SYZ mirror symmetry for del Pezzo surfaces

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Joint w/ A. Jacob and Y.-S. Lin

Outline

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- For degree 8 there are two families given by the first Hirzebruch surface $\mathrm{Bl}_p\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.
- All del Pezzo surfaces admit a smooth divisor $D \in |-K_Y|$.

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del Pezzo surfaces and rational elliptic surfaces as Calabi-Yau pairs

- Let Y be a RES or a del Pezzo surface, and $D \in |-K_Y|$ a divisor. If $s \in H^0(Y, -K_Y)$ is a holomorphic section with $\{s = 0\} = D$, then $\frac{1}{s}$ is a holomorphic $(2,0)$ form on $Y \setminus D$ with a simple pole on D.

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- Therefore, $Y \setminus D$ is a natural *non-compact* Calabi-Yau manifold.
- The existence of a Ricci-flat Kähler metric does not follow from Yau's theorem, since $Y \setminus D$ is non-compact.

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- For example, an instantiation of this principle is

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- This is a particular case of mirror symmetry for the Hodge diamonds of $X, \check{X}$.

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- The basic proposal for how to *construct* mirror symmetric pairs is due to Strominger-Yau-Zaslow (SYZ).
- There are programs of Gross-Siebert and Kontsevich-Soibelman aimed at using the SYZ philosophy to construct (often formal) algebraic mirrors.

SYZ mirror symmetry

Definition (Harvey-Lawson)

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- The notion makes sense for ω *not* the Calabi-Yau symplectic form, but in this case they are no longer volume minimizing.

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- The T-dual fibrations exchange complex and symplectic affine structures on B.

Mirror symmetry beyond CYs

- If Y is not a compact Calabi-Yau manifold, then the mirror is expected to be a (partial compactification of) a *Landau-Ginzburg* model: ie. a non-compact Kähler manifold M together with a holomorphic function $W : M \to \mathbb{C}$.

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- If Y is compact Kähler and $D \in |-K_Y|$ is a divisor, Auroux laid out a general picture for constructing the mirror to Y by applying SYZ mirror symmetry to the non-compact CY manifold $X = Y \setminus D$.

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- Lunts-Przyjalkowski proved mirror symmetry for "Hodge numbers", as proposed by Katzarkov-Kontsevich-Pantev.
- Doran-Thompson studied del Pezzo $\leftrightarrow$ RES mirror symmetry at a lattice theoretic level.

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- Explain a proof of a strong form of SYZ mirror symmetry for del Pezzo surfaces of degree k and RES with an I_k fiber.
- Explain mirror symmetry for Hodge numbers in terms of moduli of complete CY metrics.
- Describe applications to existence of some new CY metrics, a question of Yau, etc.

del Pezzo surfaces: Complete Calabi-Yau metrics

The first ingredient we need is a fundamental result of Tian-Yau, which in our case gives

Theorem (Tian-Yau)

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and such that ω_{TY} is asymptotic to the Calabi model *(with estimates...)*

Remark

The Tian-Yau theorem holds in all dimensions

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$$\mathcal{C} := \{\xi \in E : 0 < |\xi|_h < 1\}, \qquad \Omega_{\mathcal{C}} := \Omega_D \wedge \frac{dw}{w}.$$

$$\omega_{\mathcal{C}} = \frac{2}{3}\sqrt{-1}\partial\overline{\partial}\left(-\log|\xi|_h^2\right)^{\frac{3}{2}}.$$

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Then $(\mathcal{C}, \Omega_{\mathcal{C}}, \omega_{\mathcal{C}})$ is Calabi-Yau, and furthermore complete at $0 \subset E$.

del Pezzo surfaces: Complete Calabi-Yau metrics

- The Riemannian geometry of $(\mathcal{C}, \Omega_{\mathcal{C}}, \omega_{\mathcal{C}})$ can be visualized by considering the level sets

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are S^1 bundles over D, fibering $\mathcal{C}$.

