Some boundary value and mapping problems for differential forms

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PLAN of TALK.

1. Background
2. Outline of some older work in dimension $6+1$ (arxiv 1708.01649,1801.01806,1802.09694).
3. Preliminary report on work in progress in dimension $5+1$ with Fabian Lehmann (and with important input from Robert Bryant).

Part 1. Background

There are special geometric constructions in dimensions $6, 7, 8$ related to exceptional holonomy and special properties of differential forms in those dimensions.

Dimension 7

Let V be an oriented 7-dimensional real vector space and $\phi \in \Lambda^3 V^*$.

We have a quadratic form on V with values in the line Λ^7 defined by

$$v \mapsto (i_v(\phi))^2 \wedge \phi.$$

The form is called *positive* if this is positive definite.

In that case we fix the scale by requiring that $|\phi|^2 = 7$. This gives a Euclidean structure g_ϕ on V, determined by ϕ.

We have a 4-form $*\phi = *_\phi \phi$.

The stabiliser in $GL(V)$ of any positive 3-form is isomorphic to the 14-dimensional exceptional Lie group G_2.

Fernandez and Gray: *A torsion-fee G_2-structure on an oriented 7-manifold M is equivalent to a 3-form ϕ which is everywhere positive and with $d\phi = 0, d * \phi = 0$.*

One reason for being interested in these structures is that the Riemannian metric g_ϕ of a torsion-free structure has $\mathrm{Ricci} = 0$.

Hitchin's variational formulation

We have a volume function

$$\nu : \Lambda^3_+ \to \Lambda^7.$$

The derivative is

$$\delta\nu = \frac{1}{3}\delta\phi \wedge *\phi.$$

Let ϕ be a positive form on a compact M^7. It defines a volume

$$\mathrm{Vol}(\phi) = \int_M \nu(\phi).$$

Consider the volume as a functional on the closed forms ϕ in a fixed de Rham cohomology class (assuming that this space is not empty).

The condition for a critical point is

$$\int_M d\sigma \wedge *\phi = 0$$

for all 2-forms σ.

Integrating by parts, this is equivalent to $d * \phi = 0$.

So the torsion-free G_2 structures correspond to the solutions of this variational problem.

Dimension 6

Let U be a 6-dimensional oriented real vector space and $\rho \in \Lambda^3 U^*$.

Say ρ is *positive* if $i_v \rho$ has rank 4 for all $v \neq 0$. Assume this holds.

For $v \neq 0$, let $N_v = \{v' : i_{v'} i_v \rho = 0\}$. So $v \in N_v$.

Then $\dim N_v = 2$ and for each $v' \in N_v$ the form $i_{v'}\rho$ induces a symplectic form $\omega_{v'}$ on the 4-dimensional space U/N_v.

The map $v' \mapsto \omega_{v'}^2$ defines a conformal structure on N_v and, thinking about orientations, we see that there is a corresponding complex structure on N_v. So we have a way to define $I_\rho v \in N_v \subset U$. This gives a complex structure I_ρ on U.

We also get a complex 3-form $\Omega = \rho + i\tilde{\rho}$ which is of type $(3,0)$ with respect to I_ρ.

The stabiliser in $GL^+(U)$ of a positive 3-form is isomorphic to $SL(3, \mathbf{C})$.

A *complex Calabi-Yau structure* on an oriented 6-manifold Z is equivalent to a 3-form ρ which is everywhere positive and with $d\rho = d\tilde{\rho} = 0$.

Then the almost-complex structure I_ρ is integrable, so we have a complex manifold, and $\Omega = \rho + i\tilde{\rho}$ is a nowhere-vanishing holomorphic 3-form.

There is a similar Hitchin variational description, with a volume functional on the space of positive 3-forms in a fixed cohomology class.

Part 2: Boundary value problem for G_2-structures

Let M be a compact oriented 7-manifold with boundary $\partial M = Z$ and ρ a closed positive 3-form on Z.

We consider the problem of finding a torsion-free G_2-structure ϕ (in a fixed “relative” class) on M which restricts to ρ on the boundary.

