CLASSICAL PERIOD DOMAINS

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Abstract. We survey the role played by Hermitian symmetric domains in the study of variations of Hodge Structure. These are extended notes based on the lectures given by the first author in Vancouver at the "Advances in Hodge Theory" school (June 2013).

Introduction

There are two classical situations where the period map plays an essential role for the study of moduli spaces, namely the moduli of principally polarized abelian varieties and the moduli of polarized K3 surfaces. What is common for these two situations is the fact that the period domain is in fact a Hermitian symmetric domain. It is well known that the only cases when a period domain is Hermitian symmetric are weight 1 Hodge structures and weight 2 Hodge structures with $h^{2,0} = 1$.

In general, it is difficult to study moduli spaces via period maps. A major difficulty in this direction comes from the Griffiths' transversality relations. Typically, the image Z of the period map in a period domain $\mathbf{D}$ will be a transcendental analytic subvariety of high condimension. The only cases when Z can be described algebraically are when Z is a Hermitian symmetric subdomain of $\mathbf{D}$ with a totally geodesic embedding (and satisfying the horizontality relation). This is closely related to the geometric aspect of the theory of Shimura varieties of Deligne. It is also the case of unconstrained period subdomains in the sense of [GGK12]. We call this case classical, in contrast to the "non-classical" case when the Griffiths' transversality relations are non-trivial.

The purpose of this survey is to review the role of Hermitian symmetric domains in the study of variations of Hodge structure. Let us give a brief overview of the content of the paper. In Section 1, we review the basic definitions and properties of Hermitian symmetric domains (Section 1.1) and their classification (Section 1.2) following [Mil04]. The classification is done by reconstructing Hermitian symmetric domains from the associated (semisimple) Shimura data, which are also convenient for the purpose of constructing variations of Hodge structure over Hermitian symmetric domains (Section 1.3). As a digression, we also include the discussion that if the universal family of Hodge structures over a period subdomain satisfies Griffiths transversality then the subdomain must be Hermitian symmetric (i.e. unconstrained $\Rightarrow$ Hermitian symmetric). Section 2 concerns locally symmetric varieties which are quotients of Hermitian symmetric domains. We first review the basic theory of locally symmetric domains and provide some examples of algebraic

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varieties whose moduli spaces are birational to locally symmetric domains (Section 2.1), and then give a representation theoretic description of variations of Hodge structure on locally symmetric domains (Section 2.2) following [Mil13]. Using the description, we discuss the classification of variations of Hodge structure of abelian variety type and Calabi-Yau type following [Del79] and [FL13] respectively. Baily-Borel and toroidal compactifications of locally symmetric varieties and their Hodge theoretic meanings are reviewed in Section 3.

1. Hermitian Symmetric Domains

In this section, we review the basic concepts and properties related to Hermitian Symmetric domains with an eye towards the theory of Shimura varieties and Hodge theory. The standard (differential geometric) reference for the material in this section is Helgason [Hel78] (see also the recent survey [Viv13]). For the Hodge theoretic point of view, we refer to the original paper of Deligne [Del79] and the surveys of Milne [Mil04] [Mil13].

1.1. **Hermitian symmetric spaces and their automorphisms.**

1.1.1. *Hermitian symmetric spaces.* We start by recalling the definition of Hermitian symmetric spaces.

Definition 1.1. A *Hermitian manifold* is a pair (M, g) consisting of a complex manifold M together with a Hermitian metric g on M. A Hermitian manifold (M, g) is *symmetric* if additionally

1. (1) (M, g) is homogeneous, i.e. the holomorphic isometry group $\mathrm{Is}(M, g)$ acts transitively on M;
2. (2) for any point $p \in M$, there exists an involution s_p (i.e. s_p is a holomorphic isometry and $s_p^2 = \mathrm{Id}$) such that p is an isolated fixed point of s_p (such an involution s_p is called a *symmetry* at p).

A connected symmetric Hermitian manifold is called a *Hermitian symmetric space*. (If there is no ambiguity, we will use M to denote the Hermitian manifold (M, g).)

Note that if (M, g) is homogeneous, it suffices to check Condition (2) at a point (i.e. it suffices to construct a symmetry s_p at some point $p \in M$). Also, the automorphism group $\mathrm{Is}(M, g)$ consists of holomorphic isometries of M:

$$\mathrm{Is}(M, g) = \mathrm{Is}(M^\infty, g) \cap \mathrm{Hol}(M),$$

where M^∞ denotes the underlying C^∞ manifold, $\mathrm{Is}(M^\infty, g)$ is the group of isometries of (M^∞, g) as a Riemannian manifold, and $\mathrm{Hol}(M)$ is the group of automorphisms of M as a complex manifold (i.e. the group holomorphic automorphisms).

Example 1.2. There are three basic examples of Hermitian symmetric spaces:

- (a) the upper half plane $\mathfrak{H}$;
- (b) the projective line $\mathbb{P}^1$ (or the Riemann sphere endowed with the restriction of the standard metric on $\mathbb{R}^3$);
- (c) any quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by a discrete additive subgroup $\Lambda \subset \mathbb{C}$ (with the natural complex structure and Hermitian metric inherited from $\mathbb{C}$).

To illustrate the definition, we discuss the example of the upper half plane $\mathfrak{H}$. First, it is easy to see that $\mathfrak{H}$, endowed with the metric $\frac{\mathrm{d}z\mathrm{d}\bar{z}}{y^2}$, is a Hermitian manifold. Clearly, $\mathfrak{H}$ is homogeneous with respect to the natural action of $\mathrm{SL}_2(\mathbb{R})$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}, \quad \text{for } z \in \mathfrak{H}.$$

In fact, $\mathrm{Is}(\mathfrak{H}) \cong \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$. Finally, the isomorphism $z \mapsto -\frac{1}{z}$ is an involution at the point $i \in \mathfrak{H}$. Since $\mathfrak{H}$ is connected, we conclude that the upper half space $\mathfrak{H}$ is a Hermitian symmetric space.

The three examples above are representative of the three basic classes of Hermitian symmetric spaces. Specifically, we recall the following:

Definition 1.3. Let M be a Hermitian symmetric space.

(1) M is said to be of *Euclidean type* if it is isomorphic to $\mathbb{C}^n/\Lambda$ for some discrete additive subgroup $\Lambda \subset \mathbb{C}^n$.
(2) M is said to be *irreducible* if it is not of Euclidean type and can not be written as a product of two Hermitian symmetric spaces of lower dimensions.
(3) M is said to be of *compact type* (resp. *noncompact type*) if it is the product of compact (resp. noncompact) irreducible Hermitian symmetric spaces. Moreover, Hermitian symmetric spaces of noncompact type are also called *Hermitian symmetric domains*.

Every Hermitian symmetric space can be decomposed uniquely into a product of Hermitian symmetric spaces of these three types:

Theorem 1.4 (Decomposition Theorem)**.** *Every Hermitian symmetric space M decomposes uniquely as*

$$M = M_0 \times M_- \times M_+,$$

where M_0 is a Euclidean Hermitian symmetric space and M_- (resp. M_+) is a Hermitian symmetric space of compact type (resp. of noncompact type). Moreover, M_- (resp. M_+) is simply connected and decomposes uniquely as a product of compact (resp. noncompact) irreducible Hermitian symmetric spaces.

Proof. See [Hel78, Ch. VIII], especially Proposition 4.4, Theorem 4.6 and Proposition 5.5. □

In this survey, we are mostly interested in Hermitian symmetric domains (or, equivalently, Hermitian symmetric spaces of noncompact type). Note that the terminology is justified by the Harish-Chandra embedding theorem: *every Hermitian symmetric space of noncompact type can be embedded into some $\mathbb{C}^n$ as a bounded domain*. Conversely, every bounded symmetric domain $D \subset \mathbb{C}^n$ has a canonical Hermitian metric (called the Bergman metric) which makes D a Hermitian symmetric domain. For instance, the bounded realization of the upper half plane $\mathfrak{H}$ is the unit ball $\mathcal{B}_1 \subset \mathbb{C}$.

1.1.2. *Automorphism groups of Hermitian symmetric domains.* Let (D, g) be a Hermitian symmetric domain. Endowed with the compact-open topology, the group $\mathrm{Is}(D^\infty, g)$ of isometries has a natural structure of (real) Lie group. As a closed subgroup of $\mathrm{Is}(D^\infty, g)$, the group $\mathrm{Is}(D, g)$ inherits the structure of a Lie group. Let us denote by $\mathrm{Is}(D, g)^+$ (resp. $\mathrm{Is}(D^\infty, g)^+$, $\mathrm{Hol}(D)^+$) the connected component of $\mathrm{Is}(D, g)$ (resp. $\mathrm{Is}(D^\infty, g)$, $\mathrm{Hol}(D)$) containing the identity.

Proposition 1.5. *Let (D, g) be a Hermitian symmetric domain. The inclusions*

$$\mathrm{Is}(D^\infty, g) \supset \mathrm{Is}(D, g) \subset \mathrm{Hol}(D)$$

induce identities

$$\mathrm{Is}(D^\infty, g)^+ = \mathrm{Is}(D, g)^+ = \mathrm{Hol}(D)^+.$$

Proof. See [Hel78, Lemma 4.3]. □

Since D is homogeneous, one can recover the smooth structure of D as a quotient Lie group of $\mathrm{Is}(D, g)^+$ by the stabilizer of a point. Specifically,

Theorem 1.6. *Notations as above.*

1. $\mathrm{Is}(D, g)^+$ *is an* adjoint *(i.e. semisimple with trivial center) Lie group.*
2. *For any point $p \in D$, the subgroup K_p of $\mathrm{Is}(D, g)^+$ fixing p is compact.*
3. *The map*
$$\mathrm{Is}(D, g)^+/K_p \to D, \quad gK_p \mapsto g \cdot p$$
is an $\mathrm{Is}(D, g)^+$-equivariant diffeomorphism. In particular, $\mathrm{Is}(D, g)^+$ (hence $\mathrm{Hol}(D)^+$ and $\mathrm{Is}(D^\infty, g)^+$) acts transitively on D.

Proof. See [Hel78, Ch. IV], especially Theorem 2.5 and Theorem 3.3. □

In particular, every irreducible Hermitian symmetric domain is diffeomorphic to H/K for a unique pair (H, K) (obtained as above) with H a connected noncompact simple adjoint Lie group and K a maximal connected compact Lie group (cf. [Hel78, Ch. VIII, §6]). Conversely, given such a pair (H, K), we obtain a smooth homogenous manifold H/K. The natural question is how to endow H/K with a complex structure and a compatible Hermitian metric so that it is a Hermitian symmetric domain. This can be done in terms of standard Lie theory (see [Viv13, §2.1] and the references therein). However, we shall answer this question from the viewpoint of Shimura data. Specifically, we shall replace the Lie group H by an algebraic group G, replace cosets of K by certain homomorphisms $u : U_1 \to G$ from the circle group U_1 to G, and then answer the question in terms of the pairs (G, u).

To conclude this subsection (and as an initial step to produce a Shimura datum), we discuss how to associate a $\mathbb{R}$-algebraic group G to the real Lie group $\mathrm{Hol}(D)^+$ in such a way that $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$. The superscript $^+$ in $G(\mathbb{R})^+$denotes the neutral connected component relative to the real topology (vs. the Zariski topology). We shall follow [Mil11] for the terminologies on algebraic groups, and also refer the readers to it for the related background materials. For example, we say an algebraic group is *simple* if it is non-commutative and has no proper normal algebraic subgroups, while *almost simple* if it is non-commutative and has no proper normal connected algebraic subgroup (N.B. an almost simple algebraic group can have finite center).

Proposition 1.7. *Let (D, g) be a Hermitian symmetric domain, and let $\mathfrak{h} = \mathrm{Lie}(\mathrm{Hol}(D)^+)$. There is a unique connected adjoint real algebraic subgroup G of $\mathrm{GL}(\mathfrak{h})$ such that (inside $\mathrm{GL}(\mathfrak{h})$)*

$$G(\mathbb{R})^+ = \mathrm{Hol}(D)^+.$$

Moreover, $G(\mathbb{R})^+ = G(\mathbb{R}) \cap \mathrm{Hol}(D)$ (inside $\mathrm{GL}(\mathfrak{h})$); therefore $G(\mathbb{R})^+$ is the stabilizer in $G(\mathbb{R})$ of D.

Proof. We sketch the proof of the first statement here, and refer the readers to [Mil04, Prop. 1.7] and references therein for details and the second statement.

Since $\mathrm{Hol}(D)^+$ is adjoint, its adjoint representation on the Lie algebra $\mathfrak{h}$ is faithful, and thus there exists an algebraic group $G \subset \mathrm{GL}(\mathfrak{h})$ such that $\mathrm{Lie}(G) = [\mathfrak{h}, \mathfrak{h}]$ (inside $\mathfrak{gl}(\mathfrak{h})$). Because $\mathrm{Hol}(D)^+$ is semisimple, $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ and so $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$ (inside $\mathrm{GL}(\mathfrak{h})$). □

1.2. **Classification of Hermitian symmetric domains.** Consider the circle group $U_1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Motivated by the following fact, one can think of a point of D as a homomorphism $U_1 \to G$.

Theorem 1.8. *Let D be Hermitian symmetric domain. For each $p \in D$, there exists a unique homomorphism $u_p : U_1 \to \mathrm{Hol}(D)^+$ such that $u_p(z)$ fixes p and acts on T_pD as multiplication by z.*

Proof. See [Mil04, Thm. 1.9]. □

Remark 1.9. Using the uniqueness of u_p one can easily see that $\mathrm{Hol}(D)^+$ acts on the set of u_p's via conjugation. Clearly, given two different points $p \neq p'$ we choose $f \in \mathrm{Hol}(D)^+$ with $f(p) = p'$, then $f \circ u_p(z) \circ f^{-1}$ $(z \in U_1)$ satisfies the conditions in Theorem 1.8 for p', and thus $u_{p'} = f \circ u_p \circ f^{-1}$.

