RIEMANNIAN GEOMETRY AND GAUSSIAN-LIKE MEASURES ON SOME INFINITE-DIMENSIONAL CURVED SPACES

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MOTIVATION

$$\begin{tikzcd}
L^2(\mathbb{R}^n, dx) \arrow[r, "\text{ground state transformation}"] & L^2(\mathbb{R}^n, d\rho_t) \arrow[dd, "S_t"] \arrow[dr, "B"] & \\
f \arrow[r, mapsto] & \frac{f}{\varphi} & \mathfrak{F}_s \\
& \mathcal{H}L^2(\mathbb{C}^n, d\mu_t) \arrow[ur, "T"'] &
\end{tikzcd}$$

$$d\rho_t = \rho_t dx, \quad d\mu_t(z) = \frac{e^{-|z|^2/t}}{(\pi t)^n} dz$$

$\mathfrak{F}_s = \oplus_{m=0}^{\infty} H^{\otimes_s m}$ is a **bosonic Fock space**, where H is a state space of a single particle ($\mathbb{C}^n$).

$\mathcal{H}L^2(\mathbb{C}^n, \mu_t)$ ={square integrable holomorphic functions}

$$\boldsymbol{B} : \boldsymbol{H_{k_1}(x_{j_1}) \cdot \ldots \cdot H_{k_n}(x_{j_n}) \mapsto x_{j_1}^{k_1} \otimes_s \ldots \otimes_s x_{j_n}^{k_n}}$$

$$\boldsymbol{S_t} : \boldsymbol{H_{k_1}(x_{j_1}) \cdot \ldots \cdot H_{k_n}(x_{j_n}) \mapsto z_{j_1}^{k_1} \cdot \ldots \cdot z_{j_n}^{k_n}}$$

$\boldsymbol{H_k(x)}$ is an Hermite polynomial.

$$\boldsymbol{T} : \boldsymbol{z_{j_1}^{k_1} \cdot \ldots \cdot z_{j_n}^{k_n} \mapsto x_{j_1}^{k_1} \otimes_s \ldots \otimes_s x_{j_n}^{k_n}}$$

INFINITE-DIMENSIONAL LINEAR CASE

A complex Wiener space is $(\boldsymbol{W}, \boldsymbol{H}_{CM}, \boldsymbol{\mu})$, where

$\boldsymbol{W}$ is a complex separable Banach space,

$\boldsymbol{H}_{CM}$ is the Cameron-Martin subspace (a complex separable Hilbert space which is densely and continuously embedded in $\boldsymbol{W}$), and

$\boldsymbol{\mu}$ is a Gaussian measure defined by

$$\int_{\boldsymbol{W}} e^{i\varphi(z)} d\mu(z) = e^{-\frac{1}{4}\|\varphi\|^2_{H^*_{CM}}}, \quad \varphi \in \boldsymbol{W}^* \subset \boldsymbol{H}^*_{CM}.$$

$\boldsymbol{\mu}(\boldsymbol{H}_{CM}) = 0$.

Irving Segal's Work on Infinite Dimensional Integration Theory

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Irving Segal's work on infinite dimensional integration theory was driven by the mathematical needs of quantum field theory.