This has a variational description. (We consider variations $d\sigma$ with $\sigma|_Z = 0$. Then

$$\delta \mathrm{Vol} = \int_M d(*\phi) \wedge \sigma + \int_Z *\phi \wedge \sigma,$$

and the boundary term vanishes.)

Proposition *This is an elliptic boundary value problem for ϕ, modulo diffeomorphisms of M fixing the boundary.*
Brief discussion
Sketch of the standard case, for a closed manifold M.

We suppose that ϕ is a torsion-free G_2 structure and want to consider the linearised equation (equivalently, the Hessian of the volume functional).

An infinitesimal variation of ϕ has the form $d\sigma$. The variations $L_v\phi = d\ i_v\phi$ for vector fields v are "trivial".

The 2-forms on M decompose into $\Omega^2_7 \oplus \Omega^2_{14}$. This decomposition is preserved by the Laplace operator Δ.

The first factor corresponds to the $i_v\phi$. So we can suppose that $\sigma \in \Omega^2_{14}$.

Slightly over-simplifying, the linearised operator turns out to be the Laplacian on Ω^2_{14}.

The structure ϕ is a strict *local maximum* of the volume functional on the forms in the same cohomology class, modulo diffeomorphisms.

In the case of a manifold with boundary, the space Λ^2_{14} decomposes at a boundary point into a sum of 8-dimensional and 6-dimensional pieces. (The 8-dimensional piece corresponds to the Lie algebra of $SU(3)$ inside the Lie algebra of G_2.)

The linearised operator associated to the boundary value problem turns out to be the Laplacian on Ω^2_{14} with boundary conditions

- $d^*\sigma|_{\partial M} = 0$;
- $\sigma||_{\partial M,8} = 0$.

(Note that this is $6 + 8 = 14$ boundary conditions, of mixed Dirichlet and Neumann type.)

One checks that this is an elliptic boundary value set-up.

Consequence of the Proposition

For small variations of the data (*i.e.* the 3-form ρ and the cohomology class of ϕ) there is a unique small perturbation of the solution to the B.V. problem for G_2-structures **MODULO** possible obstructions in a finite-dimensional vector space H_ϕ.

This is a relatively standard application of the implicit function theorem in Banach spaces, and elliptic theory.

One can prove is certain cases, and plausibly conjecture in some generality, that these obstruction spaces H_ϕ vanish. (Related to the question whether the solutions are local maxima in the variational theory).

Example If ρ_0, ρ_1 are closed 3-forms in the same cohomology class on a 6-manifold Z which are sufficiently close to a Calabi-Yau structure then there is a "G_2-cobordism" between them, perturbing the cylinder $(\mathrm{Calabi-Yau}) \times [0, 1]$.

Riemannian geometry aspects

To have any hope of general existence theorems one perhaps needs to impose conditions on the boundary data ρ.

If ρ is any closed positive 3-form on Z^6 the 4-form $d\tilde{\rho}$ has type $(2,2)$ with respect to the almost-complex structure I_ρ.

Thus there is notion of *positivity* of $d\tilde{\rho}$ and an intrinsic numerical invariant $\det d\tilde{\rho}$.

If $d\tilde{\rho} > 0$ and ρ is the boundary value of a torsion-free G_2-structure ϕ on M then the *mean curvature* μ of Z in (M, g_ϕ) is positive and

$$\mu \geq \frac{3}{2}(\det \tilde{\rho})^{1/3} > 0.$$

This combines well with the fact that $\mathrm{Ricci}(g_\phi) = 0$.

For example, if $\mu \geq \mu_0 > 0$ then the distance of any point in M from the boundary is at most $7/\mu_0$.

(Proof similar to Myers' Theorem.)

Dimension reduction

Various interesting equations in lower dimensions arise by imposing symmetry.

Example Let $\Sigma \subset \mathbf{R}^3$ be the boundary of a convex domain U. Let T be a current of the form

$$T(f) = \int_\Sigma a\, \nabla_\nu f + b\, f,$$

for functions a, b on Σ, where ∇_ν is the normal derivative.

Problem: Minimise $T(f)$ over solutions f of the Monge-Ampère equation $\det(\nabla^2 f) = 1$ on the domain U.

