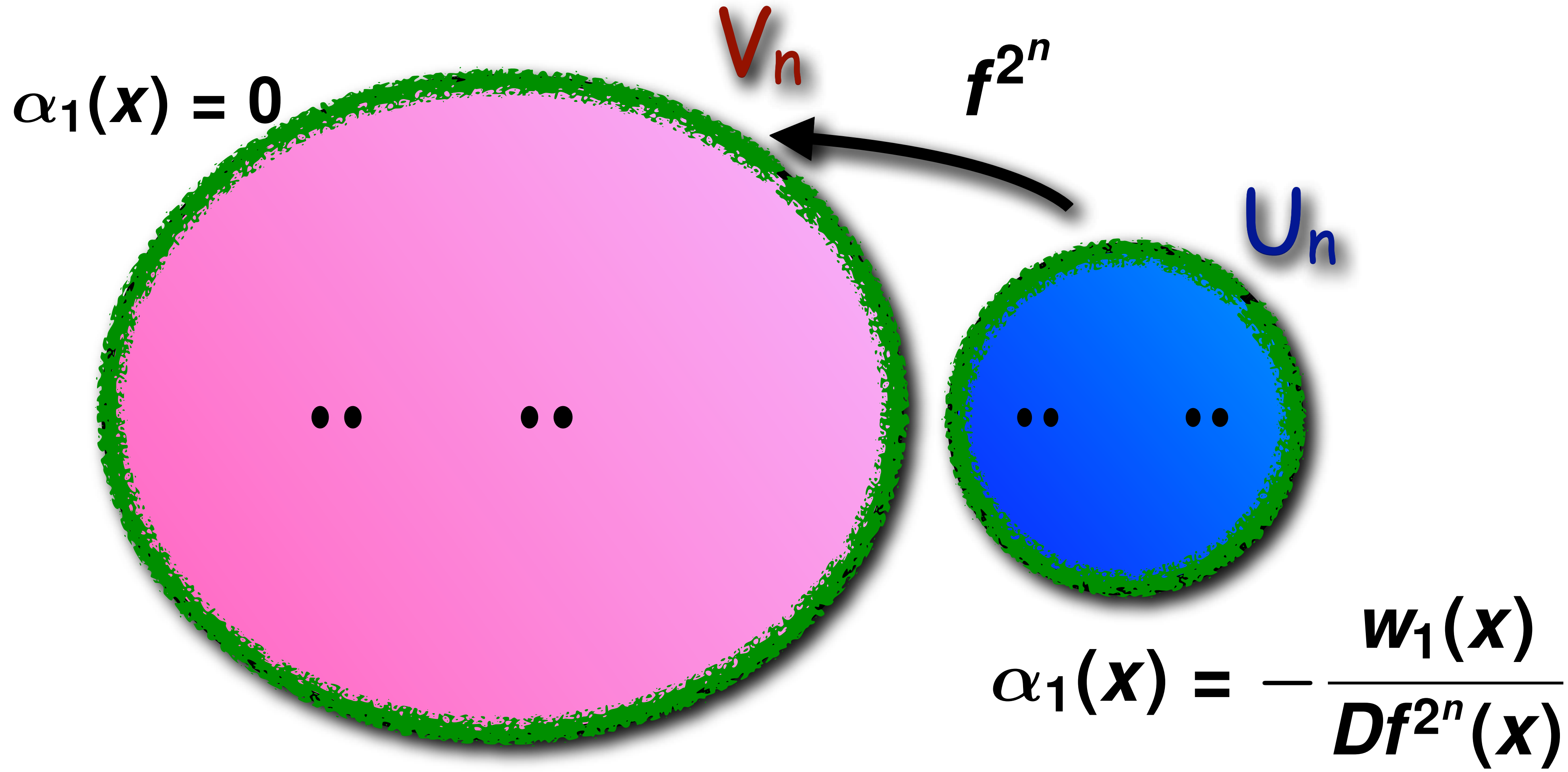
nice!! since $|D\mathcal{R}^n \cdot v| \leq C$ for every n!!

$$+ \underbrace{\partial_x f^{2^n}(y) \cdot \beta_n(y) - \beta_n(f^{2^n}(y))}_{w_2}$$