Irving's point of view on integration over an infinite dimensional real Hilbert space is easy to understand. Define Gauss measure on $\mathbb{R}^n$ by $\gamma_n = (2\pi)^{-n/2} e^{-|x|^2/2} \, dx$. Suppose that F is an n dimensional subspace of a real infinite dimensional Hilbert space H and that $P\colon H \to F$ is the orthogonal projection. If B is a Borel set in F then the set $C = P^{-1}(B) := \{x \in H;\ Px \in B\}$ is a cylinder in H with base B. Define $\gamma(C) = \gamma_n(B)$. The collection, $\mathscr{R}$, of all such cylinder sets, allowing F to run over all finite dimensional subspaces of H, is a field (but not a σ-field.) The set function γ is well defined on $\mathscr{R}$ because γ_n is a product measure in any orthogonal decomposition of $\mathbb{R}^n$. Unfortunately, although γ is finitely additive on $\mathscr{R}$, it is not countably additive (wherever possible) and consequently it has no countably additive extension to the Borel field of H. In fact the outer γ measure of H is zero. For this reason the theory of integration over an infinite dimensional Hilbert space H deviates significantly from that over $\mathbb{R}^n$. And yet the theory is also more structured than that for general finitely additive measures. Think of the set S of rational numbers in the interval $[0, 1]$ and the set function m which assigns measure $b-a$ to the subset $S \cap (a, b]$. m extends to a finitely additive measure on the field generated by these intervals. But the m outer measure of S is zero and m has no countably additive extension. Of course we know what to do about this: complete S in such a way as to obtain the real unit interval. The same formula used to define m now has a countably additive extension—Lebesgue measure on $[0, 1]$. Similarly there is a "completion" Ω of H and a countably additive probability measure Pr on Ω which extends the finitely additive measure γ in a useful way. The space Ω is not unique, however,

FINITE-DIMENSIONAL CURVED SPACES

$\mathbb{C}^n \Longrightarrow$ a Lie group G or a Riemannian manifold M

The bosonic Fock space $\mathfrak{F}_s \Longrightarrow$ a completion of the universal enveloping algebra

The bosonic Fock space $\mathfrak{F}_s$ has all the information about the Lie group G (third Lie theorem)

Geometry and Brownian motion: stochastic completeness versus geodesic completeness.

Geodesic completeness + **Ricci** bounded from below $\Longrightarrow$ stochastic completeness $\Longrightarrow$....

INFINITE-DIMENSIONAL CURVED SPACES

• **Lie algebra and Laplacian**

Different choices of a norm on a Lie algebra give different Lie algebras

$$\Updownarrow$$

the Lie algebra determines directions of differentiation

$$\Updownarrow$$

the Lie algebra and a norm on it determines a Laplacian

• **Riemannian geometry of these spaces:** exponential map, curvature.

• **What is a heat kernel measure?**

No Haar measure, no heat kernel, only a heat kernel measure.

• **Make sense of the "Gaussian measure"**

$$d\mu(x) \text{ ``}=\text{''} \frac{1}{Z_t} e^{-\int_0^t |\dot{x}(s)|^2 ds} Dx \quad \text{Feynman-Kac formula}$$

I. HILBERT-SCHMIDT GROUPS: GEOMETRY AND THE HEAT KERNEL MEASURES

A. HILBERT-SCHMIDT GROUPS.

$B(H)$	bounded linear operators on a complex Hilbert space H.
$G=GL(H)$	invertible elements of $B(H)$.
Q	a bounded linear symmetric nonnegative operator on HS.
HS	Hilbert-Schmidt operators on H with the inner product $(A,B)_{HS}=TrB^*A$.
$\mathfrak{g}=\mathfrak{g}_{CM}\subseteq HS$	an infinite-dimensional Lie algebra with a Hermitian inner product $(\cdot,\cdot)$, $\|A\|_{\mathfrak{g}}=\|Q^{-1/2}A\|_{HS}$.

$G_{CM} \subseteq GL(H)$ Cameron-Martin group $\{x \in GL(H), d(x, I) < \infty\}$

$d(x, y)$ the Riemannian distance induced by $|\cdot|$

$$d(x, y) = \inf_{\substack{g(0)=x \\ g(1)=y}} \int_0^1 |g(s)^{-1} g'(s)|_{\mathfrak{g}} ds$$

HS as infinite matrices

HS= matrices $\{a_{ij}\}$ such that $\sum_{i,j} |a_{ij}|^2 < \infty$.

$$e_{ij} = i \begin{array}{c} j \\ \begin{pmatrix} \cdots\cdots\cdots \\ \ldots \; 1 \; \ldots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{pmatrix} \end{array}, \quad Qe_{ij} = \lambda_{ij} e_{ij}, \; \lambda_{ij} \geqslant 0.$$

$$\xi_{ij} = \sqrt{\lambda_{ij}} e_{ij}.$$

Q is a trace class operator $\Longleftrightarrow \sum_{i,j} \lambda_{ij} < \infty$.