Example 1.10. Let $p = i \in \mathfrak{H}$. As previously noted, we have $\mathrm{Hol}(\mathfrak{H}) = \mathrm{PSL}_2(\mathbb{R})$. The associated real algebraic group (compare Proposition 1.7) is $(\mathrm{PGL}_2)_{\mathbb{R}}$, and it holds: $\mathrm{PGL}_2(\mathbb{R})^+ = \mathrm{PSL}_2(\mathbb{R})$ (N.B. the group PSL_2 is not an algebraic group). To define $u_i : U_1 \to \mathrm{PSL}_2(\mathbb{R})$ we first consider the homomorphism

$$h_i : U_1 \to \mathrm{SL}_2(\mathbb{R}), z = a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

It is easy to verify that $h_i(z)$ fixes i. Since

$$\frac{d}{dw}\left(\frac{aw+b}{-bw+a}\right)\bigg|_{w=i} = \frac{a^2+b^2}{(a-ib)^2} = \frac{z}{\bar{z}} = z^2,$$

$h_i(z)$ acts on the tangent space $T_i\mathfrak{H}$ as multiplication by z^2. Thus, for $z \in U_1$, we choose a square root $\sqrt{z} \in U_1$ and set

$$u_i(z) := h_i(\sqrt{z}).$$

The homomorphism $u_i : U_1 \to \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\pm I$ is independent of the choice of $\sqrt{z}$ (since $h_i(-1) = -I$). Thus, u_i satisfies the conclusion of Theorem 1.8 at the point $i \in \mathfrak{H}$.

Since $G(\mathbb{R})^+(= \mathrm{Hol}(D)^+)$ acts transitively on D, set-theoretically we can view D as the $G(\mathbb{R})^+$-conjugacy class of $u_p : U_1 \to G(\mathbb{R})$. (Later, we will see that u_p is an algebraic homomorphism). This viewpoint suggests a connection between Hermitian symmetric domains and variations of Hodge structure. Namely, recall that one can view a Hodge structure as a representation of the Deligne torus $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Then, if we define $h_p : \mathbb{S} \to G$ by $h_p(z) = u_p(z/\bar{z})$, any representation $G \to \mathrm{GL}(V)$ of G (e.g. $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$), composed with h_p for all $p \in D$, will produce a variation of Hodge structure on D.

Conversely, given an abstract pair $(G, u : U_1 \to G)$ with G a real adjoint algebraic group and u an algebraic homomorphism it is natural to ask the following questions:

Question 1.11. For a pair (G, u) as above, we let D be the $G(\mathbb{R})^+$-conjugacy class of u. Denote by K_u the subgroup of $G(\mathbb{R})^+$ fixing u. There is a bijection $G(\mathbb{R})^+/K_u \to D$ and so the space D has a natural smooth structure.

(1) Under what conditions can D be given a nice complex structure (or a Hermitian structure)? Under what additional conditions is D a Hermitian symmetric space?
(2) Under what conditions is K_u compact?
(3) Under what conditions is D be a Hermitian symmetric domain (i.e. of noncompact type)?

1.2.1. *Representations of U_1.* Let T be an algebraic torus defined over a field k, and let K be a Galois extension of k splitting T. The character group $X^*(T)$ is defined by $X^*(T) = \mathrm{Hom}(T_K, \mathbb{G}_m)$. If r is the rank of T, then $X^*(T)$ is a free abelian group of rank r which comes equipped with an action of $\mathrm{Gal}(K/k)$. In general, to give a representation ρ of T on a k-vector space V amounts to giving an $X^*(T)$-grading $V_K = \bigoplus_{\chi \in X^*(T)} V_\chi$ on $V_K := V \otimes_k K$ with the property that

$$\sigma(V_\chi) = V_{\sigma\chi}, \quad \text{all } \sigma \in \mathrm{Gal}(K/k), \quad \chi \in X^*(T).$$

Here V_χ is the K-subspace of V_K on which $T(K)$ acts through χ:

$$V_\chi = \{v \in V_K \mid \rho(t)(v) = \chi(t) \cdot v, \quad \forall\, t \in T(K)\}.$$

For instance, we can regard U_1 as a real algebraic torus. As an $\mathbb{R}$-algebraic group, the K-valued points (with K an $\mathbb{R}$-algebra) of U_1 are

$$U_1(K) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_{2\times 2}(K) \;\middle|\; a^2 + b^2 = 1 \right\}.$$

In particular, $U_1(\mathbb{R})$ is the circle group and $U_1(\mathbb{C})$ can be identified with $\mathbb{C}^*$ through

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + ib, \quad \text{conversely } z \mapsto \begin{pmatrix} \frac{1}{2}(z + \frac{1}{z}) & \frac{1}{2i}(z - \frac{1}{z}) \\ -\frac{1}{2i}(z - \frac{1}{z}) & \frac{1}{2}(z + \frac{1}{z}) \end{pmatrix}.$$

Noting that $X^*(U_1) \cong \mathbb{Z}$ and complex conjugation acts on $X^*(U_1)$ as multiplication by -1, we obtain the following proposition.

Proposition 1.12. *Consider a representation ρ of U_1 on a $\mathbb{R}$-vector space V. Then $V_\mathbb{C} = \bigoplus_{n\in\mathbb{Z}} V^n_\mathbb{C}$ with the property that $\overline{V^n_\mathbb{C}} = V^{-n}_\mathbb{C}$, where $V^n_\mathbb{C} = \{v \in V_\mathbb{C} \mid \rho(z)(v) = z^n \cdot v, \quad \forall z \in \mathbb{C}^*\}$. Moreover, if V is irreducible, then it must be isomorphic to one of the following types.*

(a) *$V \cong \mathbb{R}$ with U_1 acting trivially (so $V_\mathbb{C} = V^0_\mathbb{C}$).*
(b) *$V \cong \mathbb{R}^2$ with $z = x + iy$ acting as $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^n$ for some $n > 0$ (so $V_\mathbb{C} = V^n_\mathbb{C} \oplus V^{-n}_\mathbb{C}$).*

In particular, every real representation of U_1 is a direct sum of representations of these types.

Remark 1.13. Let V be a $\mathbb{R}$-representation of U_1 and write $V_\mathbb{C} = \bigoplus_{n\in\mathbb{Z}} V^n_\mathbb{C}$ as above. Because $\overline{V^0_\mathbb{C}} = V^0_\mathbb{C}$, the weight space $V^0_\mathbb{C}$ is defined over $\mathbb{R}$; in other words, it is the complexification of the real subspace V^0 of V defined by $V \cap V^0_\mathbb{C}$: $V^0 \otimes_\mathbb{R} \mathbb{C} = V^0_\mathbb{C}$. The natural homomorphism $V/V^0 \to V_\mathbb{C}/\bigoplus_{n\le 0} V^n_\mathbb{C} \cong \bigoplus_{n>0} V^n_\mathbb{C}$ is a $\mathbb{R}$-linear isomorphism.

The representations of U_1 have the same description no matter if we regard it as a Lie group or an algebraic group, and so every homomorphism $U_1 \to \mathrm{GL}(V)$ of Lie groups is algebraic. In particular, the homomorphism $u_p : U_1 \to \mathrm{Hol}(D)^+ \cong G(\mathbb{R})^+$ is algebraic for any $p \in D$. Let K_p be the subgroup of $G(\mathbb{R})^+$ fixing p. By Theorem 1.8, $u_p(z)$ acts on the $\mathbb{R}$-vector space

$$\mathrm{Lie}(G)/\,\mathrm{Lie}(K_p) \cong T_pD$$

as multiplication by z, and it acts on $\mathrm{Lie}(K_p)$ trivially. Suppose $T_pD \cong \mathbb{C}^k$ and identify it with $\mathbb{R}^{2k}$ by $(a_1 + ib_1, \cdots, a_k + ib_k) \mapsto (a_1, b_1, \cdots, a_k, b_k)$, then it is easy to write down the matrix of multiplication by $z = x + iy$ and conclude that T_pD (as a real representation of U_1) splits into a direct sum of $\mathbb{R}^2$'s as in the Part (b) of the previous proposition with $n = 1$.

Accordingly, we can determine the representation $\mathrm{Ad} \circ u_p : U_1 \to G \to \mathrm{GL}(\mathrm{Lie}(G))$ (because $u_p(z)$ is contained in the stabilizer K_p of p, the action of $u_p(z)$ on T_pD is induced by the adjoint representation). It splits into a direct sum of 1-dimensional real vector spaces (as in Part (a) of Proposition 1.12) and 2-dimensional spaces (as in Part (b) with $n = 1$). Taking the complexification of the representation $\mathrm{Lie}(G)$, we obtain the following proposition.

Proposition 1.14. *Notations as above. Only the characters z, 1 and z^{-1} occur in the representation of U_1 on $\mathrm{Lie}(G)_{\mathbb{C}}$ defined through u_p.*

1.2.2. *Cartan involutions.* Let G be a connected algebraic group defined over $\mathbb{R}$, and let $g \mapsto \bar{g}$ denote complex conjugation on $G(\mathbb{C})$.

Definition 1.15. An involution θ of G is said to be *Cartan* if the group

$$G^{(\theta)}(\mathbb{R}) := \{g \in G(\mathbb{C}) \mid g = \theta(\bar{g})\}$$

is compact.

Example 1.16. Let $G = \mathrm{SL}_2$, and let θ be the conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since

$$\theta\left(\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix},$$

we have

$$\mathrm{SL}_2^{(\theta)}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \middle| d = \bar{a},\ c = -\bar{b} \right\} = \mathrm{SU}_2(\mathbb{R}).$$

Clearly, the group $\mathrm{SU}_2(\mathbb{R})$ is compact, and hence θ is a Cartan involution for SL_2.

Theorem 1.17. *A connected $\mathbb{R}$-algebraic group G has a Cartan involution if and only if it is reductive, in which case any two Cartan involutions are conjugate by an element of $G(\mathbb{R})$.*

Proof. See [Sat80, Ch. I, Thm. 4.2 and Cor. 4.3]. □

Example 1.18. Let G be a connected $\mathbb{R}$-algebraic group.

(a) The identity map is a Cartan involution if and only if $G(\mathbb{R})$ is compact. Moreover, it is the only Cartan involution of G.

(b) Let $G = \mathrm{GL}(V)$ with V a real vector space of dimension n. Fix a basis of V, then G has an involution given by $\theta : M \mapsto (M^t)^{-1}$. On $G(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$, $M = \theta(\bar{M})$ if and only if $M\bar{M}^t = I$ (i.e. $M \in \mathrm{U}(n)$). Thus θ is a Cartan involution. Note that different choices of bases give different Cartan involutions, and the previous theorem says that all Cartan involutions of G arise in this way.

(c) ([Sat80, Ch. I, Cor. 4.4]) Let $G \hookrightarrow \mathrm{GL}(V)$ be a faithful representation. Then G is reductive if and only if it is stable under $g \mapsto g^t$ for a suitable choice of a basis for V, in which case the restriction of $g \mapsto (g^t)^{-1}$ to G is a Cartan involution. Furthermore, all Cartan involutions of G arise in this way from the choice of a basis of V.

(d) Let θ be an involution of G. Then there is a unique real form $G^{(\theta)}$ of $G_{\mathbb{C}}$ such that complex conjugation on $G^{(\theta)}(\mathbb{C})$ is $g \mapsto \theta(\bar{g})$. So the Cartan involutions of G correspond to the compact forms of $G_{\mathbb{C}}$.

Now let us go back to Hermitian symmetric domains. Let D be a Hermitian symmetric domain. As before, G is the associated real adjoint algebraic group (cf. Proposition 1.7), and $u_p : U_1 \to G$ is an algebraic homomorphism attached to a point $p \in D$.

Proposition 1.19. *The conjugation by $u_p(-1)$ is a Cartan involution of G.*

Proof. Let s_p be a symmetry at p. Denote by $\mathrm{Inn}(s_p)$ the conjugation of G by s_p. $\mathrm{Inn}(s_p)$ is an involution because $s_p^2 = \mathrm{Id}$. According to Section [Hel78, §V.2], the real form of $G_{\mathbb{C}}$ defined by the involution $\mathrm{Inn}(s_p)$ (cf. Example 1.18 (d)) is that associated to the compact dual of the symmetric space. As a result, a symmetry at a point of a symmetric space gives a Cartan involution of G if and only if the space is of noncompact type. In particular, $\mathrm{Inn}(s_p)$ is Cartan. On the other hand, both $u_p(-1)$ and s_p fix p and acts as multiplication by (-1) on T_pD, and hence $u_p(-1) = s_p$ (cf. [Mil04, Prop. 1.14], which also implies the uniqueness of symmetries at a point of a Hermitian symmetric domain.) □

Note that Example 1.16 is cooked up in this way.

1.2.3. *Classification of Hermitian symmetric domains in terms of real groups.* We will classify (pointed) Hermitian symmetric domain in this section. Let D be a Hermitian symmetric domain. We have already discussed the first two statements of the following theorem.

Theorem 1.20. *Let G be the associated adjoint real algebraic group of D. The homomorphism $u_p : U_1 \to G$ attached to a point $p \in D$ satisfies the following properties:*

(a) *only the character z, 1 and z^{-1} occur in the representation of U_1 on $\mathrm{Lie}(G)_{\mathbb{C}}$ defined by u_p;*

(b) *The conjugation of G by $u_p(-1)$ is a Cartan involution;*

(c) *$u_p(-1)$ does not project to 1 in any simple factor of G.*

Proof. See Proposition 1.14 and 1.19 for Part (a) and (b). Suppose $u_p(-1)$ projects to 1 for some simple factor G_1 (which corresponds to a noncompact irreducible factor of D, see Theorem 1.4), then the Cartan involution $\mathrm{Inn}(u_p(-1))$ is the identity map on G_1. But by Example 1.18 (a), this implies that $G_1(\mathbb{R})$ is compact, which is a contradiction. □

The properties (a), (b), (c) in Theorem 1.20 turns out to be an answer to Question 1.11.

Theorem 1.21. *Let G be a real adjoint algebraic group, and let $u : U_1 \to G$ be a homomorphism satisfying (a), (b) and (c) of Theorem 1.20. Then the set D of conjugates of u by elements of $G(\mathbb{R})^+$ has a natural structure of a Hermitian symmetric domain, such that $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$ and $u(-1)$ is the symmetry at u (regarded as a point of D).*

Proof. (Sketch) Let K_u be the subgroup of $G(\mathbb{R})^+$ fixing u (i.e. the centralizer of u). By (b), $\theta := \mathrm{Ad}(u(-1))$ is a Cartan involution for G. So $G^{(\theta)}(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid g = u(-1) \cdot \bar{g} \cdot u(-1)^{-1}\}$ is compact. Since $K_u \subset G(\mathbb{R})^+$, $\bar{g} = g$ for any $g \in K_u$, and so $K_u \subset G^{(\theta)}(\mathbb{R})$. As K_u is closed, it is also compact. The natural bijection $D \cong (G(\mathbb{R})^+/K_u) \cdot u$ endows D with the structure of a smooth (homogeneous) manifold.