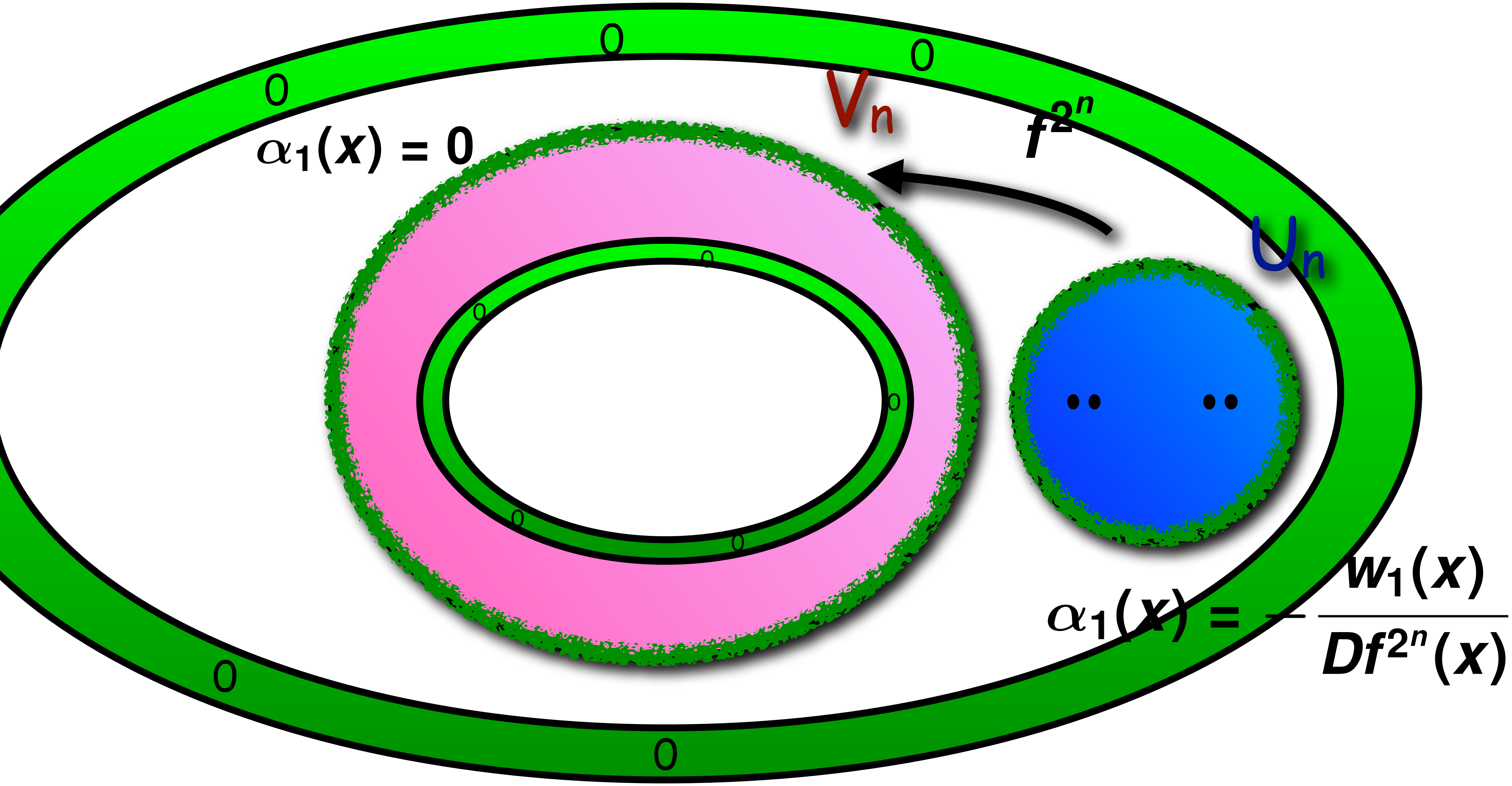
where $\beta_n(y) = \frac{\partial_t p_{n,t}}{p_{n,t}} y$

Solution of induced problem for w_1



$$w_1(x) = \alpha_1(f^{2^n}(x)) - Df^{2^n}(x) \cdot \alpha_1(x)$$

Solution of induced problem for w_1



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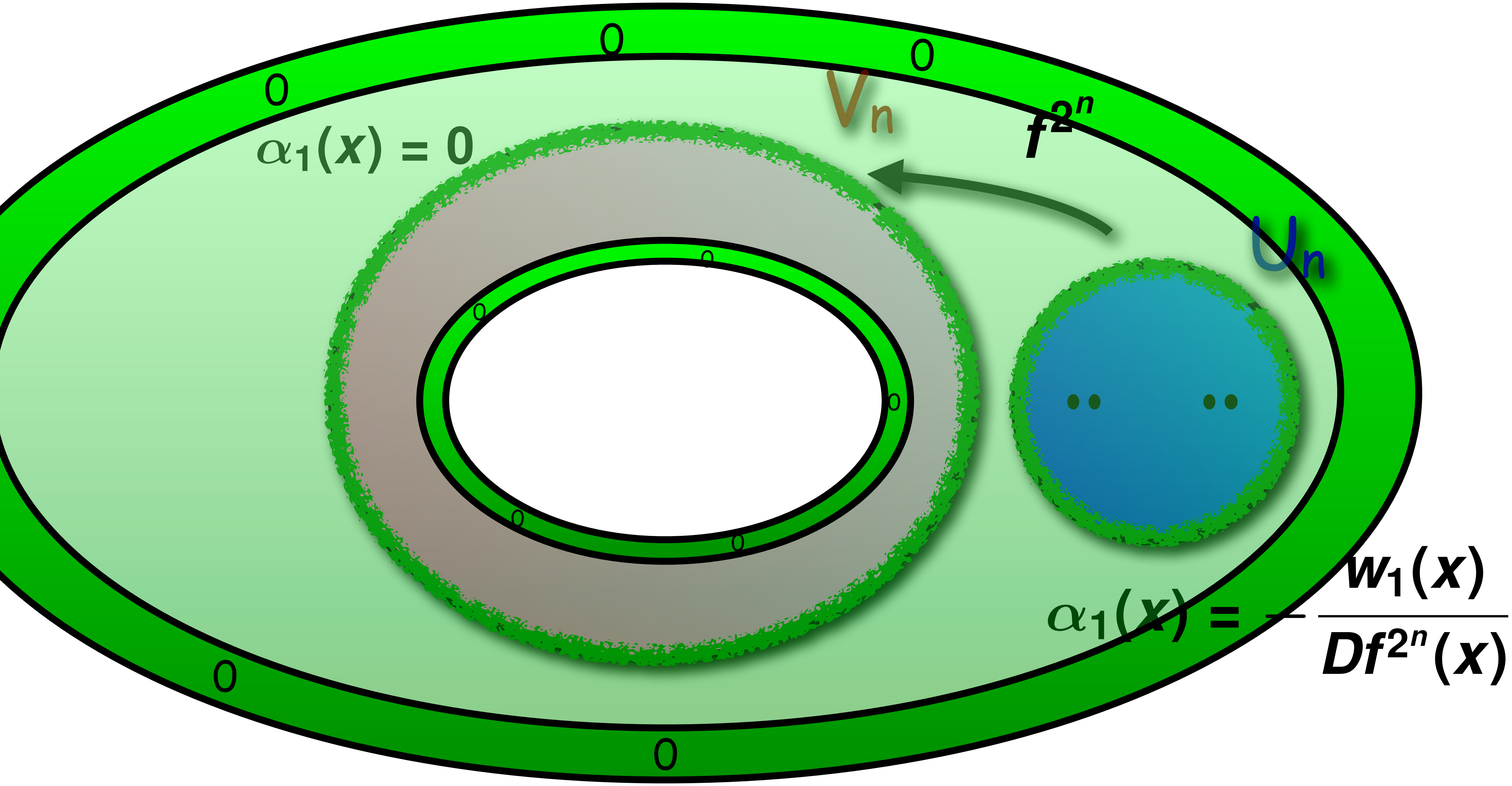
Solution of induced problem for w_1