For example, $\lambda_{ij} = r^{i+j}, \; 0 < r < 1$.

(i) The Hilbert-Schmidt general group

$$GL_{HS}=GL(H)\bigcap(I+HS),$$

Lie algebra $\mathfrak{gl}_{HS}=HS,\ \mathfrak{g}_{CM}=Q^{1/2}HS.$

(ii) The Hilbert-Schmidt orthogonal group SO_{HS} is the connected component of

$$\{B : B-I\in HS,\ \ B^TB=BB^T=I\}.$$

Lie algebra $\mathfrak{so}_{HS}=\{A : A\in HS,\ \ A^T=-A\},$

$$\mathfrak{g}_{CM}=Q^{1/2}\mathfrak{so}_{HS}.$$

(iii) The Hilbert-Schmidt symplectic group

$\boldsymbol{Sp}_{HS}=\{\boldsymbol{X} : \boldsymbol{X} - \boldsymbol{I} \in \boldsymbol{HS}, \;\; \boldsymbol{X}^T\boldsymbol{J}\boldsymbol{X}=\boldsymbol{J}\}$, where

$$\boldsymbol{J}=\begin{pmatrix} 0 & -\boldsymbol{I} \\ \boldsymbol{I} & 0 \end{pmatrix}.$$

Lie algebra $\mathfrak{sp}_{HS}=\{\boldsymbol{X} : \boldsymbol{X} \in \boldsymbol{HS}, \;\; \boldsymbol{X}^T\boldsymbol{J}+\boldsymbol{J}\boldsymbol{X}=0\}$,

$$\mathfrak{g}_{CM} = Q^{1/2}\mathfrak{sp}_{HS}.$$

B. GEOMETRY OF THE HILBERT-SCHMIDT GROUPS (G., JFA 05)

Levi-Civita connection ∇_x **for any** $x, y, z \in \mathfrak{g}$

$$\langle \nabla_x y, z \rangle = \frac{1}{2}(\langle [x,y], z\rangle - \langle [y,z], x\rangle + \langle [z,x], y\rangle)$$

Riemannian curvature tensor R

$$R_{xy} = \nabla_{[x,y]} - \nabla_x \nabla_y + \nabla_y \nabla_x, \ x, y \in \mathfrak{g}.$$

sectional curvature $K(x,y)=\langle R_{xy}(x), y\rangle$ for orthogonal x, y in $\mathfrak{g}$.

Ricci curvature

$$R(x) = \sum_{i=1}^{\infty} K(x, \xi_i) = \sum_{i=1}^{\infty} \langle R_{x\xi_i}(x), \xi_i \rangle$$

for $\{\xi_i\}_{i=1}^{\infty}$, an orthonormal basis of $\mathfrak{g}$.

truncated Ricci curvature

$$R^N(x) = \sum_{i=1}^{N} K(x, \xi_i) = \sum_{i=1}^{N} \langle R_{x\xi_i}(x), \xi_i \rangle$$

for $\{\xi_i\}_{i=1}^{\infty}$ an orthonormal basis of $\mathfrak{g}$

Theorem(G.) [Ricci curvature]. For Hilbert-Schmidt general and orthogonal groups

$$R = -\infty.$$

$$Qe_{ij} = \lambda_i \lambda_j e_{ij}, \;\; \xi_{ij} = \sqrt{\lambda_i \lambda_j} e_{ij}$$

The Hilbert-Schmidt general group

$$GL_{HS}{=}GL(H)\bigcap(I+HS), \quad \mathfrak{g}_{CM}{=}Q^{1/2}HS$$

$$R^N(\xi_{ij}) = -A_{ij}N - B_N,$$

$$0 < A_{ij}, \; 0 < B_N \xrightarrow[N\to\infty]{} B < \infty.$$

If $(\cdot,\cdot) = (\cdot,\cdot)_{HS}$, then $R^N(\xi_{ij}) = -A_{ij}N.$

The Hilbert-Schmidt orthogonal group

$SO_{HS}=\{B : B - I \in HS,\ B^T B = BB^T = I\}$.