With this structure, the (real) tangent space at u is $T_uD = \mathrm{Lie}(G)/\mathrm{Lie}(K_u)$. Note that $\mathrm{Lie}(G)$ a real representation of U_1 via $\mathrm{Ad} \circ u$. Using the notations in Proposition 1.12, (a) gives that $\mathrm{Lie}(G)_{\mathbb{C}} = \mathrm{Lie}(G)_{\mathbb{C}}^{-1} \oplus \mathrm{Lie}(G)_{\mathbb{C}}^{0} \oplus \mathrm{Lie}(G)_{\mathbb{C}}^{1}$. Clearly, $K_u = \mathrm{Lie}(G)_{\mathbb{C}}^{0} \cap \mathrm{Lie}(G)$. Using the natural isomorphism

$$\mathrm{Lie}(G)/\mathrm{Lie}(K_u) \to \mathrm{Lie}(G)_{\mathbb{C}}/\mathrm{Lie}(G)_{\mathbb{C}}^{0} \oplus \mathrm{Lie}(G)_{\mathbb{C}}^{-1} \cong \mathrm{Lie}(G)_{\mathbb{C}}^{1},$$

the tangent space T_uD can be identified with $\mathrm{Lie}(G)_{\mathbb{C}}^{1}$, the complex vector space of $\mathrm{Lie}(G)_{\mathbb{C}}$ on which $u(z)$ acts as multiplication by z. This endows T_pD with a $\mathbb{C}$-vector space structure. (In particular, the corresponding almost complex structure J is $u(i)$.) Since D is homogenous, this induces a structure of an almost complex manifold on D, which is integrable (cf. [Wol84, §8.7.9]).

The action of K_u on D induces an action of it on T_uD. As K_u is compact, there is a K_u-invariant positive definite form on T_uD (cf. Proposition 1.18 of [Mil04]), which is compatible with the complex structure J of T_uD because $J = u(i) \in K_u$. Now use the homogeneity of D to move the bilinear form to each tangent space, which will make D into a Hermitian manifold. It is not difficult to see that $u(-1)$ is the symmetric at u and D is a Hermitian symmetric space.

Finally, D is a Hermitian symmetric domain because of (b) and (c). The proof is quite similar with the one of Theorem 1.20. □

Remark 1.22. As we saw in the proof of Theorem 1.21, the condition (b) guarantees that K_u is compact. If further assuming (a) holds, then one can endow D with the structure of a Hermitian symmetric space. The space D is a Hermitian symmetric domain because of (b) and (c).

As a corollary, we can classify Hermitian symmetric domains in terms of such pairs.

Corollary 1.23. *There is a natural one-to-one correspondence between isomorphism classes of pointed Hermitian symmetric domains and pairs (G, u) consisting of a real adjoint algebraic group G and a non-trivial homomorphism $u : U_1 \to G$ satisfying (a), (b), (c) in Theorem 1.20.*

1.2.4. *Classification of Hermitian symmetric domains in terms of Dynkin diagrams.* Let us now focus on irreducible Hermitian symmetric domains. The irreducibility of the domain implies that the associated adjoint algebraic group is a simple algebraic group. Let G be a simple adjoint group over $\mathbb{R}$, and let u be a homomorphism

$U_1 \to G$ satisfying (a) and (b) of Theorem 1.20. (N.B. the condition (c) then holds trivially.)

Lemma 1.24. *Let $G_{\mathbb{C}}$ be the scalar extension of G from $\mathbb{R}$ to $\mathbb{C}$, and $\mu = u_{\mathbb{C}} : \mathbb{G}_m \to G_{\mathbb{C}}$. Then*

(1) *$G_{\mathbb{C}}$ is also simple;*
(2) *Only the characters z, 1, z^{-1} occur in the action of $\mathrm{Ad} \circ \mu : \mathbb{G}_m \to \mathrm{Lie}(G_{\mathbb{C}})$.*

Proof. See Page 21 of [Mil04] or Page 478 of [Mil13] for Part (1). Part (2) follows from Theorem 1.20 (a). □

Proposition 1.25. *The map $(G, u) \mapsto (G_{\mathbb{C}}, u_{\mathbb{C}})$ defines a bijection between the sets of isomorphism classes of pairs consisting of*

(1) *a simple adjoint algebraic group over $\mathbb{R}$ and a conjugacy class of $u : U_1 \to G$ satisfying (a) and (b) in Theorem 1.20, and*
(2) *a simple adjoint algebraic group over $\mathbb{C}$ and a conjugacy class of cocharacters satisfying (2) of Lemma 1.24.*

Proof. See [Mil04, Prop. 1.24]. □

Example 1.26. Let μ be a cocharacter of $(\mathrm{PGL}_2)_{\mathbb{C}}$. Let θ be the conjugation of $\mathrm{PGL}_2(\mathbb{C})$ by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The same computation as in Example 1.16 shows that the involution $\bar{\theta}$, defined by $g \mapsto \theta(\bar{g})$, is the identity on the compact form PGU_2 of $(\mathrm{PGL}_2)_{\mathbb{C}}$. Consider another involution on $\mathrm{PGL}_2(\mathbb{C})$ given by $g \mapsto \mu(-1) \circ \bar{\theta}(g) \circ \mu(-1)^{-1}$. By Example 1.18 ($d$), there is a real form H of $(\mathrm{PGL}_2)_{\mathbb{C}}$ such that complex conjugation on $H(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$ is the involution as above. Also define $u := \mu|_{U_1}$ which takes value in $H(\mathbb{R})$. As $\mu(-1)^2 = \mathrm{Id}$, the conjugation by $u(-1)$ is an involution of H. By construction, it is a Cartan involution. In this way, we obtain a pair (H, u) as in (1) of Proposition 1.25.

In particular, if μ is the scalar extension of u_i which is defined in Example 1.10, then

$$\mu(-1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As $\mu(-1) \circ \bar{\theta}(g) \circ \mu(-1)^{-1} = \bar{g}$, the corresponding real form H of $(\mathrm{PGL}_2)_{\mathbb{C}}$ is $(\mathrm{PGL}_2)_{\mathbb{R}}$. Also, it is clear that $u = u_i$.

Let $G_{\mathbb{C}}$ be a simple algebraic group. We choose a maximal torus T, and let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ (resp. $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$) be the character (resp. cocharacter) group. Note that there is a natural pairing $\langle -, - \rangle : X^*(T) \times X_*(T) \to \mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}$ between $X^*(T)$ and $X_*(T)$ (see Page 335 of [Mil11]). Choose a set of simple roots $(\alpha_i)_{i \in I}$. The nodes of the Dynkin diagram of $(G_{\mathbb{C}}, T)$ are also indexed by I. Recall that highest root is the unique root $\tilde{\alpha} = \sum_{i \in I} n_i \alpha_i$ such that, for any other root $\sum_{i \in I} m_i \alpha_i$, $n_i \geq m_i$. We say that an root α_i (or the corresponding node) is *special* if $n_i = 1$ in the expression of $\tilde{\alpha}$.

Theorem 1.27. *The isomorphism classes of irreducible Hermitian symmetric domains are classified by the special nodes on connected Dynkin diagrams.*

Proof. See [Mil04, Thm. 1.25]. For completeness, we include the proof here. Notations as above. By Theorem 1.21 and Proposition 1.25, it suffices to construct a bijection between the conjugacy classes of $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$ satisfying (2) of Lemma 1.24, and special nodes of the Dynkin diagram of $G_{\mathbb{C}}(\mathbb{C})$. Since all maximal tori are conjugate, we can assume that μ is in the cocharacter group $X_*(T) \subset X_*(G_{\mathbb{C}})$ of T. Moreover, there is a unique representative μ such that $\langle \alpha_i, \mu \rangle \geq 0$ for all $i \in I$ because the Weyl group acts transitively and freely on the Weyl chambers. Now (2) of Lemma 1.24 is equivalent to $\langle \alpha, \mu \rangle \in \{-1, 0, 1\}$ for all roots α. Since μ is non-trivial, not all values can be 0, so there must be a (unique) simple root α_i such that $\langle \alpha_i, \mu \rangle = 1$, which is in fact a special root (otherwise $\langle \tilde{\alpha}, \mu \rangle > 1$). The other direction easily follows from the fact that $\langle -, - \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ is a perfect pairing. $\square$

The special roots of connected Dynkin diagrams are listed in the following table.

Type	$\widetilde{\alpha}$	Special root
A_n	$\alpha_1 + \cdots + \alpha_n$	$\alpha_1, \cdots, \alpha_n$
B_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$	α_1
C_n	$2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$	α_n
D_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	α_1, α_{n-1}, α_n
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	α_1,α_6
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	α_7
E_8, F_4, G_2		none

TABLE 1. Special roots of connected Dynkin diagrams.

1.3. **Hermitian symmetric domains and Hodge structures.** The goal of this section is twofold: on one hand, given a Hermitian symmetric domain D, we show how to use the associated pair (G, u) (cf. Proposition 1.7 and Theorem 1.8) to construct variations of Hodge structure over D; on the other hand, given a real vector space V, we consider certain sets of Hodge structures on V and show that they can be endowed with structures of Hermitian symmetric domains (in fact, every irreducible Hermitian symmetric domain can be obtained in this way). We refer the readers to [GGK12, Chap. 1,2] for the background of Hodge structures and variations of Hodge structures.

1.3.1. *A closer look at Condition* (*a*) *of Theorem 1.20.* Given a pair (G, u) as in Theorem 1.20, we have seen in Theorem 1.21 that the $G(\mathbb{R})^+$-conjugacy class of u has a natural structure of a Hermitian symmetric domain. In this subsection, we would like to consider a more general situation.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ be the Deligne torus. We consider the following pairs $(\mathbf{G}, h)$ where $\mathbf{G}$ is a reductive (see for example Page 16 of [Mil11]) algebraic group over $\mathbb{R}$ and $h : \mathbb{S} \to \mathbf{G}$ is an algebraic homomorphism. We denote by X the conjugacy class of h by elements of $\mathbf{G}(\mathbb{R})$ (not $\mathbf{G}(\mathbb{R})^+$). Note that one can produce such a pair from a Hermitian symmetric domain D. Specifically, we set $\mathbf{G}$ to be the adjoint algebraic group G in Proposition 1.7 and define h by $h(z) = u(z/\bar{z})$ with $u = u_p$ (cf. Theorem 1.8) for some $p \in D$ (see also Example 1.10). In this case, D will be a connected component of X, the $G(\mathbb{R})$-conjugacy class of h (cf. [Mil04, Prop. 4.9]).

Let Z_h be the centralizer of h in $\mathbf{G}(\mathbb{R})$. Then the orbit map identifies X with $\mathbf{G}(\mathbb{R})/Z_h$. We view X as a homogenous manifold via this identification.

For any real representation $\rho : \mathbf{G} \to \mathrm{GL}(V)$ and any $h' \in X$, the composition $\rho \circ h' : \mathbb{S} \to \mathbf{G} \to \mathrm{GL}(V)$ defines a real Hodge structure on V (e.g. Page 26 of [Mil04]). In other words,

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} V_{h'}^{p,q},$$

where $V_{h'}^{p,q} = \{v \in V_{\mathbb{C}} \mid (\rho(h'(z)))(v) = z^{-p}\bar{z}^{-q} \cdot v,\ \forall z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*\}$. In particular, over $\mathbb{R}$ we have the weight space decomposition:

$$V = \bigoplus_{n \in \mathbb{Z}} V_{n,h'}, \quad V_{n,h'} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} V_{h'}^{p,q}$$

(i.e. $v \in V_{n,h'}$ if and only if $(\rho(h'(r)))(v) = r^n \cdot v$ for all $r \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^*$, here $\mathbb{G}_m$ is mapped into $\mathbb{S}$ via $\mathbb{G}_m \xrightarrow{w} \mathbb{S}$, $r \mapsto r^{-1}$).

Remark 1.28. For a Hodge structure $\varphi : \mathbb{S} \to \mathrm{GL}(V)$, the standard convention in the theory of Shimura variety is $\varphi_{\mathbb{C}}(z_1, z_2)(v^{p,q}) = z_1^{-p} z_2^{-q} \cdot v^{p,q}$ (cf. [Del79, (1.1.1.1)]). Meanwhile, a different convention $\varphi_{\mathbb{C}}(z_1, z_2)(v^{p,q}) = z_1^{p} z_2^{q} \cdot v^{p,q}$ is largely used in Hodge theory (e.g. Page 31 of [GGK12]). We shall use different conventions in different contexts.

Lemma 1.29. *The following statements are equivalent.*

(1) *For all representations (V, ρ) of $\mathbf{G}$, the weight space decomposition of V induced by $h' \in X$ is independent of the h'.*
(2) *For any $h' \in X$, the real Hodge structure on* $\mathrm{Lie}(\mathbf{G})$ *defined by* $\mathrm{Ad} \circ h'$ *is pure of weight* 0.

Proof. See [Del79, 1.1.13(α)]. See also [Con, Lemma 5.1]. ☐

Assume X satisfies one of the properties in the previous lemma, then for any representation (V, ρ) the weight spaces $V_{n,h'}$ are independent of $h' \in X$, and so we have a trivial vector bundle $X \times (V_n)_{\mathbb{C}} \to X$ for every weight n. Furthermore, the Hodge filtration $F^{\bullet}_{n,h'}$ on $(V_n)_{\mathbb{C}}$ induced by $\rho \circ h'$ defines (as h' varies) a filtration on $X \times (V_n)_{\mathbb{C}}$ by subbundles $\mathcal{F}_n^{\bullet}$.

We want to put a complex structure on X such that (1) $\mathcal{F}_n^p$ will be a holomorphic subbundle ($0 \le p \le n$); (2) $\mathcal{F}_n^p$'s satisfy Griffiths transversality for the natural connection on $X \times (V_n)_{\mathbb{C}}$. To do this, we need the following axiom. Recall that the *type* of a Hodge structure $V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ is the set of (p, q) such that $V^{p,q}$ is non-empty.

(**Axiom I**) The Hodge structure on $\mathrm{Lie}(\mathbf{G})$ given by $\mathrm{Ad} \circ h'$ for any $h' \in X$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$.