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SYZ fibrations on del Pezzo surfaces

Using this model geometry we prove:

Theorem (C.-Jacob-Lin)

Let Y be a del Pezzo surface, D a smooth elliptic curve, and equip $X = Y \setminus D$ with the Tian-Yau metric ω_{TY}.

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Remark

In fact, X admits countably many distinct special Lagrangian fibrations, one for each choice of simple closed loop $\gamma \in H_1(D, \mathbb{Z})$.

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1. Fix a simple closed geodesic (ie. a special Lagrangian) γ in D.

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4. Run the Lagrangian mean curvature flow, and show convergence to a fibration in a neighborhood of infinity.

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7 The essential point is to show that only finitely many singular holomorphic curves can appear as limits of curves obtained from deforming the fibration near infinity.

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The idea of the proof:

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6. Now we can use ideas from J-holomorphic curve theory (deformation theory, moduli, compactness etc.) to extend the local fibration to a global fibration (this takes some work!).
7. The essential point is to show that only finitely many singular holomorphic curves can appear as limits of curves obtained from deforming the fibration near infinity.

Applications of SYZ fibrations on del Pezzo surfaces

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There is a rational elliptic surface $\check{\pi} : \check{Y} \to \mathbb{P}^1$ with an I_k singular fiber $\check{D}$ such that

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Question (Yau $\sim$ 80s): What is the symplectic structure of this hyperKähler rotation?

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The model geometry: Let $\Delta^* = \{z \in \mathbb{Z} : 0 < |z| < 1\}$, consider

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$$\Omega = \frac{\kappa(z) dx \wedge dz}{z} \qquad \kappa(z) : \Delta \to \mathbb{C} \text{ hol'c}, \quad \kappa(0) \neq 0$$

Let's assume: $\kappa = 1$ for simplicity.

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$$\omega_{sf,\varepsilon} = \sqrt{-1}\frac{k|\log|z||}{2\pi\varepsilon}\frac{dz \wedge d\bar{z}}{|z|^2} + \frac{\sqrt{-1}}{2}\frac{2\pi\varepsilon}{k|\log|z||}(dx + B(x,z)dz) \wedge \overline{(dx + B(x,z)dz)}$$

where $B(x,z) = -\frac{\mathrm{Im}(x)}{\sqrt{-1}z|\log|z||}$, $\varepsilon =$ volume of the fibers.

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- The metric $\omega_{sf,\sigma,\varepsilon} = F_\sigma^* \omega_{sf,\varepsilon}$ is Ricci-flat and complete near the I_k fiber. It is the Standard semi-flat metric with respect to σ.

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For all $\alpha > \alpha_0$ there exists a CY metric in the Bott-Chern cohomology class of ω_0 converging exponentially fast to $\alpha\omega_{sf,\sigma,\frac{\varepsilon}{\alpha}}$ at infinity (with very precise estimates to all orders)

This applies, for example, to Kähler metrics restricted from Y.

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Main problem: If we want to understand the Kähler moduli (to do mirror symmetry, define Hodge numbers etc.) on a RES pair (Y, D) in terms of moduli of Calabi-Yau metrics, then we need a parameter space.

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Outstanding questions from Hein's theorem:

- $H^2_{dR}(X, \mathbb{R})$ has dimension $11 - k$.

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- Recall that Bott-Chern cohomology is a refinement of de Rham cohomology given by

$$H^{p,q}_{BC} := \frac{\{\mathrm{Ker}\, d : \Lambda^{p,q} \to \Lambda^{p+1,q} \oplus \Lambda^{p,q+1}\}}{\{\mathrm{Im}(\sqrt{-1}\partial\overline{\partial} : \Lambda^{p-1,q-1} \to \Lambda^{p,q})\}}$$

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  Hein's theorem depends on a construction which leaves open the possibility of (infinitely many) distinct CY metrics *even in a fixed Bott-Chern class*.