$$w_1(x) = \alpha_1(f^{2^n}(x)) - Df^{2^n}(x) \cdot \alpha_1(x)$$

Solution of induced problem for w_2

$$w_2(x) = Df^{2^n}(x) \cdot \beta_n(x) - \beta_n(f^{2^n}(x))$$

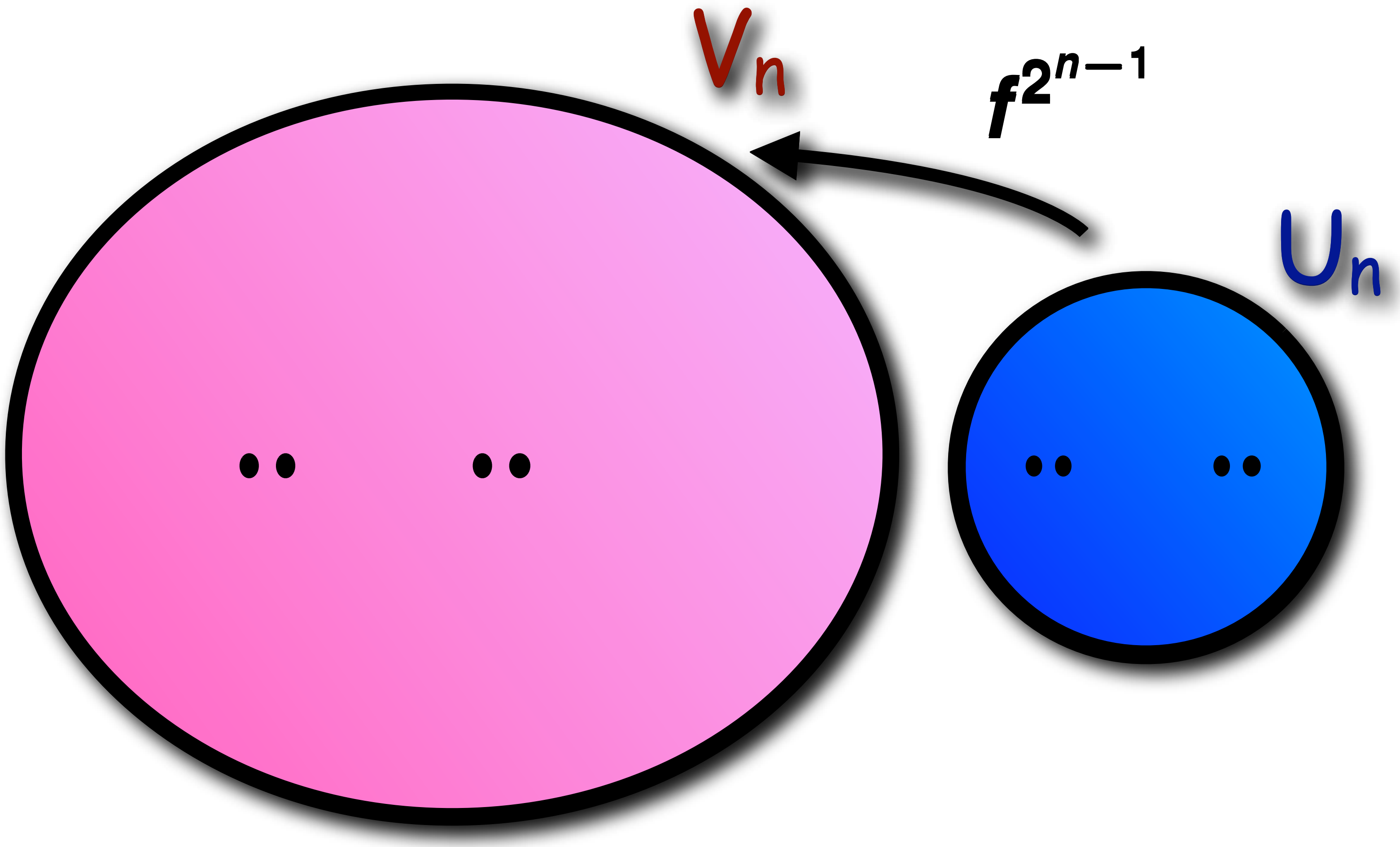
$$\beta_n(x) = \frac{\partial p_{n,t}}{p_{n,0}} \cdot x = c_n \cdot x$$