Note that pairs $(\mathbf{G}, h)$ coming from Hermitian symmetric domains clearly satisfy Axiom I (cf. Theorem 1.20 (a)). Also, if Axiom I is satisfied, then Lemma 1.29 (2) automatically holds.

Assuming Axiom I, we can endow X with a complex structure as follows (compare to Theorem 1.21). Let $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. Also fix $h' \in X$ and denote the Hodge structure on $\mathfrak{g}$ induced by $h' \in X$ by $\{\mathfrak{g}_{\mathbb{C}}^{p,q}\}$. Then there is a natural isomorphism $T_{h'}X \cong \mathfrak{g}/\mathfrak{g}^{0,0}$, where $\mathfrak{g}^{0,0}$ is the real descent of $\mathfrak{g}_{\mathbb{C}}^{0,0}$. Because $T_{h'}X = \mathfrak{g}/\mathfrak{g}^{0,0} \subset \mathfrak{g}_{\mathbb{C}}^{1,-1} \oplus \mathfrak{g}_{\mathbb{C}}^{-1,1}$, $\mathrm{Ad}(h'(i))$ acts on $T_{h'}X$ as multiplication by -1. Define

$J_{h'} = \mathrm{Ad}(h'(\mathrm{e}^{\frac{\pi \mathrm{i}}{4}}))$. Since $J_{h'}^2 = -\mathrm{Id}$, this defines a complex structure on $T_{h'}X$. Moving $J_{h'}$ around using the homogeneity of X, we obtain an almost complex structure on X.

Theorem 1.30. *Let $\mathbf{G}$ be a reductive group over $\mathbb{R}$ and let X be the $\mathbf{G}(\mathbb{R})$-conjugacy class of an algebraic homomorphism $h : \mathbb{S} \to \mathbf{G}$. If $(\mathbf{G}, X)$ satisfies Axiom I, then the almost complex structure defined by $\{J_{h'}\}$ is integrable.*

For any representation V of $\mathbf{G}$ and any integers n and p, $\mathcal{F}_n^p$ is a holomorphic vector bundle on X with respect to this complex structure. Moreover, $\mathcal{F}_n^\bullet$ satisfies Griffiths transversality for the connection $\nabla = 1 \otimes \mathrm{d} : (V_n)_\mathbb{C} \otimes \mathcal{O}_X \to (V_n)_\mathbb{C} \otimes \Omega_X^1$. (In other words, $(X \times (V_n)_\mathbb{R}, \nabla, \mathcal{F}_n^\bullet)$ forms a real variation of Hodge structure of weight n over X.)

Proof. See [Del79, Prop. 1.1.14] or [Mil04, Prop. 5.9]. See also [Con, Prop. 5.3] and [Lev, Thm. 3.7]. ☐

Remark 1.31. Axiom I is one of Deligne's axioms in the definition of a Shimura datum, for which we refer the readers to [Mil04, Def. 5.5]. Also, see [Mil04, Def. 4.22] for the definition of a connected Shimura datum. For the connections between (connected) Shimura data and Hermitian symmetric domains, see Proposition 4.8, Proposition 5.7 and Corollary 5.8 of op. cit..

1.3.2. *Parameter spaces for certain Hodge structures.* Fix a real vector space V and an integer n. Let T be a set of tensors (i.e. multilinear maps $V \otimes \cdots \otimes V \to \mathbb{R}$) including a nondegenerate bilinear form t_0, and let $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ be a function with the property that (1) $d(p, q) = 0$ unless $p + q = n$; (2) $d(p, q) = d(q, p)$.

For a real Hodge structure V of weight m, a tensor $V^{\otimes 2r} \to \mathbb{R}(-mr)$ of V is said to be a Hodge tensor if it is a morphism of Hodge structures (of type $(0, 0)$). Define $S(d, T)$ be the set of all Hodge structures φ of weight n on V such that

(1) $\dim V_\varphi^{p,q} = d(p, q)$;
(2) each $t \in T$ is a Hodge tensor for φ;
(3) t_0 is a polarization (e.g. Page 32 of [GGK12]) for φ.

For every $\varphi \in S(d, T)$, we consider the corresponding Hodge filtrations $F_\varphi^\bullet$. Denote by $\mathbf{d}$ the sequence of dimensions of F_φ^i for $0 \le i \le n$. Because $\dim V_\varphi^{p,q} = d(p, q)$, $\mathbf{d}$ is independent of φ. Consider the flag variety $\mathrm{Fl}(\mathbf{d}, V_\mathbb{C})$ of type $\mathbf{d}$, then $S(d, T)$ acquires a topology as a subspace of $\mathrm{Fl}(\mathbf{d}, V_\mathbb{C})$.

Note that by construction there is an universal family of Hodge structures $\{\varphi\}_{\varphi \in S(d,T)}$ over $S(d, T)$.

Theorem 1.32. *Let S^+ be a connected component of $S(d, T)$.*

(1) *If nonempty, then S^+ has a unique complex structure for which $\{\varphi\}_{\varphi \in S^+}$ is a holomorphic family of Hodge structures. (Alternatively, the map $S^+ \to$ $\mathrm{Fl}(\mathbf{d}, V_\mathbb{C})$, $\varphi \mapsto F_\varphi^\bullet$ is holomorphic.)*
(2) *With this complex structure, S^+ is a Hermitian symmetric domain if $\{\varphi\}_{\varphi \in S^+}$ is a variation of Hodge structure.*
(3) *Every irreducible Hermitian symmetric domain is of the form S^+ for a suitable V, n, d and T.*

Proof. (Sketch) See [Mil04, Thm. 2.14]. We sketch the proof of the statement (2) here and refer the readers to op. cit. for the proof of the statements (1) and (3). For simplicity, we further assume that $T = \{t_0\}$. Choose a point $\varphi \in S^+$, and denote by h_0 the corresponding homomorphism $h_0 : \mathbb{S} \to \mathrm{GL}(V)$. Let $\mathbf{G}$ be the algebraic subgroup of $\mathrm{GL}(V)$ whose elements fix t_0 up to scalar. Then h_0 factors through $\mathbf{G}$: $h_0 : \mathbb{S} \to \mathbf{G} \hookrightarrow \mathrm{GL}(V)$. Let $w : \mathbb{G}_m \to \mathbb{S}$ be the homomorphism defined by $r \mapsto r^{-1}$. Because V has a single weight n, $h_0 \circ w$ maps $\mathbb{G}_m$ into the center of $\mathbf{G}$. As a result, there exists a homomorphism $u_0 : U_1 \to \mathbf{G}^{\mathrm{ad}}$ such that $h_0(z) = u_0(z/\bar{z})$ modulo the center $Z(\mathbf{G})(\mathbb{R})$. (Note that there is an exact sequence $0 \to \mathbb{G}_m \xrightarrow{w} \mathbb{S} \to U_1 \to 0$, where w is defined as above and the other homomorphism $\mathbb{S} \to U_1$ is defined by $z \mapsto z/\bar{z}$.)

We now show that $(\mathbf{G}^{\mathrm{ad}}, u_0)$ satisfies Theorem 1.20 (a), (b) and (c). Let $\mathfrak{g}$ be $\mathrm{Lie}(\mathbf{G})$ with the Hodge structure provided by $\mathrm{Ad} \circ h_0$. One can verify that $\mathfrak{g}$ is a sub-Hodge structure of $\mathrm{End}(V)$ (with the natural Hodge structure of weight 0 induced from (V, φ)). Note that we have

$$\mathfrak{g}/\mathfrak{g}^{0,0} \cong T_\varphi S^+ \subset T_\varphi(\mathrm{Fl}(\mathbf{d}, V_\mathbb{C})) \cong \mathrm{End}(V_\mathbb{C})/F^0\,\mathrm{End}(V_\mathbb{C}),$$

where $\mathfrak{g}^{0,0}$ is the real descent of $\mathfrak{g}_\mathbb{C}^{0,0} \subset \mathfrak{g}_\mathbb{C}$. If the universal family $\{\varphi\}_{\varphi \in S^+}$ satisfies Griffiths transversality, then $\mathfrak{g}/\mathfrak{g}^{0,0} \subset F^{-1}\,\mathrm{End}(V_\mathbb{C})/F^0\,\mathrm{End}(V_\mathbb{C})$. This implies that $\mathfrak{g}$ is of type $\{(1,-1),(0,0),(-1,1)\}$, and so u_0 satisfies (a).

Let $\mathbf{G}^1$ be the subgroup of $\mathbf{G}$ whose elements fix t_0 ($(\mathbf{G}^1)^{\mathrm{ad}} \cong \mathbf{G}^{\mathrm{ad}}$). Let $C = h_0(i) = u_0(-1)$ be the Weil operator. As t_0 is a polarization of the Hodge structure (V, φ), t_0 is $\mathbf{G}^1$-invariant and $(2\pi i)^n t_0(-, C-)$ is symmetric and positive definite. By [Mil04, Prop. 1.20], this implies that u_0 satisfies (b).

The set S^+ can be viewed as a connected component of the space of homomorphisms $U_1 \to (\mathbf{G}^1)^{\mathrm{ad}}$. By [Del79, 1.1.12], it is equal to the set of conjugates of u_0 by elements of $(\mathbf{G}^1)^{\mathrm{ad}}(\mathbb{R})^+$. Now discard any compact factors of $(\mathbf{G}^1)^{\mathrm{ad}}$ and apply Theorem 1.21. □

When $T = \{t_0\}$, $S(d, t_0)$ is the Griffiths period domain. In general, $S(d, T)$ is a subdomain of $S(d, t_0)$.

As a result of the theorem, if the universal family of Hodge structures on a period subdomain satisfies Griffiths transversality, then the domain must be a Hermitian symmetric domain.

2. Locally symmetric varieties and Hodge theory

In this section, we review the basic theory of locally symmetric varieties. We also explore the role of locally symmetric varieties in studying moduli spaces and variations of Hodge structure.

2.1. **Locally symmetric varieties.**

2.1.1. *Motivations from algebraic geometry.* One of the most important applications of Hodge theory in algebraic geometry is to study moduli spaces via *period maps*. Let $\mathcal{M}$ be a moduli space of certain smooth complex algebraic varieties X, choose an integer $0 \le k \le \dim X$ (typically $k = \dim X$) and let $\mathbf{D}$ be a period domain parametrizing polarized Hodge structure of weight k which has the same Hodge numbers as $H^k_{\mathrm{prim}}(X, \mathbb{Q})$. A period map

$$\mathcal{P} : \mathcal{M} \to \Gamma\backslash\mathbf{D}$$

is defined by associating to X the polarized Hodge structure of weight k on the primitive cohomology group $H^k_{\mathrm{prim}}(X, \mathbb{Q})$ (Γ is a suitable discrete group acting properly and discontinuously on $\mathbf{D}$). The ideal situation is when $\mathcal{P}$ is birational.

For that, one needs to prove that $\mathcal{P}$ is injective (Torelli type results) and $\mathcal{P}$ is dominant. However, due to Griffiths transversality, except for principally polarized abelian varieties and K3 type situations, $\mathcal{P}$ is never dominant. Because of this, we consider subdomains of period domains $\mathbf{D}$. In general, the periods lie in Mumford-Tate subdomains of $\mathbf{D}$, and the image Z of $\mathcal{P}$ is typically highly transcendental. If the periods satisfy enough algebraic relations, then they belong to a Hermitian symmetric domain.

Theorem 2.1. *Let Z be a closed horizontal subvariety of a classifying space $\mathbf{D} = \mathbf{G}(\mathbb{R})/\mathbf{K}$ for Hodge structures and let $\Gamma = \mathrm{Stab}_Z \cap\ \mathbf{G}(\mathbb{Z})$. Assume that*

(i) *$\Gamma\backslash Z$ is strongly quasi-projective;*
(ii) *Z is semi-algebraic in $\mathbf{D}$ (i.e. open in its Zariski closure in the compact dual $\check{\mathbf{D}}$).*

Then Z is a Hermitian symmetric domain $G(\mathbb{R})/K$, whose embedding in $\mathbf{D}$ is an equivariant, holomorphic, horizontal embedding.

Proof. See [FL13, Thm. 1.4]. □

In other words, the only case when the image of a period map can be described purely algebraically is when it is a locally symmetric domain. They are slight generalizations of the classical cases of principally polarized abelian varieties and K3 surfaces. A number of such semi-classical examples have been constructed, including n points on $\mathbb{P}^1$ with $n \leq 12$ [DM86], algebraic curves of genus 3 or 4 [Kon00], cubic surfaces [ACT02], cubic threefolds [ACT11] and some examples of Calabi-Yau varieties [Voi93], [Bor97], [Roh09], [Roh10], [GvG10] (See also [DK07]). In such situations, one can use the rich structures of arithmetic locally symmetric domains (e.g. the existence of natural compactifications, the theory of automorphic forms) to study the moduli spaces, see for example [Laz14, §2.2] and [Loo14].

Example 2.2. We briefly review the example of Kondō [Kon00] realizing the moduli space of smooth genus 3 curves as a ball quotient. A canonical genus 3 curve C is a plane quartic. To it one can associate a quartic K3 surface S by taking the cyclic μ_4-cover of $\mathbb{P}^2$ branched along C. Specifically, if $C = V(f_4)$, then $S = V(f_4(x_0, x_1, x_2) + x_3^4) \subset \mathbb{P}^3$. In this way, one gets a period map

$$\mathcal{P} : \mathcal{M}_3^{nh} \to \mathcal{F}_4 \cong \mathcal{D}/\Gamma$$

from the moduli of non-hyperelliptic genus 3 curves to the period domain of degree 4 K3 surfaces. Since the resulting K3 Hodge structures are special (they have multiplication by μ_4), the image of $\mathcal{P}$ will lie in a Mumford-Tate subdomain. In this situation, the subdomain will be a 6-dimensional complex ball $\mathcal{B}$ embedded geodesically into the 19-dimensional Type IV domain $\mathcal{D}$. In conclusion, one gets

$$\mathcal{P} : \mathcal{M}_3^{nh} \to \mathcal{B}/\Gamma' \subset \mathcal{D}/\Gamma,$$

which turns out to be birational.