$\mathfrak{so}_{HS} = \{A : A \in HS, A^T = -A\}$, $\mathfrak{g}_{CM} = Q^{1/2}\mathfrak{so}_{HS}$.

$$R^N(\xi_{ij}) = -A_{ij}N + B_N^{ij}.$$

If $(\cdot,\cdot) = (\cdot,\cdot)_{HS}$, then $R^N(\xi_{ij}) = +\frac{N}{2}$.

The Hilbert-Schmidt upper triangular group

$$R^N(\xi_{ij}) = -A_{ij}N + B_N.$$

If $(\cdot,\cdot) = (\cdot,\cdot)_{HS}$, then $R^N(\xi_{ij}) = -\frac{N}{2}$.

CAMERON-MARTIN GROUP AND EXPONENTIAL MAP.

Definition. The Cameron-Martin group is

$G_{CM} = \{x \in GL(H),\ d(x, I) < \infty\}$, where d is the Riemannian distance induced by $|\cdot|$:

$$d(x,y) = \inf_{\substack{g(0)=x \\ g(1)=y}} \int_0^1 |g(s)^{-1} g'(s)|_{\mathfrak{g}} ds$$

Finite dimensional approximations:

$\mathfrak{g}_n$ ascending finite dimensional Lie subalgerbas of $\mathfrak{g}$,

G_n Lie groups with Lie algebras

Assumption: all $\mathfrak{g}_n$ are invariant subspaces of Q.

Theorem(G.) If $|[X, Y]| \leqslant c|X||Y|$ then

1. $\mathfrak{g} = \overline{\bigcup \mathfrak{g}_n}$, and the exponential map is a local diffeomorphism from $\mathfrak{g}$ to G_{CM}.

2. $\bigcup G_n$ is dense in G_{CM} in the Riemannian distance induced by $|\cdot|$.

D. STOCHASTIC DIFFERENTIAL EQUATIONS ON HS, HEAT KERNEL MEASURES (G., JFA 2000)

W_t a Brownian motion in HS with the covariance operator Q, that is,

$$W_t = \sum_{i=1}^{\infty} W_t^i \xi_i,$$

W_t^i one-dimensional independent real Brownian motions

$\{\xi_j\}_{j=1}^{\infty}$ an orthonormal basis of $\mathfrak{g}$ as a real space

T $= \frac{1}{2} \sum_{j=1}^{\infty} \xi_j^2$

Theorem(G.) Suppose that Q is a trace-class operator. Then
•[SDEs] the stochastic differential equations

$$dG_t = TG_t dt + dW_t G_t,\ G_0 = X,$$

$$dZ_t = Z_t T dt - Z_t dW_t,\ Z_0 = Y$$

have unique solutions in HS.

•[Inverse]. The solutions of these SDEs with $G_0 = Z_0 = I$ satisfy

$$Z_t G_t = I \text{ with probability } 1 \text{ for any } t > 0.$$

• [Kolmogorov's backward equation] The function $v(t, X) = P_t \varphi(X)$ is a unique solution to the parabolic type equation

$$\frac{\partial}{\partial t} v(t, X) = \frac{1}{2} \sum_{j=1}^{\infty} D^2 v(t, X)(\xi_j X \otimes \xi_j X) + (TX, Dv(t, X))_{HS}.$$

Kolmogorov's backward equation=the group heat equation.

Definition. **The heat kernel measure** μ_t on HS is the transition probability of the stochastic process G_t, that is, $\mu_t(A) = P(G_t \in A)$.

Open question: quasi-invariance of the heat kernel measures.