Because $|D\mathcal{R}_f^n \cdot v| < C$ it follows that

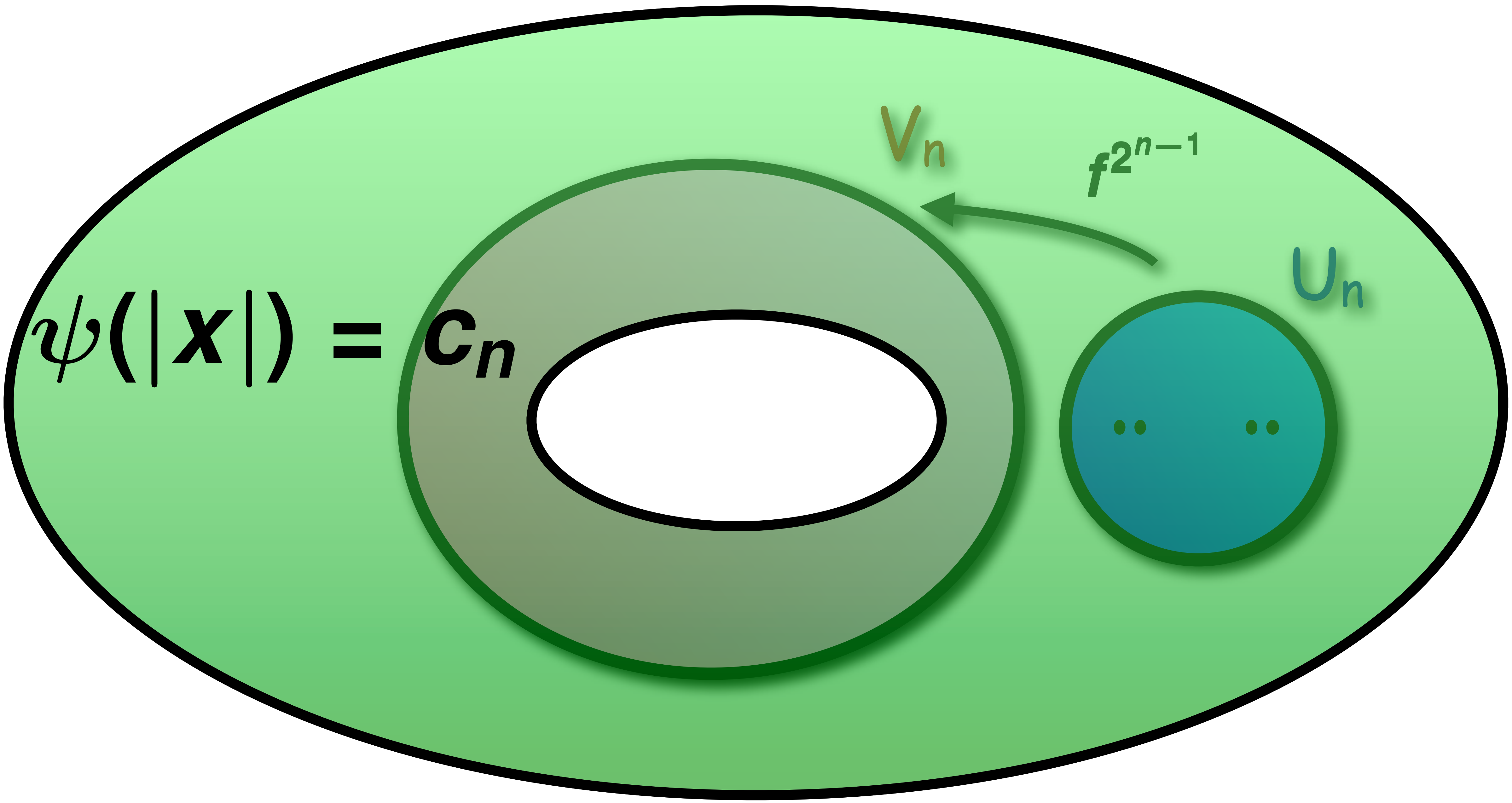
$$|c_{n+1} - c_n| < C$$

Define $\alpha_2(x) = \psi(|x|) \cdot x$