2.1.2. *Quotients of Hermitian symmetric domains.* We discuss locally symmetric domains in this section. Let us start by defining some special (discrete) subgroups of an algebraic group or a Lie group. Let G be an algebraic group over $\mathbb{Q}$. For an injective homomorphism $r : G \to \mathrm{GL}_n$, we let

$$G(\mathbb{Z})_r = \{g \in G(\mathbb{Q}) \mid r(g) \in \mathrm{GL}_n(\mathbb{Z})\}.$$

Note that $G(\mathbb{Z})_r$ is independent of r up to commensurability (cf. [Bor69, Cor. 7.13]), so r can sometimes be omitted from the notation. A subgroup Γ of $G(\mathbb{Q})$ is *arithmetic* if it is commensurable with $G(\mathbb{Z})_r$ for some r. (In other words, $\Gamma\cap G(\mathbb{Z})_r$ has finite index in both Γ and $G(\mathbb{Z})_r$.) Note that every arithmetic subgroup Γ contains a torsion free subgroup of $G(\mathbb{Q})$ of finite index (cf. [Bor69, Prop. 17.4]).

As an example, let us consider

$$\Gamma(N) := r(G(\mathbb{Q})) \cap \{A \in \mathrm{GL}_n(\mathbb{Z}) \mid A \equiv I \bmod N\},$$

and define a congruence subgroup of $G(\mathbb{Q})$ to be any subgroup containing $\Gamma(N)$ as a subgroup of finite index. Although $\Gamma(N)$ depends on the choice of the embedding r, congruence subgroups do not. Every congruence subgroup is an arithmetic subgroup.

Recall that a *lattice* of a Lie group is a discrete subgroup of finite covolume with respect to an equivariant measure. Consider a connected adjoint Lie group H with no compact factors (e.g. $\mathrm{Hol}(D)^+$ for a Hermitian symmetric domain D), and let Γ be a subgroup of H. If there exists a simply connected (cf. Page 199 of [Mil11]) algebraic group G over $\mathbb{Q}$ and a surjective homomorphism $\varphi : G(\mathbb{R}) \to H$ with compact kernel such that Γ is commensurable with $\varphi(G(\mathbb{Z}))$, then we also say that Γ is an arithmetic subgroup of H. In fact, such a subgroup is a lattice of H (cf. Page 484 of [Mil13]), and so Γ is an *arithmetic lattice*.

We now discuss the quotient of a Hermitian symmetric domain D by a certain discrete subgroup Γ of $\mathrm{Hol}(D)^+$ (e.g. a lattice or an arithmetic lattice). If Γ is torsion free, then it acts freely on D, and there is a unique complex structure on $\Gamma\backslash D$ such that the natural quotient map $D \to \Gamma\backslash D$ is holomorphic. In this case, D is also the universal covering space of $\Gamma\backslash D$ with Γ the group of deck transformations; the choice of a point of D determines an isomorphism of Γ with the fundamental group of $\Gamma\backslash D$. Moreover, it is easy to see that for each $p \in \Gamma\backslash D$, there is an involution s_p defined in a neighborhood of p having p as an isolated point. (In other words, $\Gamma\backslash D$ is "locally symmetric".)

Note that a discrete group Γ of $\mathrm{Hol}(D)^+$ is a lattice (i.e. $\Gamma\backslash\mathrm{Hol}(D)^+$ has finite volume) if and only if $\Gamma\backslash D$ has finite volume.

Let H be a connected semisimple Lie group with finite center. We say that a lattice Γ in H is *irreducible* if $\Gamma \cdot N$ is dense in H for every noncompact closed normal subgroup N of H. If we further assume that H has trivial center and no compact factor, then any lattice in H decomposes into irreducible lattices as in [Mil13, Thm. 3.1]. In particular, one can decompose locally symmetric domains as follows.

Theorem 2.3. *Let D be a Hermitian symmetric domain with $H = \mathrm{Hol}(D)^+$. Let Γ be a lattice in H. Then D can be written uniquely as a product $D = D_1 \times \cdots \times D_r$ of Hermitian symmetric domains such that $\Gamma_i := \Gamma \cap \mathrm{Hol}(D_i)^+$ is an irreducible lattice in $\mathrm{Hol}(D_i)^+$ and $\Gamma_1\backslash D_1 \times \cdots \times \Gamma_r\backslash D_r$ is a finite covering of $\Gamma\backslash D$.*

Proof. See [Mil13, Thm 3.2]. □

Recall that a connected semisimple algebraic group can be written as an almost direct product of its almost simple subgroups (called almost simple factors) (cf. [Mil11, Thm. 17.16]). We say a simply connected or adjoint algebraic group G over $\mathbb{Q}$ is *of compact type* (resp. *of noncompact type*) if $G_i(\mathbb{R})$ is compact (resp. noncompact) for every almost simple factor G_i of G (see also [Mil13, Def. 3.7]).

Recall also that the rank of a semisimple algebraic group G over $\mathbb{R}$ is the dimension of a maximal split torus in G.

Theorem 2.4. *Let D be a Hermitian symmetric domain with $H = \mathrm{Hol}(D)^+$. Let Γ be a lattice in H. If $\mathrm{rank}(\mathrm{Hol}(D_i)^+) \geq 2$ in Theorem 2.3, then there exists a simply connected algebraic group G of noncompact type over $\mathbb{Q}$ and a surjective homomorphism $\varphi : G(\mathbb{R}) \to H$ with compact kernel such that Γ is commensurable with $\varphi(G(\mathbb{Z}))$. (In particular, Γ is an arithmetic lattice of H.) Moreover, such a pair (G, φ) is unique up to a unique isomorphism.*

Proof. See [Mil13, Thm. 3.13]. We also include the proof here. By Theorem 2.3 we can assume that the lattice Γ is irreducible. The existence of (G, φ) just means that Γ is arithmetic. By Margulis arithmeticity theorem (see for example [Mil13, Thm. 3.12]), the only possibility one has to rule out is that H is isogenous to $\mathrm{SU}(1, n)$ or $\mathrm{SO}(1, n)$, which can not happen because of the assumption that $\mathrm{rank}(H) \geq 2$. See Page 485 of op. cit. for the proof that G is of noncompact type.

Because Γ is irreducible, G is almost simple (cf. Theorem 3.9 of op. cit.). Let (G_1, φ_1) be a second pair. Because the kernel of φ_1 is compact, its intersection with $G_1(\mathbb{Z})$ is finite, and so there exists an arithmetic subgroup Γ_1 of $G_1(\mathbb{Q})$ such that $\varphi_1|_{\Gamma_1}$ is injective. Because $\varphi(G(\mathbb{Z}))$ and $\varphi_1(\Gamma_1)$ are both commensurable with Γ, they are commensurable, which implies that there exists an arithmetic subgroup Γ' of $G(\mathbb{Q})$ such that $\varphi(\Gamma') \subset \varphi_1(\Gamma_1)$. By Margulis superrigidity theorem (e.g. Theorem 3.10 of op. cit.), the homomorphism $\Gamma' \xrightarrow{\varphi} \varphi_1(\Gamma_1) \cong \Gamma_1$ can be lifted uniquely to a homomorphism $\alpha : G \to G_1$ such that $\varphi_1(\alpha(\gamma)) = \varphi(\gamma)$ for all γ in a subgroup $\Gamma'' \subset \Gamma'$ of finite index. By Borel density theorem (e.g. Theorem 3.8 of op. cit.), the subgroup $\Gamma' \subset G(\mathbb{Q})$ is Zariski dense in G, and so $\varphi_1 \circ \alpha(\mathbb{R}) = \varphi$. Since G and G_1 are almost simple, α is an isogeny. Because G_1 is simply connected, α is an isomorphism. It is unique because it is uniquely determined on an arithmetic subgroup of G. □

A few remarks on the algebraic structure of locally symmetric domains. Recall that there is a functor $X \mapsto X^{\mathrm{an}}$ associating to a smooth complex algebraic variety X a complex manifold X^{an}. This functor is faithful, but far from surjective both on objects and on arrows. However, if we restrict the functor to closed subvarieties of the projective spaces $\mathbb{P}^n_{\mathbb{C}}$, then it produces an equivalence of categories between smooth projective complex varieties and closed submanifolds of $(\mathbb{P}^n_{\mathbb{C}})^{\mathrm{an}}$ (Chow's theorem). By the Baily-Borel theorem, every quotient $\Gamma\backslash D$ of a Hermitian symmetric domain D by a torsion free arithmetic subgroup Γ of $\mathrm{Hol}(D)^+$ can be realized (canonically) as a Zariski open subvariety of a projective variety and hence has a canonical structure of an algebraic variety. Now by a *locally symmetric variety* we mean a smooth complex algebraic variety X such that X^{an} is isomorphic to $\Gamma\backslash D$ for a Hermitian symmetric domain D and a torsion free subgroup $\Gamma \subset \mathrm{Hol}(D)^+$ (see also Footnote 15 on Page 488 of [Mil13]).

To obtain an interesting arithmetic theory, one needs to put further restrictions on a locally symmetric variety X. When $X^{\mathrm{an}} \cong \Gamma\backslash D$ for an arithmetic subgroup Γ of $\mathrm{Hol}(D)^+$, we call X an *arithmetic locally symmetric variety*. The group Γ is usually a lattice, so by Margulis arithmeticity theorem nonarithmetic locally symmetric varieties can only occur in very few cases. For an arithmetic locally symmetric variety X with $X^{\mathrm{an}} \cong \Gamma\backslash D$, we let (G, φ) be the pair associated to $\Gamma\backslash D$ as in Theorem 2.4. If there exists a congruence subgroup Γ_0 of $G(\mathbb{Z})$ such that Γ

contains $\varphi(\Gamma_0)$ as a subgroup of finite index, then X will have very rich arithmetic structures; such arithmetic locally symmetric varieties are called *connected Shimura varieties*.

We refer the readers to [Mil04, Chap. 4, 5] for the formal definitions of connected Shimura varieties and Shimura varieties.

2.2. **Variations of Hodge structure on locally symmetric domains.** We first describe general variations of Hodge structure over locally symmetric domains following [Del79] and [Mil13, Chap. 8], and then turn to the discussion of two special types of variations of Hodge structure, namely those of abelian variety type and Calabi-Yau type. In what follows, we shall always let D be a Hermitian symmetric domain and let Γ be an torsion free arithmetic lattice of $\mathrm{Hol}(D)^+$, and use $D(\Gamma)$ to denote the arithmetic locally symmetric variety.

2.2.1. *Description of the variations of Hodge structure on* $D(\Gamma)$. According to Theorem 2.3, D decomposes uniquely into a product $D = D_1 \times \cdots \times D_r$ such that $\Gamma_i = \Gamma \cap \mathrm{Hol}(D_i)^+$ is an irreducible lattice of $\mathrm{Hol}(D_i)^+$ and the map $D(\Gamma_1) \times \cdots \times D(\Gamma_r) \to D(\Gamma)$ is a finite covering. We further assume that

$$\mathrm{rank}(\mathrm{Hol}(D_i)^+) \geq 2 \tag{2.5}$$

for each $1 \leq i \leq r$. According to Margulis arithmeticity theorem, there exists a pair (G, φ) where G is a simply connected $\mathbb{Q}$-algebraic group and $\varphi : G(\mathbb{R}) \to \mathrm{Hol}(D)^+$ is a surjective homomorphism with compact kernel such that $\varphi(G(\mathbb{Z}))$ is commensurable with Γ; moreover, such a pair is unique up to a unique isomorphism. (cf. Theorem 2.4.)

We also fix a point $o \in D$. By Theorem 1.8, there exists a unique homomorphism $u : U_1 \to \mathrm{Hol}(D)^+$ such that $u(z)$ fixes o and acts on T_oD as multiplication by z.

Let

$$G^{\mathrm{ad}}_{\mathbb{R}} = G_{\mathrm{c}} \times G_{\mathrm{nc}},$$

where $G^{\mathrm{ad}}_{\mathbb{R}}$ is the quotient of $G_{\mathbb{R}}$ by its center and G_{c} (resp. G_{nc}) is the product of the compact (resp. noncompact) simple factors of $G^{\mathrm{ad}}_{\mathbb{R}}$. The homomorphism $\varphi : G(\mathbb{R}) \to \mathrm{Hol}(D)^+$ factors through G_{nc} and defines an isomorphism of Lie groups $G_{\mathrm{nc}}(\mathbb{R})^+ \to \mathrm{Hol}(D)^+$. Now we define $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ by

$$\bar{h}(z) = (h_{\mathrm{c}}(z), h_{\mathrm{nc}}(z)) \in G_{\mathrm{c}}(\mathbb{R}) \times G_{\mathrm{nc}}(\mathbb{R}), \tag{2.6}$$

where $h_{\mathrm{c}}(z) = 1$ and $h_{\mathrm{nc}}(z) = u(z/\bar{z})$ in $G_{\mathrm{nc}}(\mathbb{R})^+ \cong \mathrm{Hol}(D)^+$. Note that $\mathbb{G}_m$ can be embedded into $\mathbb{S}$ via the exact sequence

$$0 \to \mathbb{G}_m \xrightarrow{w} \mathbb{S} \to U_1 \to 0,$$

which is defined on the real valued points by $r \overset{w}{\mapsto} r^{-1}$and $z \mapsto z/\bar{z}$ respectively. It is clear that $\bar{h}$ factors through $\mathbb{S}/\mathbb{G}_m$. Moreover, the $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugates of $\bar{h}$ can be identified with D through $g\bar{h}g^{-1} \mapsto g \cdot o$.

Proposition 2.7. *Notations as above. The pair* $(G, \bar{h})$ *associated to the arithmetic locally symmetric domain* $D(\Gamma)$ *and a point* $o \in D$ *satisfies the following properties.*

(1) *The Hodge structure on* $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})$ *defined by* $\mathbb{S} \xrightarrow{\bar{h}} G^{\mathrm{ad}}_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{GL}(\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}}))$ *is of type* $\{(1,-1),(0,0),(-1,1)\}$*;*
(2) *The conjugation by* $\bar{h}(i)$ *is a Cartan involution of* $G^{\mathrm{ad}}_{\mathbb{R}}$.

Proof. By definition, $h_{\mathrm{nc}}(z) = u(z/\bar{z})$ under the identification $G_{\mathrm{nc}}(\mathbb{R})^+ \cong \mathrm{Hol}(D)^+$. Because $G^{\mathrm{ad}}_{\mathbb{R}}$ has trivial center, $\bar{h}$ satisfies (1) and (2) if and only if u satisfies (a) and (b) of Theorem 1.20. □

Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$ and let $h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ be a homomorphism. To state the main results of this subsection, we define the weight homomorphism

$$w_h := h \circ w$$

where $w : \mathbb{G}_m \to \mathbb{S}$ is given as above by $r \mapsto r^{-1}$ (N.B. to give a Hodge structure on a $\mathbb{Q}$-vector space V amounts to giving a homomorphism $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ such that w_h is defined over $\mathbb{Q}$). We also consider the following condition on h.