This arises from the G_2 problem in 7 dimensions by imposing symmetry under an action of $\mathbf{R}^3 \times S^1$.

Part 3: Boundary values of complex Calabi-Yau structures

Let Y be an compact oriented 5-manifold and ψ a closed 3-form on Y.

If Y is the boundary of a 6-manifold Z then we can seek a complex Calabi-Yau structure ρ on Z with boundary value ψ, in the manner above.

This is *not* an elliptic boundary value problem.

Note that, unlike the G_2-case, complex Calabi-Yau structures are locally trivial.
We focus on a variant of the problem, which is to seek Z as a domain in some fixed complex Calabi-Yau manifold Z_+, with holomorphic 3-form Ω. For example $Z_+ = \mathbf{C}^3$.

Then we have a *mapping problem*: Is there an embedding

$$F : Y \to Z_+$$

such that

$$F^*(\mathrm{Re}\,\Omega) = \psi?$$

Informal parameter count.

A closed 3-form in 5 dimensions is given by $10 - 5 + 1 = 6$ functions,which is the same number as a map into a 6-manifold Z_+.

This is special to the dimension: for example,a closed 4-form on a 7-manifold is given by $35 - 21 + 7 - 1 = 20$ functions, which is much more than 8.

A dimensionally reduced problem
Take $Y = \Sigma \times \mathbf{R}^3_y$ where Σ is diffeomorphic to S^2 and

$$\psi = \omega_1 dy_1 + \omega_2 dy_2 + \omega_3 dy_3,$$

where $\omega_1, \omega_2, \omega_3$ are 2-forms on Σ and y_a are co-ordinates on $\mathbf{R}^3_y$.

Take F to have the form

$$F(u, (y_1, y_2, y_3)) = f(u) + (iy_1, iy_2, iy_3)$$

for a map $f : \Sigma \to \mathbf{R}^3_x \subset \mathbf{C}^3$.

Then the problem is to find f such that

$$f^*(dx_a dx_b) = \omega_c,$$

for (abc) cyclic permutations of (123), where x_a are co-ordinates on $\mathbf{R}^3_x$.

Clearly we need to assume that

$$\int_\Sigma \omega_a = 0$$

Choose an area form σ on Σ. We have $\omega_a = h_a\sigma$ for functions h_a on Σ. Write $\underline{h} = (h_1, h_2, h_3) : \Sigma \to \mathbf{R}^3_x$.

Assume that $\underline{h}$ nowhere vanishes and that it induces a diffeomorphism $g = h/|h| : \Sigma \to S^2$.

Then finding the map f is equivalent to the classical Minkowski problem, solved by Nirenberg in 1953.

For, without loss of generality we can suppose that $|h| = 1$ and that $\Sigma = S^2$ with g the identity map. Then ω_a are the 2-forms on S^2 determined by a positive function K:

$$\omega_a = K^{-1} x_a \, dA,$$

where dA is the standard area form on S^2.

The condition on the map $f : S^2 \to \mathbf{R}^3$ is that the normal vector to the image at $f(x)$ is x and that the Gauss curvature is $K(x)$.

Closed 3**-forms in dimension** 5
(Incorporating suggestions of Robert Bryant.)

Let ψ be a closed 3-form on the oriented Y^5. At each point ψ defines a skew form on cotangent vectors with values in Λ^5:

$$(a, b) \mapsto a \wedge b \wedge \psi.$$

Assumption 1 This has maximal rank, 4, at each point. (This is necessary for a form induced from $Y \subset (Z_+, \Omega)$.
The 1-dimensional kernel in T^*Y correspond to a field of 4-dimensional subspaces $H \subset TY$.

Assumption 2 H is a contact structure on Y.
There is a contact 1-form θ with $(d\theta)^2 \wedge \theta > 0$. This is not unique; we could change θ to $f\theta$ for any positive function f.
By construction $\psi = \theta \wedge \alpha$ for a 2-form α.

Assumption 3

$$\theta \wedge \alpha^2 > 0.$$

(Assumptions 2 and 3 imply that if ψ is induced from $Y \subset (Z_+, \Omega)$ bounding a domain Z then, with the right choice of orientation, the boundary is pseudoconvex.)