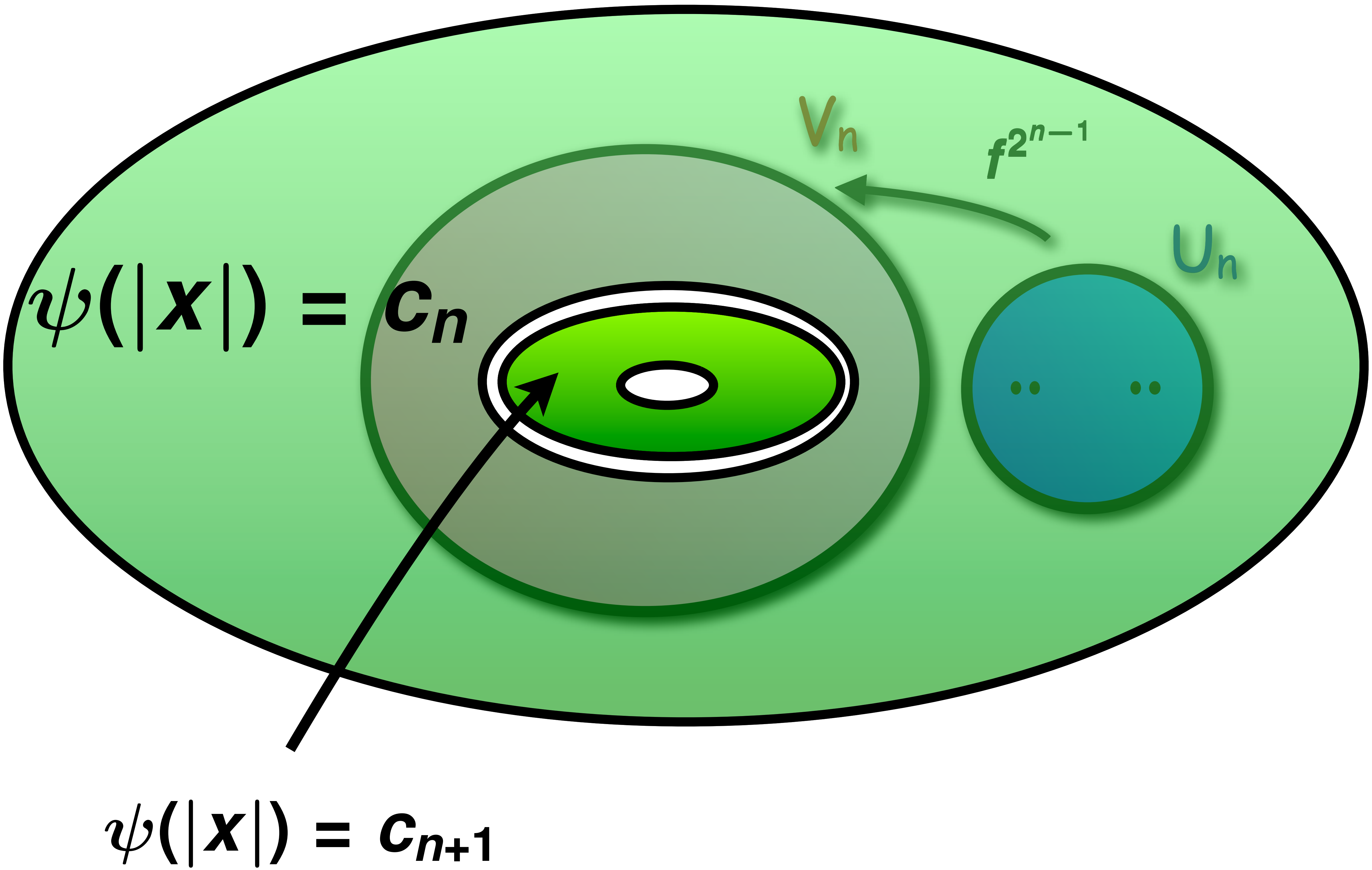
Solution of induced problem for w_2



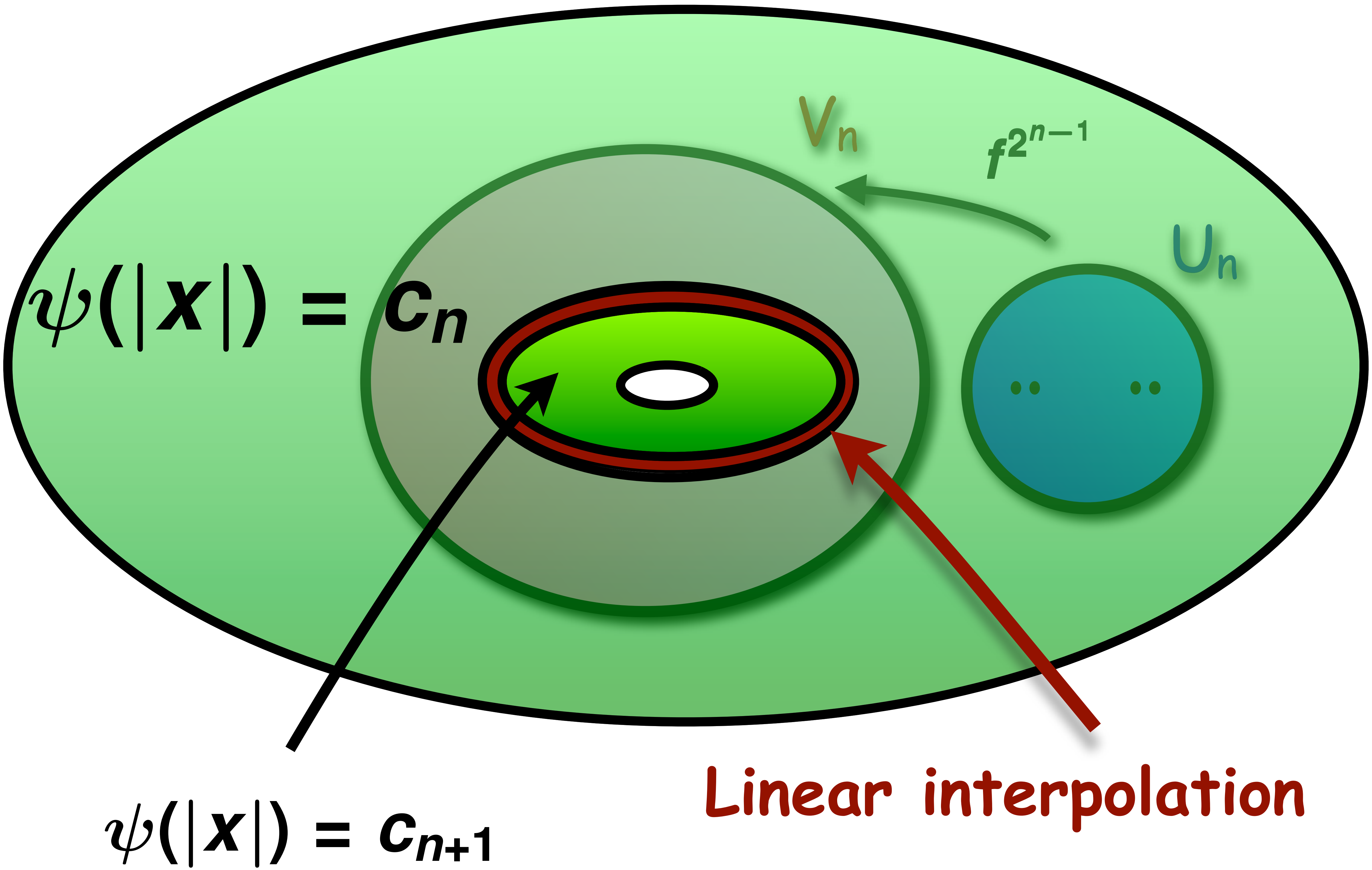