(**Axiom II***) The conjugation by $h(i)$ is a Cartan involution of $\mathbf{G}_{\mathbb{R}}/w_h(\mathbb{G}_m)$.

Axiom II* can be motivated from the following fact. Let V be a faithful representation of $\mathbf{G}$, if w_h is defined over $\mathbb{Q}$, then the homomorphism $h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ defines a rational Hodge structure on V; assume that $\mathbf{G}$ is the Mumford-Tate group of V, then V is polarizable if and only if $(\mathbf{G}, h)$ satisfies Axiom II*. (cf. [Del79, 1.1.18(a)] and [Mil13, Prop. 6.4]. Roughly speaking, a Cartan involution produces a bilinear form invariant under the group action, but the Mumford-Tate group $\mathbf{G}$ only preserves a polarization up to scalar, so we consider a Cartan involution on the quotient of the Mumford-Tate group by $w_h(\mathbb{G}_m)$.)

A Hodge structure is said *of CM type* if it is polarizable and its Mumford-Tate group is a torus. Also, by a variation of integral Hodge structure we mean a variation of rational Hodge structure that admits an integral structure (i.e. the local system of $\mathbb{Q}$-vector spaces comes from a local system of free $\mathbb{Z}$-modules). The underlying local system of a variation of Hodge structure is determined by the monodromy representation. See for example [Voi03, §3.1] for the definition and properties of the monodromy representation. Finally, we denote by $\mathbf{G}^{\mathrm{der}}$ the derived subgroup (cf. Page 187 of [Mil11]) of $\mathbf{G}$.

Theorem 2.8. *Let $D(\Gamma)$ be an arithmetic locally symmetric domain satisfying (2.5). Let G be the simply connected $\mathbb{Q}$-algebraic group associated to $D(\Gamma)$ as in Theorem 2.4. Choose a point $o \in D$ and define $\bar{h}$ as in (2.6). To give*

a polarizable variation of integral Hodge structure on $D(\Gamma)$ such that some fiber is of CM type and the monodromy representation has finite kernel

is the same as giving

a triple $(\mathbf{G}, h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}, \rho : G \to \mathrm{GL}(V))$, where V is a $\mathbb{Q}$-representation of G and $\mathbf{G} \subset \mathrm{GL}(V)$ is a reductive algebraic group defined over $\mathbb{Q}$, such that

1. *The homomorphism h satisfies Axiom II* and w_h is defined over $\mathbb{Q}$;*
2. *The representation ρ factors through $\mathbf{G}$ and $\rho(G) = \mathbf{G}^{\mathrm{der}}$;*
3. *The composition $\mathrm{Ad} \circ h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \to \mathbf{G}^{\mathrm{ad}}_{\mathbb{R}} \cong G^{\mathrm{ad}}_{\mathbb{R}}$ is equal to $\bar{h}$.*

Proof. See [Mil13, Summary 8.6]. □

Remark 2.9. The reductive group $\mathbf{G}$ should be thought of as the generic Mumford-Tate group of a polarizable variation of Hodge structure on $D(\Gamma)$. Also, we need to assume the variation of Hodge structure is polarizable and integral so that $\rho(G) = \mathbf{G}^{\text{der}}$ (cf. [Mil13, Thm. 6.22]).

For every arithmetic locally symmetric variety, there exists a triple $(\mathbf{G}, h, \rho)$ satisfying the conditions in Theorem 2.8, and hence there is a polarizable variation of integral Hodge structure on the variety. See Pages $512-514$ of [Mil13] for details.

2.2.2. *Symplectic representations.* In this subsection, we show how to construct a family of abelian varieties (equivalently, polarizable variations of integral Hodge structure of length 1) on an arithmetic locally symmetric variety $D(\Gamma)$ following [Del79, §1.3] and [Mil13, Chap. 10, 11]. For simplicity, we assume that D is irreducible. Also, we assume that $\operatorname{rank}(\operatorname{Hol}(D)^+) \geq 2$ as in (2.5).

According to Theorem 2.4, there is a unique simply connected $\mathbb{Q}$-algebraic group G of non-compact type and a surjective homomorphism $\varphi : G(\mathbb{R}) \to \operatorname{Hol}(D)^+$ with compact kernel such that $\varphi(G(\mathbb{Z}))$ is commensurable with Γ. Note that φ factors through $G^{\text{ad}}_{\mathbb{R}}(\mathbb{R})$ and induces an isomorphism of Lie groups $G^{\text{ad}}_{\mathbb{R}}(\mathbb{R})^+ \to \operatorname{Hol}(D)^+$. Fix a point $o \in D$ and let $\bar{h} : \mathbb{S} \to G^{\text{ad}}_{\mathbb{R}}$ be as defined in (2.6), then $\varphi(\bar{h}(z))$ fixes $o \in D$ and acts on T_oD as multiplication by $z/\bar{z}$.

By Theorem 2.8, variations of Hodge structure on $D(\Gamma)$ corresponds to certain representations of G. We now define symplectic representations and show that they corresponds to families of abelian varieties on $D(\Gamma)$.

Let V be a rational vector space and ψ be a nondegenerate alternating form on V. Denote by $\operatorname{GSp}(V, \psi)$ the group of symplectic similitudes (the algebraic subgroup of $\operatorname{GL}(V)$ whose elements preserves ψ up to scalar). The derived subgroup of $\operatorname{GSp}(V, \psi)$ is the symplectic group $\operatorname{Sp}(V, \psi)$. Also, let $D(\psi)$ be the set of Hodge structures which are of type $\{(-1, 0), (0, -1)\}$ and are polarized by $2\pi i\psi$.

Definition 2.10. A homomorphism $G \to \operatorname{GL}(V)$ with finite kernel is a *symplectic representation* of $(G, \bar{h} : \mathbb{S} \to G^{\text{ad}}_{\mathbb{R}})$ if there exists a pair $(\mathbf{G}, h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}})$ consisting of a reductive $\mathbb{Q}$-algebraic group $\mathbf{G}$ and a homomorphism h, a nondegenerate alternating form ψ on V, and a factorization of $G \to \operatorname{GL}(V)$ through $\mathbf{G}$:

$$G \xrightarrow{\phi} \mathbf{G} \xrightarrow{\xi} \operatorname{GL}(V)$$

such that

(1) $\xi \circ h \in D(\psi)$;
(2) $\phi(G) = \mathbf{G}^{\text{der}}$ and $\xi(\mathbf{G}) \subset \operatorname{GSp}(V, \psi)$;
(3) The composition $\operatorname{Ad} \circ h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \to \mathbf{G}^{\text{ad}}_{\mathbb{R}} \cong G^{\text{ad}}_{\mathbb{R}}$ is equal to $\bar{h}$.

Recall that a family on a connected complex manifold is said to be faithful if the monodromy representation is injective.

Theorem 2.11. *Let $D(\Gamma)$ be an arithmetic locally symmetric variety with D irreducible (for simplicity only) and* $\operatorname{rank}(\operatorname{Hol}(D)^+) \geq 2$*, and let $(G, \bar{h})$ be the pair associated to $D(\Gamma)$ and a point $o \in D$ as above. There exists a faithful family of abelian varieties on $D(\Gamma)$ having a fiber of CM type if and only if $(G, \bar{h})$ admits a symplectic representation.*

Proof. (Sketch) This is [Mil13, Thm. 11.8]. Let $g : \mathcal{A} \to D(\Gamma)$ be a faithful family of abelian varieties. By Theorem 2.8, the polarizable variation of integral Hodge structure $R^1g_*\mathbb{Q}$ produces a certain triple $(\mathbf{G}, h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}, \rho : G \to \operatorname{GL}(V))$ with $\mathbf{G}$ its generic Mumford-Tate group. Because the family is faithful, ρ has finite kernel.

Also, by [Mil13, Lemma 10.15] there exists an alternating form ψ on V such that ρ induces a homomorphism $G \to \mathbf{G} \hookrightarrow \mathrm{GSp}(V, \psi)$. Then one can easily check that ρ is a symplectic representation of $(G, \bar{h})$.

Conversely, given a symplectic representation $\rho : G \to \mathrm{GL}(V)$ of $(G, \bar{h})$ as in Definition 2.10, using the notations there, the triple $(\xi(\mathbf{G}), \xi_{\mathbb{R}} \circ h, \xi \circ \phi)$ satisfies the conditions (1) (cf. also [Mil13, Prop. 6.4]), (2) and (3) in Theorem 2.8. In this way, we get a polarizable variation of integral Hodge structure of type $\{(-1, 0), (0, -1)\}$ which must come from a family of abelian varieties (see for example [Moo99, Thm. 2.2]). $\square$

In the rest of this subsection we study symplectic representations. Because D is irreducible, there is no harm to study symplectic representations over $\mathbb{R}$. Let $\rho : G \to \mathrm{GL}(V)$ be a symplectic representation of $(G, \bar{h})$. After scalar extension to $\mathbb{R}$, we assume that G is an almost simple and simply connected $\mathbb{R}$-algebraic group without compact factors and view V as a real representation of G. If V is irreducible, then $\mathrm{End}_G(V)$ is a division algebra over $\mathbb{R}$ (Shur's Lemma), and so there are three possibilities:

$$\mathrm{End}_G(V) = \begin{cases} \mathbb{R} & \text{(real type)}, \\ \mathbb{C} & \text{(complex type)}, \\ \mathbb{H} & \text{(quaternionic type)}. \end{cases}$$

Accordingly, we have ($V_{\mathbb{C}} := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$)

$$V_{\mathbb{C}} = \begin{cases} V_+ & \text{(real type)}, \\ V_+ \oplus V_-, V_+ \ncong V_- & \text{(complex type)}, \\ V_+ \oplus V_-, V_+ \cong V_- & \text{(quaternionic type)}, \end{cases}$$

where $V_\pm$ are irreducible complex $G(\mathbb{C})$-representations and $V_+^\vee \cong V_-$. In practice, one can use [GGK12, Theorem (IV.E.4)] to distinguish these cases.

We now classify the irreducible real symplectic representations of the pairs $(G, \bar{h})$. Define $\bar{\mu} : \mathbb{G}_m \to G_{\mathbb{C}}^{\mathrm{ad}}$ by $\bar{\mu}(z) = \bar{h}_{\mathbb{C}}(z, 1)$, where $\bar{h}_{\mathbb{C}} : \mathbb{G}_m \times \mathbb{G}_m \to G_{\mathbb{C}}^{\mathrm{ad}}$ is the complexification of $\bar{h}$. Let $u : U_1 \to \mathrm{Hol}(D)^+ \cong G^{\mathrm{ad}}(\mathbb{R})^+$ be the homomorphism associated to the point $o \in D$ as in Theorem 1.8. Because $\bar{h}_{\mathbb{C}}(z_1, z_2) = u_{\mathbb{C}}(z_1/z_2)$ as in (2.6), the homomorphism $\bar{\mu}$ is the scalar extension of u: $\bar{\mu}(z) = \bar{h}_{\mathbb{C}}(z, 1) = u_{\mathbb{C}}(z)$.

Fix a maximal torus T of $G_{\mathbb{C}}^{\mathrm{ad}}$, and let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ (resp. $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$) be the character (resp. cocharacter) group. There is a natural pairing $\langle -, - \rangle : X^*(T) \times X_*(T) \to \mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}$ between $X^*(T)$ and $X_*(T)$. Let $R \subset X^*(T)$ (resp. $R^\vee \subset X_*(T)$) be the corresponding root system (resp. coroot system). We also denote by $Q(R)$ the lattice generated by R. (In this case $Q(R) = X^*(T)$, but we will not use this.)

Recall that the lattice of weights is $P(R) = \{\varpi \in X^*(T)_{\mathbb{Q}} \mid \langle \varpi, \alpha^\vee \rangle \in \mathbb{Z} \text{ all } \alpha^\vee \in R^\vee\}$. Choose a set $B = \{\alpha_1, \cdots, \alpha_n\}$ of simple roots such that $\langle \alpha, \bar{\mu} \rangle \geq 0$ for all $\alpha \in B$, then the fundamental weights are the dual basis $\{\varpi_1 \cdots, \varpi_n\}$ of $\{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$, and the dominant weights are the elements $\sum n_i \varpi_i$ with $n_i \in \mathbb{N}$.

Also, there is a unique permutation τ of simple roots (or the corresponding Dynkin diagram or the fundamental weights) such that $\tau^2 = \mathrm{Id}$ and the map $\alpha \mapsto -\tau(\alpha)$ extends to the action of the Weyl group. Usually τ is called the opposition involution. Explicitly, τ acts nontrivially on the root systems of type

A_n ($\alpha_i \leftrightarrow \alpha_{n+1-i}$), D_n with n odd ($\alpha_{n-1} \leftrightarrow \alpha_n$) and E_6 ($\alpha_1 \leftrightarrow \alpha_6$), and trivially on the other root systems.

Theorem 2.12. *Notations as above. Let V be an irreducible real representation of G, and ϖ be the highest weight of an irreducible summand W (e.g. V_+ or V_-) of $V_{\mathbb{C}}$. The representation V is a symplectic representation of $(G, \bar{h})$ if and only if*

$$\langle \varpi + \tau(\varpi), \bar{\mu} \rangle = 1. \tag{2.13}$$

Proof. (Sketch) (Step 1) By [Del79, Lemma 1.3.3] or [Mil13, Prop. 10.4], a representation $\rho : G \to \mathrm{GL}(V)$ is a symplectic representation if there exist a pair $(\mathbf{G}, h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}})$ and a factorization $\rho = \xi \circ \phi$ of ρ as in Definition 2.10, such that (1) $\xi \circ h$ is of type $\{(-1, 0), (0, -1)\}$; (2) $\phi(G) = \mathbf{G}^{\mathrm{der}}$; and (3) $\mathrm{Ad} \circ h = \bar{h}$. In other words, the nondegenerate alternating form ψ is not needed in the first place.

(Step 2) Consider the projective system $(T_n, T_{nd} \to T_n)$, where the index set is $\mathbb{N} - \{0\}$ (ordered by divisibility), $T_n = \mathbb{G}_m$, and $T_{nd} \to T_n$ is given by $z \mapsto z^d$. Denote by $\tilde{\mathbb{G}}_m$ its inverse limit.