We can then normalise θ by the requirement that $\alpha^2 \wedge \theta = (d\theta)^2 \wedge \theta$. Write $\omega = d\theta$ and fix the volume form $\mu = \omega^2 \wedge \theta$.
We have a Reeb vector field v defined by the conditions that $i_v\omega = 0$ and $\theta(v) = 1$.

This gives a decomposition $TY = H \oplus \mathbf{R}v$ and a subspace of forms $\Omega^p_H \subset \Omega^p$ so that

$$\Omega^p = \Omega^p_H \oplus \theta \wedge \Omega^{p-1}_H.$$

We have

$$d_H : \Omega^p_H \to \Omega^{p+1}_H.$$

The square d_H^2 is the wedge product with ω.
The choice of α can be fixed by requiring $\alpha \in \Omega^2_H$.

We have an indefinite inner product of signature $(3,3)$ on $\Lambda^2 H^*$ defined by

$$(\gamma_1.\gamma_2)\,\mu = \gamma_1 \wedge \gamma_2 \wedge \theta.$$

By construction $d_H\omega = 0$ and $\omega.\omega = \alpha.\alpha = 1$. The conditions that $\psi = \alpha \wedge \theta$ is closed is equivalent to $\alpha.\omega = 0$ and $d_H\alpha = 0$.

The orthonormal pair ω, α defines a complex structure J on H.

Let L_v be the Lie derivative along the Reeb field v. There is an invariant $\chi = L\alpha$ which satisfies $\chi.\alpha = \chi.\omega = 0$, so χ has type $(1, 1)$ with respect to J.

Thus there is a notion of "positivity" of χ.

There is also a numerical invariant $\chi.\chi$ (a function on Y).

(Robert Bryant informs us that the tensor χ is the only second order invariant of closed 3-forms ψ, satisfying our assumptions.)

One question is: which contact structures on 5-manifolds are compatible with a closed 3-form in this way?

Suppose now that ψ is obtained from an embedding $Y \subset Z_+$. The restriction of $\mathrm{Im}\,\Omega$ can be written as $\beta \wedge \theta$ for $\beta \in \Omega^2_H$ with $d_H\beta = 0$ and ω, α, β form an orthonormal triple with respect to the inner product. The pair α, β defines another complex structure I on H, which is the usual CR-structure given by the embedding.
There is a unique metric on H with volume form ω^2 and self-dual space spanned by ω, α, β.

Another question is: given $\psi = \alpha \wedge \theta$ as above, can we find a $\beta \in \Omega^2_H$ with $d_H \beta = 0$ such that (ω, α, β) is an orthonormal triple, and if so is the solution unique? (Bryant)

This could be seen as a “contact” version of the Calabi-Yau problem in complex dimension 2 (the existence of a hyperkähler structure).

Given θ, α, β as above we have anti-self-dual forms Ω^-_H and a complex

$$\Omega^0_H \xrightarrow{d_H} \Omega^1_H \xrightarrow{d^-_H} \Omega^-_H. \qquad (****)_H$$

Denote the cohomology by $\mathcal{H}^p$.

The linearised embedding problem

For definiteness, take Y diffeomorphic to S^5 and $Z_+ = \mathbf{C}^3$.
A solution of the embedding problem will never be unique because it can be changed by a holomorphic volume-preserving diffeomorphism of $\mathbf{C}^3$.
Suppose that ψ is induced by an embedding $Y \subset \mathbf{C}^3$ as the boundary of a pseudoconvex domain Z. The linearised problem can be expressed in terms of a complex

$$E_0 \xrightarrow{D_1} E_1 \xrightarrow{D_2} E_2,$$

where E_0 is the space of divergence-free holomorphic vector fields on Z, E_1 is the space of sections of $T\mathbf{C}^3|_Y$ and E_2 is the space of closed 3-forms on Y.

We have the data (θ, α, β) on Y, as above.

Proposition Suppose that $\mathcal{H}^2 = 0$. Then D_2 is surjective and $\mathrm{Ker}\, D_2 = \mathrm{Im}\, D_1$.