II. INFINITE-DIMENSIONAL HEISENBERG GROUP

(joint with B.Driver, JFA'08, JDG'09, PTRF'10)

$(\boldsymbol{W}, \boldsymbol{H}, \boldsymbol{\mu})$ an abstract Wiener space

$\omega : \boldsymbol{W} \times \boldsymbol{W} \to \mathbb{C}$ a bi-linear quadratic form on $\boldsymbol{W}$.

$\boldsymbol{G} = \boldsymbol{G}(\omega) = \boldsymbol{W} \times \mathbb{C}$ with group multiplication

$$(\boldsymbol{w}, \boldsymbol{c}) \cdot (\boldsymbol{w}', \boldsymbol{c}') = \left(\boldsymbol{w} + \boldsymbol{w}', \boldsymbol{c} + \boldsymbol{c}' + \frac{1}{2}\omega(\boldsymbol{w}, \boldsymbol{w}')\right).$$

$\boldsymbol{G}_{CM} = \boldsymbol{H} \times \mathbb{C}$ the Cameron-Martin subgroup of $\boldsymbol{G}$.

$\boldsymbol{B(t)}$ a $\boldsymbol{W}$–valued Brownian motion with variance determined by $\boldsymbol{H}$,

$\boldsymbol{B_0(t)}$ an independent $\mathbb{C}$-valued Brownian motion

$$\boldsymbol{g(t) = \left(B(t), B_0(t) + \frac{1}{2}\int_0^t \omega(B(s), \circ dB(s)) \right).}$$

- Quasi-invariance: the path space measure and the heat kernel measure;
- Ricci curvature: bounded from below;
- log Sobolev inequality;
- Taylor isomorphism.

III. THE VIRASORO CASE: $\mathrm{DIFF}(\mathrm{S}^1)/\mathrm{S}^1$

(M. Gordina, P.Lescot JFA 2006)

A.RIEMANNIAN GEOMETRY OF $\mathrm{DIFF}(\mathrm{S}^1)/\mathrm{S}^1$

M. Bowick, S.Rajeev ('87), B. Zumino ('87), A. A. Kirillov, D. V. Yur'ev ('87): $\mathbf{Diff}(\mathbf{S}^1)/\mathbf{S}^1$ as a Kähler manifold.

$\mathbf{Diff}(\mathbf{S}^1)$ smooth orientation-preserving diffeomorphisms of S^1

$\mathbf{diff}(\mathbf{S}^1)$ $\varphi(t)\frac{d}{dt}$, where φ is a smooth periodic function

$\{f_k, g_k\}$ $f_k = \sin(kt)\frac{d}{dt}$, $g_k = \cos(kt)\frac{d}{dt}$, an orthonormal basis of the Lie algebra $\mathbf{diff}(\mathbf{S}^1)$

$$\mathbf{diff}_0(\mathbf{S}^1) = \{\varphi(t)\frac{d}{dt} \in \mathbf{diff}(\mathbf{S}^1), \int_0^{2\pi} \varphi(\mathrm{t})\mathrm{dt} = 0\}$$

J $\quad Jf_k = f_k$, $Jg_k = -g_k$, an almost complex structure on $\mathrm{Diff}(\mathrm{S}^1)/\mathrm{S}^1$

$$\omega_{c,h}(f,g) = \int_0^{2\pi} \left((2h - \tfrac{c}{12}) f'(t) - \tfrac{c}{12} f^{(3)}(t) \right) \frac{g(t)dt}{2\pi},$$ a cocycle on $\mathrm{diff}(\mathrm{S}^1)$,

$B(f,g) =$ $\quad \omega_{c,h}(f,Jg) = \omega_{c,h}(g,fg)$, an inner product on $\mathrm{diff}(\mathrm{S}^1)$,

∇ $\quad$ the covariant derivative determined by the inner product $B(f,g)$.