Solution of induced problem for w_2



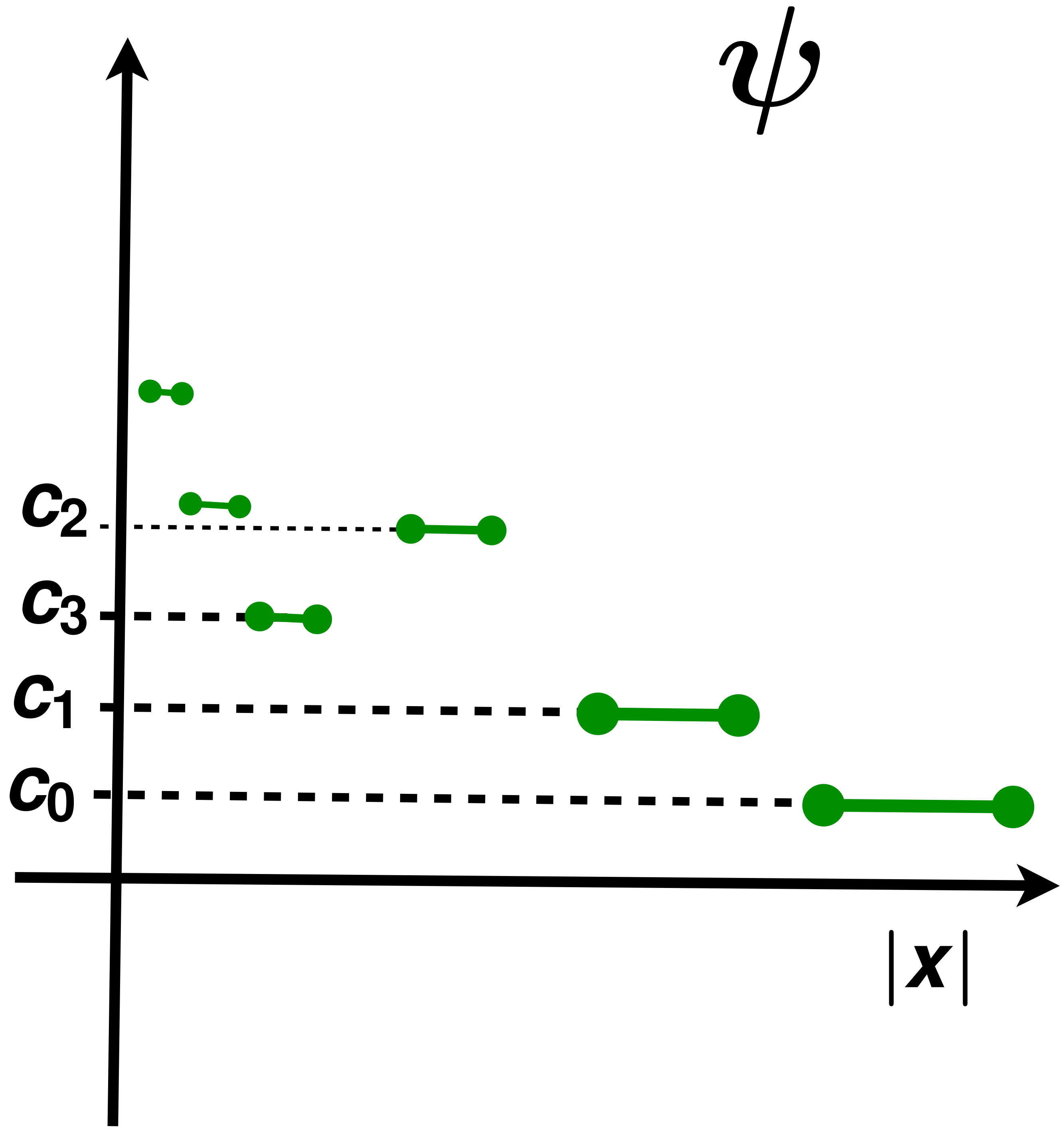
Solution of induced problem for w_2



Solution of induced problem for w_2