This makes it plausible that, when $\mathcal{H}^2 = 0$, any small deformation of ψ can be realised by a small deformation of the embedding, unique up to volume-preserving holomorphic diffeomorphisms. In other words, the isomorphism class of $(Z, \Omega|_Z)$ would be uniquely determined by the boundary value ψ (with respect to small variations).

We expect/hope that Nash-Moser theory can be applied to prove such a result.

Proof of the assertion $\mathcal{H}^2 = 0$ implies that D_2 is surjective.

The restriction of the tangent bundle of $\mathbf{C}^3$ to Y is $H \oplus \mathbf{C}v$. For any section w let $K(w)$ be the restriction of the 2-form $i_w(\mathrm{Re}\Omega)$ to Y. Then then the map D_2 is

$$D_2(w) = d(K(w)).$$

Any closed 3-form on Y can be written as $d\sigma$ and σ is unique up to $d\Omega^1_Y$. It follows that the cokernel of D_2 is isomorphic to the cokernel of

$$d \oplus K : \Omega^1_Y \oplus \Gamma(T\mathbf{C}^3|_Y) \to \Omega^2_Y.$$

The image of the bundle map K is the span of α, β plus $\Omega^1_H \wedge \theta$.

Since $d(f\theta) = f\omega$ modulo $\Omega^1_H \wedge \theta$, the cokernel of $d \oplus K$ is isomorphic to the quotient of the cokernel of $d_H : \Omega^1_H \Omega \Omega^2_H$ by the subspace generated by α, β, ω. This is the same as the cokernel of d^-_H.

The proof of the other assertion in the Proposition involves an integration argument to show that any $w \in \mathrm{ker} D_2$ is a holomorphic section in the sense of the $\overline{\partial}_b$ operator on Y, which then extends holomorphically over Z by a theorem of Hartogs type.

Analysis on S^5

It seems likely that the theory of hypoelliptic operators can be applied to these questions. In the case when $Y \subset \mathbf{C}^3$ is the standard sphere S^5 with induced 3-form ψ_0 one can proceed in a more elementary way.

We have an S^1 action with quotient $\mathbf{CP}^2$ and H is the pull-back of the tangent space of $\mathbf{CP}^2$. Let $L \to \mathbf{CP}^2$ be the Hopf line bundle. It has a connection with curvature a self-dual form on $\mathbf{CP}^2$, which lifts to ω on S^5.

For each integer k there is a complex over $\mathbf{CP}^2$:

$$\Omega^0(L^k) \xrightarrow{d_k} \Omega^1(L^k) \xrightarrow{d_k^-} \Omega^-(L^k). \qquad (****)_k$$

of a kind which is well-known in the deformation theory of Yang-Mills instantons.

Set $\Delta_k = d_k^- \left(d_k^-\right)^*$.

On a general Riemannian 4-manifold, the Weitzenbock formula on Ω^- is

$$\Delta_k = \frac{1}{2}\nabla^*\nabla + (W_- + S/3),$$

where W_- is the anti-self-dual part of the Weyl curvature. For $\mathbf{CP}^2$ this vanishes, the scalar curvature is positive and we have

$$\Delta_k = \frac{1}{2}\nabla^*\nabla + 4.$$

This implies that d_k^- is surjective.

(Remark: On self-dual manifolds like $\mathbf{CP}^2$ the complex above has a “twistor” description.)

Returning to S^5, we can decompose Ω^*_H into Fourier components with respect to the circle action and we find that the complex $(****)_H$ on S^5 can be identified with the sum over $k \geq 0$ of the complexes $(****)_k$ on $\mathbf{CP}^2$.

The preceding discussion then implies that for ψ_0 we have $\mathcal{H}^2 = 0$.

Going further, one can show that $\Delta_k \geq ck$ for some $c > 0$ and use this to show that for any closed 3-form sufficiently close to ψ_0 we also have $\mathcal{H}^2 = 0$ and there is a right inverse to the linearised operator d^-_H satisfying explicit estimates in Sobolev spaces. This is the crucial requirement to apply the Nash-Moser theory.