Brooklyn 99 Riddle

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1 The Riddle

This is a riddle taken from Brooklyn Nine Nine; it is from season 2, episode 18. The essence of the riddle is:

"There are 12 men on an island. 11 of them weigh the same and one of them is either slightly heavier or slightly lighter. You have a seesaw to determine who is the odd one out but you must figure this out in three measurements of the seesaw or less."

Well, the essence of the riddle doesn't need the men to be on an island. And though it was posed originally as 12 (angry?) men, it's more realistic that it's 12 marbles, coins, balls, or something else.

Anyways, let's say they are coins and make some preliminary remarks.

- Let us, at the start, divide the coins into three groups of 4 and label them: ABCD and EFGH and IJKL.
- We will assume that what constitutes a measurement depends on the final arrangement of the coins. To explain what I mean, I will demonstrate by saying what does **not** count as a single measurement. Suppose we wish to weigh AB against CD. We cannot put A on one side and then C on the other, make a note of what happens to the seesaw, before placing B and D. That would count as 2 measurements rather than one: weighing A and C before weighing AB and CD. Similarly, if AB and CD are on the seesaw and we wish to reset the seesaw for the next measurement, we cannot take off B and D first, make an observation, and then take off A and C. That would count as a second measurement.
- If the riddle said that the odd coin out is known to be heavier, then note that we can easily solve this by putting 6 coins on each side. Then take the coins that are collectively heavier and split them into two groups of 3 to weigh. Again, take the heavier set and weigh any 2 of them. If they are the same, the remaining one is the odd coin. If they are not the same, then it is obvious which coin is the odd one out.

  Unfortunately, the riddle says that the odd coin out may be heavier or lighter. Thus, our strategy above fails because the heavier side may consist entirely of normal coins while the lighter side contains a lighter coin. Therefore, the fact that the odd coin may be **heavier or lighter** is a crucial subtlety. Moreover, the riddle doesn't require us to say whether the odd coin is heavier or lighter, merely that it is the odd one out.
- In our measurements, we wish to gain as much information as possible. We can therefore, observe the state of the seesaw. Suppose we weigh coins A,B,C, and D with A and B on one side and C and D on the other. Suppose AB is lighter. So we can conclude that the

odd coin is among ABCD. Let's swap B and D so that the arrangement is now AD and CB. If AD is lighter, this means that A was on the lighter side and remains on the lighter side. Similarly, C was and still is on the heavier side. Therefore, swapping B and D did not change the states of A and C. This doesn't tell us which coin is the odd one out but we have narrowed it down to A and C.

If the situation had been reversed: AD is now heavier, this tells us that the swapping of B and D did change the state of A: it was on the lighter side and is now on the heavier side. This tells us that either B or D is the odd coin out.

The point of this remark is to show that we gain information by seeing how the seesaw dips (or is balanced) and whether that changes.

2 A Solution with Many Cases

We weigh ABCD against EFGH. There are **three** possibilities here: the seesaw could be level, ABCD is heavier, or EFGH is heavier.

Suppose it is level. Thus, the odd coin must be among IJKL and therefore, all the rest are normal. Pick two among this third set of 4, say I and J and weigh them against each other. This is our **second** measurement. Again, there are three possibilities but let's think of it simply as two possibilities: they weight the same or they don't.

- In the case they weigh the same, then either K or L is the odd one out. For our **third** measurement, weigh K against B. B is a normal coin so if the seesaw is level, then K is normal and L is odd. In this case, we don't know if L is heavier or lighter. If the seesaw is not level, then K is odd and we can determine whether it is heavier or lighter.

- In the case that I and J do not weigh the same, then one of them is odd. Say, I is lighter. Now for the **third** measurement, weigh it against a standard coin, say, B. We use the reasoning as the prior case to determine whether it is I or J that is odd. If I and B match, then J is the odd coin and it is heavier (because we now know I is a standard coin and J was heavier than it). If I and B do not match, then it must be that I is the odd coin and it is lighter.

Now we get to the trickier bit. Suppose that after our first measurement from above, ABCD is **lighter** than EFGH. For our **second** measurement, we remove ABC, move FGH to join D and have IJK join E. That is, we'll now be measuring DFGH against EIJK. Here is an easier way to remember this. We originally had: ABCD and EFGH and IJKL. We cyclically permute ABC and EFG and IJK via (1 3 2).

Let's make a few remarks. First, IJK are all normal coins. Secondly, we want to remember that after our first measurement, D was in the lighter group ABCD. For our second measurement, D is now placed with FGH which were in the heavier group EFGH. Again, we look at the three possibilities:

- Suppose the seesaw is now balanced; so all the coins are the normal. This means that when we removed ABC, we removed the odd coin. Not only this, ABCD was lighter compared to EFGH. Therefore, not only is the odd coin among ABC, we know it is **lighter**.

  We make our **third** measurement: weigh A and B against each other; if they are balanced, then C is the odd coin out and it is lighter than all the rest. If they are not balanced, the lighter one is easily found.

- Suppose the seesaw is configured such that DFGH is **heavier** than EIJK. We know from the first measurement that FGH belonged to the heavier of the two past groups: ABCD and EFGH. Well, coin E cannot be lighter since it was in the heavy group and it cannot be heavier now that we see DFGH is heavier in our second measurement. The heaviness cannot be attributed to coin D because it was originally on the lighter side after the first measurement.

  Therefore, the odd coin is among FGH and we know that it is heavier. We can use similar reasoning with a **third** measurement as above (with the case of ABC) to single out the odd coin.

- Lastly, consider the configuration in which DFGH is **lighter** than EIJK. Coin E was originally on the heavier side and is again on the heavier side after making our new arrangement. Similarly, D was on the lighter side before and is now again on the lighter side. On the other hand, FGH were originally in the heavier group and are now on the lighter group. We may conclude that FGH are normal coins. Combined with the fact that IJK are also normal coins, we conclude that either D is lighter or E is heavier. Our **third** measurement will be used to determine which is the odd coin.

  We weigh D against A. If they are balanced, then E is the odd coin out and is heavier. If they are not balanced, then it must be that D is lighter than A and thus we've found that the odd coin is D. Of course, D cannot be heavier than A by prior reasoning.

3 Comments

- Note that in the later steps of the problem, we were able to, under certain conditions, deduce that the odd coin was among FGH and it was heavier. It was crucial that we knew that the odd coin had to be heavier. If we only knew that the odd coin is among FGH but not whether it is heavier or lighter, we would not have been able to solve the problem at that point.

- Also, note that in 11 of the 12 cases, not only can we say whether the coin is the odd one out, we can say if it is heavier or lighter. It was only in the case of coin L that we cannot say whether it is heavier or lighter.

- Within this solution, we also took advantage of standard coins. This takes a bit of a shift in thinking. We're focused on finding the odd coin and may thus, ignore the standard coins. But in the steps, using the standard coins to test against was prominent.

- I quite enjoyed this riddle though I needed hints to solve it. Part of the fun is figuring out how to squeeze out as much information as we can from our observations. For example, we could consider the situations where the seesaw is either balanced or it is not. But we retain more information if we consider if the seesaw is balanced, dips to the left, or to the right. This conceptual step is crucial because we look back on previous configurations and compare them to current configurations as we measure.