The curvature tensor for $x, y \in \mathrm{diff}(\mathrm{S}^1)_{\mathbb{C}}$

$$\tilde{R}_{xy} = \tilde{\nabla}_x \tilde{\nabla}_y - \tilde{\nabla}_y \tilde{\nabla}_x - \tilde{\nabla}_{[x,y]_{\mathfrak{m}_{\mathbb{C}}}} - ad([x,y]_{\mathbb{C} f_0});$$

The Ricci tensor $\mathrm{Ric}(x,y)$ is the trace of the map $z \mapsto \tilde{R}_{zx} y$.

An orthonormal basis of $\mathfrak{m}_{\mathbb{C}}$

$L_n = f_n + i g_n, n > 0$,

$L_n = f_{-n} - i g_{-n}, n < 0$.

Theorem. The only non-zero components of the Ricci tensor are

$$\mathbf{Ric(L_n, L_{-n}) = -\frac{13n^3 - n}{6},\ n \in \mathbb{Z}, n \neq 0.}$$

- The parameters c and h do not appear in the Ricci curvature;
- the Ricci tensor for the original covariant derivative ∇ diverges;
- $\tilde{\nabla}$ is the Levi-Civita covariant derivative, that is, it is metric compatible and torsion-free, and it is not a Hilbert-Schmidt operator.

B.HEAT KERNEL MEASURES ON $\mathrm{Diff}(S^1)/S^1$ (H.AIRAULT, P. MALLIAVIN, S. FANG, M. WU)

• Represent $\mathrm{Diff}(S^1)$ as a certain space of univalent functions, and then as the infinite-dimensional group Sp_{HS};

• construct the Brownian motion on Sp_{HS} as the solution to the stochastic differential equation

$$dG_t = \frac{1}{2}\sum_{j=1}^{\infty} \xi_j^2 G_t dt + dW_t G_t$$

with $Qe_{ij} = r^{i+j} e_{ij}$, $0 < r < 1$.

• This choice of renormalization forces the Brownian motion G_t to live in the group Sp_{HS}, but it also changes the geometry of the group.(H.Airault, P. Malliavin)

- $\boldsymbol{H^{3/2}}$ metric forces the Brownian motion to live in Hölderian homeomorphisms of $\boldsymbol{S^1}$. (P. Malliavin, S. Fang)

- a stronger metric forces the Brownian motion to live in $\mathbf{Diff(S^1)/S^1}$. (M. Wu '11)

KÄHLERIAN STRUCTURE ON LOOP GROUPS. (I.SHIGEKAWA, S. TANIGUCHI '96)

(1) $LG = \{g : [0,1] \to G, g(0) = g(1) = e\}$,
$H_0 = \{h : [0,1] \to \mathfrak{g}, h(0) = h(1) = 0\}$
with $\|h'\|_{L^2} < \infty$;

(2) the Riemannian structure on LG can be described in terms of a "good" basis (D. Freed, B.Driver);

(3) an almost complex structure on LG (A. Pressley), then the Riemannian tensor can be computed using a Kählerian metric on LG (I.Shigekawa, S. Taniguchi);

(4) the Ricci is defined as $\mathbf{Ric} = \mathbf{dd}^* + \mathbf{d}^*\mathbf{d} - \nabla^*\nabla$.

IV. CAMERON-MARTIN GROUP AND ISOMETRIES.

$\mathcal{HL}^2(\mu_t)$ the closure in $L^2(\mu_t)$ of holomorphic polynomials $\mathcal{HP}$ on HS.

Theorem(G.) [Holomorphic skeletons].

For any $f \in \mathcal{HL}^2(\mu_t)$ there is a holomorphic function $\tilde{f}$ on G_{CM} such that for any $x \in G_{CM}$ and $p_m \in \mathcal{HP}$

$$\text{if } \; p_m \xrightarrow{L^2(\mu_t)} f, \quad \text{then } \; p_m(x) \longrightarrow \tilde{f}(x).$$

This **skeleton** $\tilde{f}$ is given by the formula

$$\tilde{f}(x) = \sum_{k=0}^{\infty} \int\limits_{0 \leqslant s_1 \leqslant \ldots \leqslant s_k \leqslant 1} (D_I^k f)(c(s_1) \otimes \ldots \otimes c(s_k)) d\vec{s},$$

where $D^k f$ is the kth derivative of function f, and $c(s) = g(s)^{-1} g'(s)$ for any smooth path $g(s)$ from I to x.