By [Del79, 1.3.4], $\tilde{\mathbb{G}}_m$ is the algebraic universal covering of $\mathbb{G}_m$, so we can lift $\bar{\mu} : \mathbb{G}_m \to G^{\mathrm{ad}}_{\mathbb{C}}$ to $\tilde{\mu} : \tilde{\mathbb{G}}_m \to G_{\mathbb{C}}$. Then $W \subset V_{\mathbb{C}}$ is a representation of $\tilde{\mathbb{G}}_m$. According to [Mil13, §10.2], such a representation $\tilde{\mathbb{G}}_m \to \mathrm{GL}(W)$ can be represented by a homomorphism $f : T_n \to \mathrm{GL}(W)$ and defines a gradation $W = \oplus W_r$ ($r \in (1/n)\mathbb{Z}$) with $f(z)$ acting on W_r by multiplication by z^{nr}. We call the r for which $W_r \neq 0$ the weights of the representation of $\tilde{\mathbb{G}}_m$ on W. One can check that the weights do not depend on the representative f.

The most important observation here is as follows: the nontrivial irreducible representation W occurs in a symplectic representation if and only if $\tilde{\mu}$ has exactly two weights a and $a+1$ on W (cf. [Del79, Lemma 1.3.5]).

We show the "only if" direction here. For $h : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$, we define $\mu_h : \mathbb{G}_m \to \mathbf{G}_{\mathbb{C}}$ by $\mu_h(z) = h_{\mathbb{C}}(z, 1)$ as in [Del79, 1.1.1, 1.1.11]. Because $\mathrm{Ad} \circ h = \bar{h}$ as in Definition 2.10 (3), we have $\phi_{\mathbb{C}} \circ \tilde{\mu} = \mu_h \cdot \nu$ with ν in the center of $\mathbf{G}_{\mathbb{C}}$. On W, μ_h has weights 0 and 1 (see Definition 2.10 (1)). If a is the unique weight of ν on W, then the only weights of $\tilde{\mu}$ on W is a and $a+1$. We need the observation in Step 1 for the other direction, see [Mil13, Lemma 10.6].

(Step 3) Note that the differential of $\tilde{\mu}$ equals the differential of $\bar{\mu}$. The conclusion in Step 2 can be rephrased as follows: if ϖ is the highest weight of W, then the representation W occurs in a symplectic representation if and only if $\langle \varpi + \tau(\varpi), \bar{\mu} \rangle = 1$. This is [Del79, (1.3.6.1)]. In fact, the lowest weight of W is $-\tau(\varpi)$, and the weights β of W are of the form ϖ + (a $\mathbb{Z}$-linear combination of roots $\alpha \in R$). Because $\langle \alpha, \bar{\mu} \rangle \in \mathbb{Z}$ for all roots α, $\langle \beta, \bar{\mu} \rangle$ takes values a and $a+1$ if and only if $\langle -\tau(\varpi), \bar{\mu} \rangle = \langle \varpi, \bar{\mu} \rangle - 1$, which is clearly equivalent to (2.13). $\square$

To apply (2.13), we make the following two observations. Because $\varpi + \tau(\varpi) \in Q(R)$, $\langle \varpi + \tau(\varpi), \bar{\mu} \rangle \in \mathbb{Z}$ for every dominant weight ϖ. Moreover, $\langle \varpi + \tau(\varpi), \bar{\mu} \rangle > 0$. So only the fundamental weights $\{\varpi_1, \cdots, \varpi_n\}$ can satisfy (2.13) ([Del79, Lemma 1.3.7]).

Also, by the proof of Theorem 1.27 there exists a special node α_s (determined by the irreducible Hermitian symmetric domain D) such that, for simple roots $\alpha \in B = \{\alpha_1, \cdots, \alpha_n\}$,

$$\langle \alpha, \bar{\mu} \rangle = \begin{cases} 1 & \text{if } \alpha = \alpha_s, \\ 0 & \text{if } \alpha \neq \alpha_s. \end{cases}$$

Express a weight ϖ as a $\mathbb{Q}$-linear combination of the simple roots $\{\alpha_i\}$ (cf. [Bou02]), then $\langle \varpi + \tau(\varpi), \bar{\mu}\rangle = 1$ if and only if the coefficient of α_s in $\varpi + \tau(\varpi)$ equals 1.

Example 2.14. (Type A_{n-1}) In this case,

$$\varpi_i = \frac{n-i}{n}\alpha_1 + \frac{2(n-i)}{n}\alpha_2 + \cdots + \frac{i(n-i)}{n}\alpha_i + \frac{i(n-i-1)}{n}\alpha_{i+1} + \cdots + \frac{i}{n}\alpha_{n-1},$$

for $1 \le i \le n-1$. The opposition involution τ switches the nodes i and $n-i$: $\tau(\varpi_i) = \varpi_{n-i}$, and so

$$\tau(\varpi_i) = \frac{i}{n}\alpha_1 + \frac{2i}{n}\alpha_2 + \cdots + \frac{(n-i)i}{n}\alpha_{n-i} + \frac{(n-i)(i-1)}{n}\alpha_{n+1-i} + \cdots + \frac{n-i}{n}\alpha_{n-1}.$$

If $i \le n-i$, one can easily compute the coefficient of a simple root α_j in $\varpi_i + \tau(\varpi_i)$:

$$\text{the coefficient of } \alpha_j \text{ in } \varpi_i + \tau(\varpi_i) = \begin{cases} j & \text{if } 1 \le j \le i, \\ i & \text{if } i \le j \le n-i, \\ n-j & \text{if } n-i \le j \le n-1. \end{cases}$$

The special root α_s can be any α_j with $1 \le j \le n-1$. We drop the cases that $\alpha_s = \alpha_1$ and $\alpha_s = \alpha_{n-1}$ so that the assumption (2.5) holds. Choose a special root $\alpha_s = \alpha_j$ for $2 \le j \le n-2$. It is easy to see that for the coefficient to be 1, ϖ_i must be ϖ_1. Similarly, if $i > n-i$, only the fundamental weight ϖ_{n-1} satisfies (2.13).

Example 2.15. (Type E_6 and E_7) In the E_6 case, the special root $\alpha_s = \alpha_1$ or α_6, and the opposite involution switches α_1 and α_6. We seek a fundamental weight ϖ such that $\varpi = a\alpha_1 + \cdots + b\alpha_6$ with $a + b = 1$. But there is no such a fundamental weight for the root system E_6, and hence there is no corresponding symplectic representation.

Similarly, there is no symplectic representation associated to the Hermitian symmetric domains of type E_7. In fact, $\alpha_s = \alpha_7$ and the opposite involution is trivial in the E_7 case. Therefore, a fundamental weight ϖ satisfies (2.13) if and only if $\varpi = \cdots + \frac{1}{2}\alpha_7$, but no fundamental of E_7 is of this form.

If a fundamental weight satisfies (2.13), then we call the corresponding node a *symplectic node*. They are listed as follows.

Type	Symplectic node
(A_n, α_1)	$\varpi_1, \cdots, \varpi_n$
(A_n, α_i), $1 < i < n$	ϖ_1, ϖ_n
(B_n, α_1), $n \ge 2$	ϖ_n
(C_n, α_n)	ϖ_1
(D_n, α_1), $n \ge 4$	ϖ_{n-1}, ϖ_n
(D_4, α_4)	ϖ_1, ϖ_3
(D_n, α_n), $n \ge 5$	ϖ_1
(E_6, α_1)	none
(E_7, α_7)	none

TABLE 2. List of symplectic nodes.

Remark 2.16. As discussed in [Mil13, Chap. 9], one needs to take motives and Hodge classes (v.s. algebraic varieties and algebraic classes) into consideration in order to realize all but a small number of Shimura varieties as moduli varieties.

Let us also mention the celebrated theorem of Deligne saying that Hodge classes are the same as absolute Hodge classes for abelian varieties. The readers can find further discussions in [Del82] and [CS11].

Example 2.17. Consider the special root α_n of the root system A_{2n-1} with $n \geq 2$. By Theorem 1.27, it corresponds to an irreducible Hermitian symmetric domain D.

Let $K = \mathbb{Q}(\sqrt{-d})$ ($d \in \mathbb{Z}^+$) be an imaginary quadratic field extension of $\mathbb{Q}$. Denote by V a K-vector space of dimension $2n$, and set $H : V \times V \to K$ to be a K-Hermitian form on V whose signature is (n, n). There exists a K-basis on V, on which H is given by

$$H(z, w) = az_1\bar{w}_1 + \cdots + z_n\bar{w}_n - z_{n+1}\bar{w}_{n+1} - \cdots - z_{2n}\bar{w}_{2n},$$

where $a \in \mathbb{Q}^+$. Let us assume $a = 1$ for simplicity. With respect to such a basis, we define a $\mathbb{Q}$-algebraic group G by

$$G(R) = \left\{ A \in \mathrm{GL}_{2n}(K \otimes_{\mathbb{Q}} R) \,\middle|\, \overline{A}^t \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} A = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \right\},$$

where R is a $\mathbb{Q}$-algebra (the matrix $\overline{A}$ is obtained by taking the conjugate of every entry in A by $\overline{k \otimes r} := \bar{k} \otimes r$). The algebraic group G is simply connected (cf. Page 232 of [Mil11]), and $G(\mathbb{R})$ is isomorphic to the special unitary group $\mathrm{SU}(n, n)$.

Since $\mathrm{Hol}(D)^+ \cong \mathrm{PSU}(n, n)$, there exists a natural homomorphism $\varphi : G(\mathbb{R}) \twoheadrightarrow \mathrm{Hol}(D)^+$. We choose an irreducible lattice Γ in $\mathrm{PSU}(n, n) \cong \mathrm{Hol}(D)^+$ which is commensurable with $\varphi(G(\mathbb{Q}) \cap \mathrm{GL}_{2n}(\mathbb{Z}))$. The pair (G, φ) then satisfies the conditions in Theorem 2.4 for the arithmetic locally symmetric variety $D(\Gamma)$.

Consider the natural representation $\rho : G \to \mathrm{GL}(V)$. Let $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$, then $\rho_{\mathbb{R}} : G_{\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ is isomorphic to the standard representation of $\mathrm{SU}(n, n)$. As representations of $G(\mathbb{C}) \cong \mathrm{SL}_{2n}(\mathbb{C})$, we have $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \cong V_+ \oplus V_-$ where V_+ has highest weight ϖ_1 and $V_- \cong V_+^\vee$ has highest weight $\tau(\varpi_1) = \varpi_{2n-1}$. In this case, both ϖ_1 and ϖ_{2n-1} correspond to symplectic nodes, and hence ρ is a symplectic representation. So by Theorem 2.11 we obtain a family of abelian varieties on $D(\Gamma)$ (or on D by pulling back via $D \to D(\Gamma)$).

This is a family of abelian varieties of Weil type over D. An abelian variety of Weil type consists of an abelian variety X of dimension $2n$ and an imaginary quadratic field extension $K \hookrightarrow \mathrm{End}(X) \otimes \mathbb{Q}$, such that for all $k \in K$ the action of k on T_0X has n eigenvalues $\sigma(k)$ and n eigenvalues $\overline{\sigma(k)}$ (here we fix an embedding $\sigma : K \subset \mathbb{C}$). We refer the readers to [vG94, §5.3−§5.12] for the explicit constructions of families of abelian varieties of Weil type over D.

2.2.3. *Hermitian variations of Hodge structure of Calabi-Yau type.* In this subsection, we consider Hodge structures of Calabi-Yau type.

Definition 2.18. A rational (resp. real) Hodge structure V of Calabi-Yau (CY) type is an effective rational (resp. real) Hodge structure[1] of weight n such that $V^{n,0}$ is 1-dimensional.

Based on earlier work of Gross ([Gro94]) and Sheng-Zuo ([SZ10]), Friedman and Laza classified $\mathbb{R}$-variations of Hodge structures of CY type over irreducible Hermitian symmetric domains in [FL13] and [FL14]. In this subsection, we only discuss Friedman-Laza's classification for irreducible Hermitian symmetric domains

[1]In this subsection, we switch to the other convention: if $\varphi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ defines a Hodge structure of weight n, then $h(z)$ ($z \in \mathbb{S}(\mathbb{R})$) acts on $V^{p,q}$ as multiplication by $z^p\bar{z}^q$.

of tube type. All the irreducible tube domains are tabulated as follows (the first column is standard Siegel's notation; the second column lists the corresponding Dynkin diagrams and special roots (cf. Theorem 1.27); the third column gives the real simply connected algebraic groups for the unique simple adjoint algebraic groups associated to Hermitian symmetric domains (cf. Proposition 1.7); the fourth column lists the corresponding maximal compact subgroup; the last column gives the real ranks of tube domains).

Label	(R, α_s)	$G(\mathbb{R})$	K	$\mathbb{R}$-rank
$\mathrm{I}_{n,n}$	(A_{2n-1}, α_n)	$\mathrm{SU}(n,n)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))$	n
II_{2n}	(D_{2n}, α_{2n})	$\mathrm{Spin}^*(4n)$	$U_1 \times_{\mu_n} \mathrm{SU}(2n)$	n
III_n	(C_n, α_n)	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{U}(n)$	n
IV_{2n-1}	(B_n, α_1)	$\mathrm{Spin}(2, 2n-1)$	$\mathrm{Spin}(2)\times_{\mu_2}\mathrm{Spin}(2n-1)$	2
IV_{2n-2}	(D_n, α_1)	$\mathrm{Spin}(2, 2n-2)$	$\mathrm{Spin}(2)\times_{\mu_2}\mathrm{Spin}(2n-2)$	2
EVII	(E_7, α_7)	$\mathrm{E}_{7,3}$	$\mathrm{U}(1)\times_{\mu_3}\mathrm{E}_6$	3

TABLE 3. Hermitian symmetric domains of tube type

Let $\mathcal{D}$ be an irreducible Hermitian symmetric domain, and let $\mathbf{D}$ be a classifying space of polarized rational Hodge structures with fixed Hodge numbers. Following Definition 2.1 of [FL13], we call the variations of Hodge structure induced by an equivariant, holomorphic and horizontal embedding of $\mathcal{D} \hookrightarrow \mathbf{D}$ a *Hermitian variation of Hodge structure*. They are the variations of Hodge structure parameterized by Hermitian symmetric domains considered by Deligne [Del79]. In the terminology of [GGK12], $\mathcal{D} \subset \mathbf{D}$ is an unconstrained Mumford-Tate domain (and hence also a Hermitian symmetric domain).