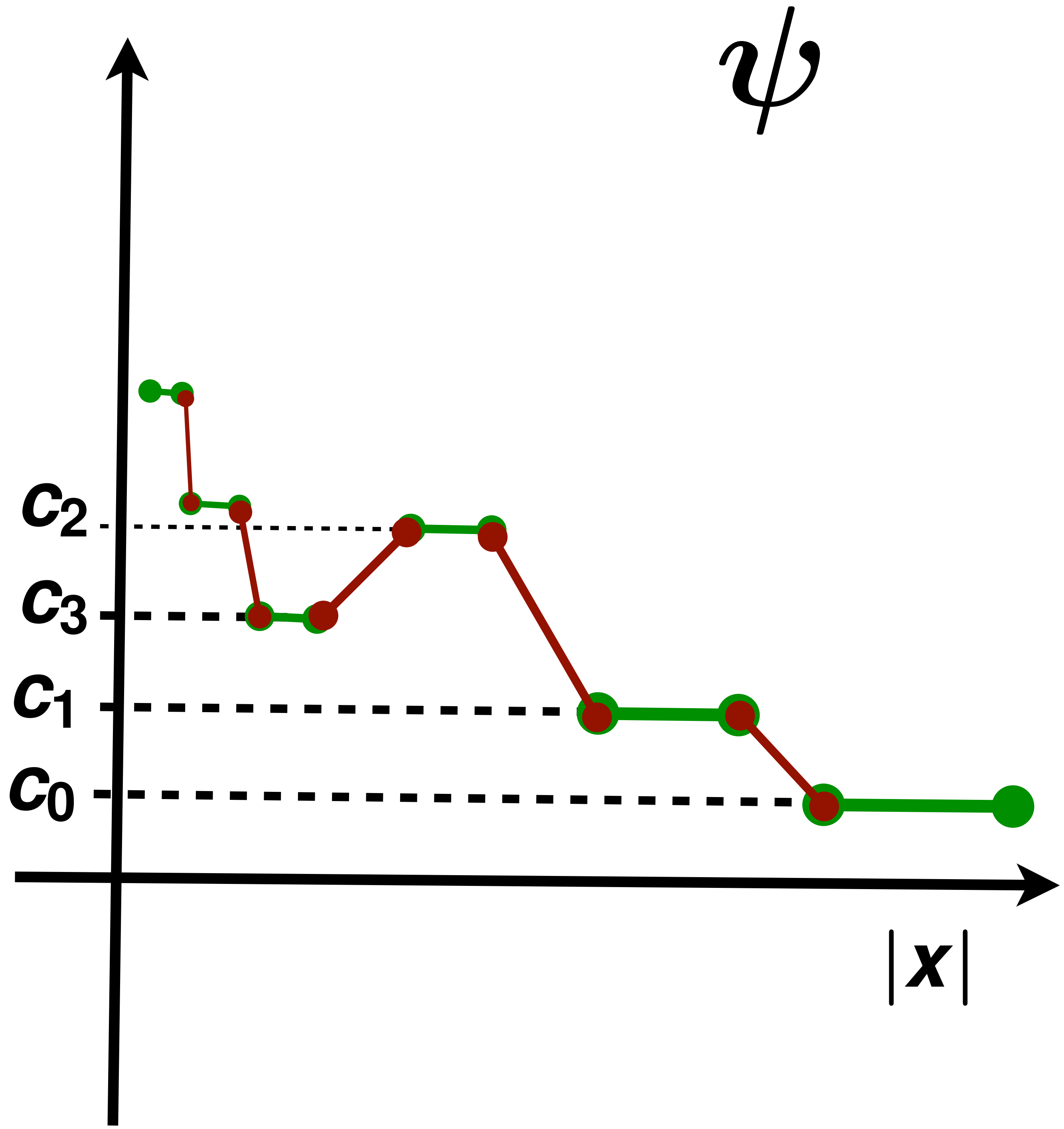
Solution of induced problem for w_2



$$\alpha_2(x) = \psi(|x|) \cdot x$$

is a quasiconformal vector field!!