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- By Leray spectral sequence calculations one can show that

$$H^{1,1}_{BC}(X) \sim H^2_{dR}(X) \times H^0(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$$

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- Lunts-Przyjalkowski computed these Hodge numbers for del Pezzos of degree k and RES with an I_k fiber, and obtained $10 - k$ on both sides (proving mirror symmetry at the level of KKP Hodge numbers).

A brief digression on sections

- a section $\sigma : \Delta^* \to X_{mod}$ can be written as

$$\sigma(z) = h(z) + \frac{a}{2\pi\sqrt{-1}}\log z + \frac{b}{(2\pi\sqrt{-1})^2}(\log(z))^2$$

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- Key point: Pulling back the semi-flat metric by a *multivalued* section still yields a well-defined, semi-flat, Calabi-Yau metric. We call these non-standard semi-flat metrics and say they are quasi-regular in the $\mathbb{Q}$ case, and irregular in the $\mathbb{R}$ case.

Calabi-Yau metrics on a RES with an I_k fiber

- If $\omega_{\sigma,sf,\varepsilon}$ is a quasi-regular semi-flat metric, then there is still a family of special Lagrangian "bad cycles" $C_r \subset X_{mod}$, $r \in (0,1)$, but C_r covers the circle $|z| = r$ in the base more than once.

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- Shows that the parameter space for CY metrics asymptotic to semi-flat metrics is the cone of Kähler classes in $H^{1,1}_{BC}(X, \mathbb{R})$ (still infinite dimensional....).
- For various reasons, the quasi-regular metrics are important for mirror symmetry.

an Application to a question of Yau

Theorem (C.-Jacob-Lin)

Let (Y, D) be a del Pezzo pair, $\gamma \in H_1(D, \mathbb{Z})$, and $(X, g_{TY}, \omega_{TY}, J)$ be the Tian-Yau Ricci-flat Kähler structure on X.

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- *ω_γ is the symplectic form of the Ricci-flat metric produced by the previous theorem.*
- *ω_γ is asymptotic to a non-standard semi-flat metric* unless *D is the torus with fundamental domain determined by the lattice $\mathbb{Z} + \sqrt{-1}\lambda\mathbb{Z}$ for $\lambda \in \mathbb{R}_{>0}$, and γ is one of the cycles generating the lattice. Generically, ω_γ is irregular.*

More applications of the uniqueness result

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Corollary

If we define the Kähler moduli to be

$$\mathcal{M}_{K\ddot{a}h} := \{ \text{ CY metrics asymptotic to } \omega_{sf,\varepsilon}\} / \mathrm{Aut}_0(X, \mathbb{C})$$

where $\mathrm{Aut}_0(X)$ are fiber preserving automorphisms homotopic to the identity. Then $\mathcal{M}_{K\ddot{a}h}$ is a cone with non-empty interior in $H^2_{dR}(X, \mathbb{R}) \sim \mathbb{R}^{11-k}$.

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If we define the Kähler moduli to be

$$\mathcal{M}_{K\ddot{a}h} := \{ \text{ CY metrics asymptotic to } \omega_{sf,\varepsilon} \} / \mathrm{Aut}_0(X, \mathbb{C})$$

where $\mathrm{Aut}_0(X)$ are fiber preserving automorphisms homotopic to the identity. Then $\mathcal{M}_{K\ddot{a}h}$ is a cone with non-empty interior in $H^2_{dR}(X, \mathbb{R}) \sim \mathbb{R}^{11-k}$.

$\Rightarrow$ can define the Hodge numbers in terms of moduli of Calabi-Yau metrics

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- argument is the same as that for del Pezzo surfaces using the "bad cycle" of the quasi-regular semi-flat metric as the model.

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- Comparing the *complex* moduli of rational elliptic surfaces with the *symplectic* moduli of del Pezzos is an interesting question for future work.

Thanks!