A larger space of holomorphic functions:

$\mathcal{H}^t(G_{CM})$ holomorphic functions on G_{CM} such that
$\|f\|_{t,\infty} = \lim_{n\to\infty} \|f\|_{t,n} =$

$$\lim_{n\to\infty} \int\limits_{G_n} |f(z)|^2 d\mu_t^n(z) < \infty.$$

Theorem(G.) [Pointwise estimates].

For any $f \in \mathcal{H}^t(G_{CM})$, $g \in G_{CM}$, $0 < s < t$

$$|(D^k f)(g)|^2_{(\mathfrak{g}^*)^{\otimes k}} \leqslant \frac{k!}{s^k} \|f\|^2_{t,\infty} \exp\left(\frac{d^2(g,I)}{t-s}\right).$$

Theorem(G.). $Q : HS \to HS$ (or $\mathfrak{so}_{HS}$ or $\mathfrak{sp}_{HS}$).

- If Q is trace class, then the heat kernel measure lives in GL_{HS} (or SO_{HS} or Sp_{HS}), and $\mathcal{H}^t(G_{CM})$ is an infinite dimensional Hilbert space.

- If the covariance operator Q is the identity operator, then $\mathcal{H}^t(G_{CM})$ contains only constant functions.

Theorem(G.) [ISOMETRIES].

• The skeleton map is an isometry from $\mathcal{H}L^2(\mu_t)$ to $\mathcal{H}^t(G_{CM})$ (the restriction map on holomorphic polynomials $\mathcal{HP}$ extends to an isometry between the spaces $\mathcal{H}L^2(\mu_t)$ and $\mathcal{H}^t(G_{CM})$).

• If Q is a trace class operator, then $\mathcal{H}L^2(\mu_t)$ is an infinite-dimensional Hilbert space.

• The Taylor map $f \mapsto \sum_{k=0}^{\infty} D^k f(I)$ is an isometry from $\mathcal{H}^t(G_{CM})$ to the Fock space $\mathfrak{F}_t(\mathfrak{g})$, a subspace of the dual of the tensor algebra of $\mathfrak{g}$ with the norm

$$|\alpha|_t^2 = \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|^2.$$

$$\mathcal{HP} \xrightarrow{\substack{\text{inclusion}\\ \text{map}}} \mathcal{H}L^2(\mu_t) \xrightarrow{\substack{\text{skeleton}\\ \text{“restriction”}\\ \text{map}}} \mathcal{H}^t(G_{CM}) \xrightarrow{\substack{\text{Taylor}\\ \text{map}}} \mathfrak{F}_t(\mathfrak{g})$$

$\mathfrak{F}_t(\mathfrak{g})$ a Hilbert space in the dual of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$

$T(\mathfrak{g})$ the tensor algebra over $\mathfrak{g}$

J the two-sided ideal in $T(\mathfrak{g})$ generated by $\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta];\ \xi, \eta \in \mathfrak{g}\}$.

$T'(\mathfrak{g})$ $= \sum_{k=0}^{\infty} (\mathfrak{g}^{\otimes k})^*$, the algebraic dual of the tensor algebra $T(\mathfrak{g})$.

J^0 the annihilator of J in the dual space $T'(\mathfrak{g})$.

$$\|\alpha\|_t^2 = \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|^2_{(\mathfrak{g}^{\otimes k})^*},\ \alpha = \sum_{k=0}^{\infty} \alpha_k,\ \alpha_k \in (\mathfrak{g}^{\otimes k})^*, k=0,1,2,\ldots$$

The generalized bosonic Fock space is $\mathfrak{F}_t(\mathfrak{g}) = \{\alpha \in J^0 : \|\alpha\|_t^2 < \infty\}$

This is the space of Taylor coefficients of functions from $\mathcal{H}^t(G_{CM})$.