Solution of induced problem for w_2



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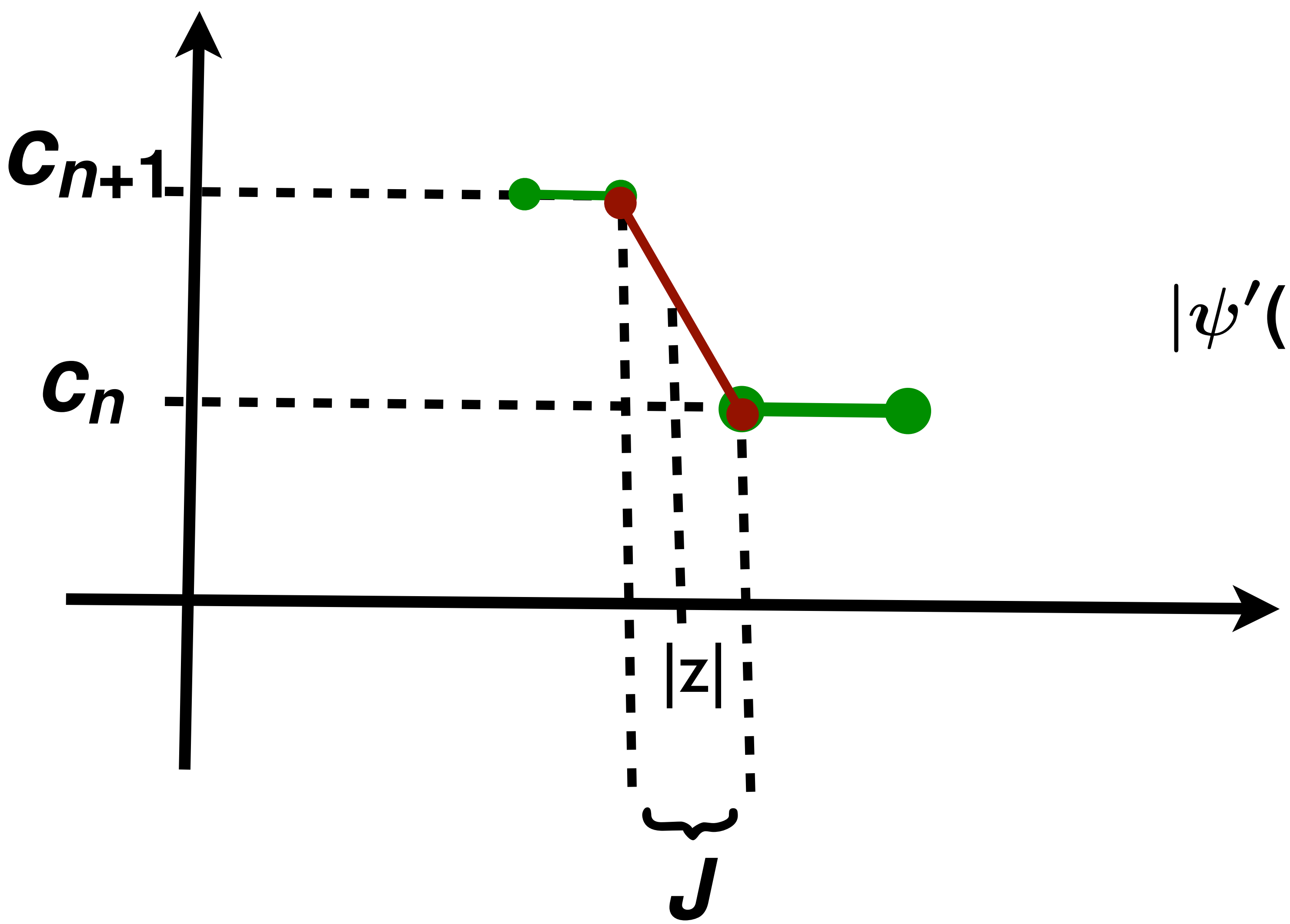
Solution of induced problem for w_2

$$\alpha_2(x) = \psi(|x|) \cdot x$$

$$|\bar{\partial}\alpha_2(z)| = \frac{|z\psi'(|z|)|}{2} < C$$

Motivation: If $\psi(|z|) = \ln|z|$. then $|\bar{\partial}\alpha_2| = 1/2$



$$|J| > \epsilon_0 |z|$$

$$|\psi'(|z|)| = \frac{|c_{n+1} - c_n|}{|J|} \leq \frac{C}{\epsilon_0 |z|}$$

Why is this proof nice? :-)

1. Proof is mostly infinitesimal!!
2. Sacker&Sell, Steps 3 and 4 do not use complex dynamics/ structure.
3. In most of the proof of Key Lemma the critical point disappears.
4. Proof is a kind of linear version of the original Lyubich(1999) argument.

Why is this proof nice? :-)

1. Proof is mostly infinitesimal!!
2. Sacker&Sell, Steps 3 and 4 do not use complex dynamics/structure. Hope for $|x|^d + c$, $d \notin 2\mathbb{N}$.
3. In most of the proof of Key Lemma the critical point disappears.
4. Proof is a kind of linear version of the original Lyubich(1999) argument.