By Proposition 1.7, there is a unique simple adjoint real algebraic group associated to $\mathcal{D}$; we denote by G its algebraic universal covering (N.B. these simply connected algebraic groups are listed in Table 3, see also [Gro94, §1]). Fix a reference point $o \in \mathcal{D}$. According to Theorem 1.8, there is a homomorphism $u : U_1 \to G^{\mathrm{ad}}$. We define $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}$ by $\bar{h}(z) = u(z/\bar{z})$ as in (2.6).

Choosing a suitable arithmetic subgroup of $\mathrm{Hol}(\mathcal{D})^+$, we assume that there is an algebraic group $G_{\mathbb{Q}}$ is defined over $\mathbb{Q}$ with $G_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \cong G$. To give a Hermitian rational variation of Hodge structure over $\mathcal{D}$, one must give a $\mathbb{Q}$-representation $\rho_{\mathbb{Q}} : G_{\mathbb{Q}} \to \mathrm{GL}(V)$ satisfying the conditions in Theorem 2.8. Following [FL13, §2.1], we assume that the induced real representation $\rho : G \to \mathrm{GL}(V_{\mathbb{R}})$ is irreducible. As variations of real Hodge structure are mainly concerned in this section, we shall focus on the representation ρ.

The question is which irreducible representations of G correspond to Hermitian variations of Hodge structure of CY type over D. As in Theorem 1.27, the Hermitian symmetric domain D determines a root system R together with a special root α_i. We call the corresponding fundamental weight ϖ_i (i.e. $\varpi_i(\alpha_j^\vee) = \delta_{ij}$) a *cominuscule weight*, and call the irreducible representation V_{ϖ_i} of $G(\mathbb{C})$ with highest weight ϖ_i a *cominuscule representation*.

Let $V_{\mathbb{R}}$ be an irreducible G-representation. Recall that $V_{\mathbb{R}}$ may be of real type, complex type or quaternionic type. Specifically, $V_{\mathbb{C}}$ may be irreducible (real type) or reducible (complex type or quaternionic type); if $V_{\mathbb{C}}$ is reducible, then we can write $V_{\mathbb{C}} = V_+ \oplus V_-$, where V_+ and V_- are irreducible representations of $G(\mathbb{C})$ and $V_+^\vee \cong V_-$. We distinguish the complex case from the quaternionic case depending on whether $V_+ \cong V_-$ (quaternionic type) or not (complex type). We now show that if $V_\mathbb{R}$ induces a CY Hermitian variation of Hodge structure over D, then the highest weight of V_+ or V_- ($V_+ = V_\mathbb{C}$ in the real case) must be a multiple of the corresponding cominuscule weight.

In what follows, let us focus on tube domains D.

Lemma 2.19. *Let D be an irreducible Hermitian symmetric domain of tube type, and let G be defined as above. Suppose D corresponds to (R, α_i) (so ϖ_i is the corresponding cominuscule weight), and let $V_{n\varpi_i}$ be the irreducible representation of $G(\mathbb{C})$ with highest weight $n\varpi_i$ ($n \in \mathbb{N}^+$). Then there exists a real G-representation $V_\mathbb{R}$ such that $V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = V_{n\varpi_i}$.*

Proof. The condition that D is of tube type is equivalent to that $\tau(\alpha_i) = \alpha_i$ where τ is the opposition involution. Let V_{ϖ_i} be a cominuscule representation. Because the dual representation has highest weight $\tau(\varpi_i)$, we have $V_{\varpi_i} \cong V_{\varpi_i}^\vee$. Now one can verify the reality of the representation using [GGK12, Thm. (IV.E.4)]. The same argument works for $V_{n\varpi_i}$. □

Example 2.20. Let D be an tube domain corresponding to (A_5, α_3). Then $G = \mathrm{SU}(3,3)$ and the cominuscule weight is ϖ_3. Because $\tau(\varpi_3) = \varpi_3$, the cominuscule representation can not be of complex type. We now determine whether it is of real type or quaternionic type using [GGK12, Thm. (IV.E.4)].

In the root system A_5, we have

$$2\varpi_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5.$$

(Denote the coefficients of α_i by m_i.) The only noncompact root in this case is α_3 (cf. Page 335 of [Kna02]). Because

$$\sum_{\alpha_i \text{ compact}} m_i = 1 + 2 + 2 + 1 = 6$$

is even, the cominuscule representation is of real type.

In the proof of Theorem 1.27 we see that there is a $\mu \in X_*(G_\mathbb{C}^{\mathrm{ad}})$ ($\mu = u_\mathbb{C}$) such that

$$\langle \alpha, \mu \rangle = \begin{cases} 1 & \text{if } \alpha = \alpha_i, \\ 0 & \text{if } \alpha \neq \alpha_i, \end{cases}$$

where α_i is the special root associated to the domain D. Following [FL13], we shall use H_0 to denote μ, and use $\varpi(H_0)$ to denote the pairing $\langle \varpi, H_0 \rangle$.

Proposition 2.21. *Notations as above. Let $\rho : G \to \mathrm{GL}(V_\mathbb{R})$ be an irreducible representation and λ be the highest weight of an irreducible factor V_+ of $V_\mathbb{C}$. Possibly replace V_+ with V_-, we can assume that $\tau\lambda(H_0) \leq \lambda(H_0)$. Then a necessary condition for ρ to arise from a CY Hermitian variation of Hodge structure over D is*

$$\varpi(H_0) < \lambda(H_0) \text{ for all weights } \varpi \neq \lambda \text{ of } V_+.$$

Furthermore, this condition implies that λ is a multiple of the fundamental cominuscule weight ϖ_i associated to the domain D. In particular, if D is a tube domain, then such representations $V_\mathbb{R}$ are of real type.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} U_1 \arrow[r, "i"] \arrow[d, "2:1"'] & \mathbb{S} \arrow[r, "h"] \arrow[d, "p"'] & \mathbf{G}_{\mathbb{R}} \arrow[r] \arrow[d, "\mathrm{Ad}"'] & \mathrm{GL}(V_{\mathbb{R}}) \\ U_1 \arrow[r, equal] & U_1 \arrow[r, "u"] & G^{\mathrm{ad}} & \end{tikzcd}$$

where $i : U_1 \hookrightarrow \mathbb{S}$ is the kernel of the norm map $\mathrm{Nm} : \mathbb{S} \to \mathbb{G}_m$ ($\mathrm{Nm}(z) = z\bar{z}$), and $p : \mathbb{S} \to U_1$ is defined by $z \mapsto z/\bar{z}$. (Note also that $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}} \cong G^{\mathrm{ad}}$.) In the situation considered here, the Hodge decomposition on $V_{\mathbb{C}}$ is the weight decomposition of $V_{\mathbb{C}}$ with respective to $U_1(\mathbb{C}) \cong \mathbb{C}^*$ via $h_{\mathbb{C}} \circ i_{\mathbb{C}}$: $V^{p,q}$ corresponds to the eigenspace for the character z^{p-q}. If $V_{\mathbb{R}}$ is of real type, then by the above diagram the weights of $\mathbb{G}_m$ on $V_{\mathbb{C}}$ via $h_{\mathbb{C}} \circ i_{\mathbb{C}}$ are $\{2\varpi(H_0) \mid \varpi \in \mathfrak{X}(V_+)\}$, where $\mathfrak{X}(V_+)$ denotes the weights of the irreducible $G(\mathbb{C})$-representation V_+ (note that $V_+ = V_{\mathbb{C}}$ in this case). If $V_{\mathbb{R}}$ is of complex or quaternionic case, then $V_{\mathbb{C}} = V_+ \oplus V_-$, and the weights of $h \circ i$ on $V_{\mathbb{C}}$ are $\{\pm 2(\varpi(H_0) - c) \mid \varpi \in \mathfrak{X}(V_+)\}$, where the constant c comes from the action of the center of $\mathbf{G}_{\mathbb{R}}$ on V_+ (cf. [FL13, §2.1.2]).

Since all the other weights of V_+ are obtained from λ by subtracting positive roots, it follows that

$$\max_{\varpi \in \mathfrak{X}(V_+)} \varpi(H_0) = \lambda(H_0).$$

Using the description of the weights of $h_{\mathbb{C}} \circ i_{\mathbb{C}}$ on $V_{\mathbb{C}}$, we see that the CY condition ($\dim_{\mathbb{C}} V^{n,0} = 1$) implies that the above maximal is attained only for the highest weight λ. In other words, for other weights $\varpi \neq \lambda$ of V_+, $\varpi(H_0) < \lambda(H_0)$.

Let α_i be the special root associated to D. By applying the reflection in another simple root $\alpha_j \neq \alpha_i$, we get

$$s_{\alpha_j}(\lambda)(H_0) = (\lambda - \lambda(\alpha_j^\vee) \cdot \alpha_j)(H_0) = \lambda(H_0) - \lambda(\alpha_j^\vee) \cdot \alpha_j(H_0) = \lambda(H_0).$$

Because $s_{\alpha_j}(\lambda) \in \mathfrak{X}(V_+)$, $s_{\alpha_j}(\lambda) = \lambda$, which is equivalent to $\lambda(\alpha_j^\vee) = 0$. Now we can conclude that $\lambda = n\varpi_i$. The last assertion follows from the previous lemma. $\square$

Let D be an irreducible Hermitian symmetric domain of tube type, and let α_i (resp. ϖ_i) be the corresponding special root (resp. cominuscule weight). As shown in [Gro94] and [FL13], if $V_{\mathbb{R}}$ is an irreducible representation of G such that $V_{\mathbb{C}}$ has highest weight $n\varpi_i$ ($n \in \mathbb{N}^+$) as a $G(\mathbb{C})$-representation, then $V_{\mathbb{R}}$ induces a CY $\mathbb{R}$-variation of Hodge structures over D. If $V_{\mathbb{C}}$ is the cominuscule representation (i.e. the highest weight is ϖ_i), then we call the induced CY $\mathbb{R}$-variation of Hodge structure the *canonical $\mathbb{R}$-variation of Hodge structure of CY type* over D.

Theorem 2.22. *For every irreducible Hermitian symmetric domain D of tube type, there exists a canonical $\mathbb{R}$-variation of Hodge structure $\mathcal{V}$ of CY type parameterized by D. Any other irreducible CY $\mathbb{R}$-variation of Hodge structure can be obtained from the canonical one by taking the unique irreducible factor of* $\mathrm{Sym}^\bullet \mathcal{V}$ *of CY type.*

Proof. See [Gro94, §3] and [FL13, Thm. 2.22]. $\square$

Remark 2.23. For an irreducible tube domain D, the weight of the canonical $\mathbb{R}$-variation of Hodge structure is also equal to the real rank of D which can be found in Table 3.

Remark 2.24. One may wonder what happens if the irreducible domain D is not of tube type. Let V_+ be a cominuscule representation of $G(\mathbb{C})$. Sheng and Zuo [SZ10] have noted V_+ carries a $\mathbb{C}$-Hodge structure of CY type, and so $V_+ \oplus V_+^\vee$ will carry a $\mathbb{R}$-Hodge structure. However, this Hodge structure is typically not of CY type. To fix this, one needs to apply the operation "half twist" defined by van Geemen [vG01]. See [FL13, §2.1.3] for details.

3. Compactifications of locally symmetric varieties and their Hodge theoretic meanings

Let $\Gamma\backslash D$ be a quotient of a Hermitian symmetric domain D by an arithmetic subgroup Γ of $\mathrm{Hol}(D)^+$. In this section, we shall review the Baily-Borel compactification and toroidal compactification of $\Gamma\backslash D$ and discuss their Hodge theoretic meanings. The emphasis will be on examples, especially when D is a classical period domain.

3.1. **The Baily-Borel compactifications.**

The Baily-Borel compactification $(\Gamma\backslash D)^*$ of $\Gamma\backslash D$ is defined by

$$(\Gamma\backslash D)^* := \mathrm{Proj}(\bigoplus_{n\geq 0} A_n),$$

where A_n is the vector space of automorphic forms on D for the n-th power of the canonical automorphy factor. The graded $\mathbb{C}$-algebra $A = \bigoplus_{n\geq 0} A_n$ is finitely generated and hence we are allowed to take the associated projective scheme. The canonical map

$$\Gamma\backslash D \to (\Gamma\backslash D)^*$$

realizes $\Gamma\backslash D$ as a Zariski-open subvariety of the normal projective algebraic variety $(\Gamma\backslash D)^*$. The readers can find the details in [BB66].

We briefly recall the (set-theoretical) construction of the Baily-Borel compactifications for irreducible locally symmetric domains. Suppose D is irreducible and write $D = H/K$, where $H = \mathrm{Hol}(D)^+$ and K is a maximal compact subgroup of H. Then the boundary components (a.k.a. *cusps*) of $(\Gamma\backslash D)^*$ are determined by rational maximal parabolic subgroups of H, which can usually be described in combinatorial terms. More specifically, let F_P be a boundary component of the closure of D in the Harish-Chandra embedding (it is itself a Hermitian symmetric space associated with a certain parabolic subgroup P of H), then the normalizer $\mathcal{N}(F_P)$, defined by $\mathcal{N}(F_P) := \{g \in H \mid g(F_P) = F_P\}$, is a maximal parabolic subgroup of H. We say such a boundary component F_P is *rational* if its normalizer $\mathcal{N}(F_P)$ can be defined over $\mathbb{Q}$. The arithmetic subgroup Γ preserves the rational boundary components, and so we obtain the Baily-Borel compactification by taking the quotient of $D \bigsqcup \partial D$, where ∂D is the disjoint union of all rational boundary components, by the action of Γ. Moreover, every boundary component of $(\Gamma\backslash D)^*$ has a structure of a locally symmetric variety of lower dimension.

Here are some properties of the Baily-Borel compactification.

(1) The Baily-Borel compactification is canonical. In other words, the construction does not depend on any choice.
(2) The boundary components usually have high codimension (≥ 2 except for some low dimensional locally symmetric domains).

(3) The Baily-Borel compactification is minimal in the following sense. Let S be a smooth variety and $\overline{S}$ a smooth simple normal crossing (partial) compactification of S. Then any locally liftable map $S \to \Gamma\backslash D$ extends to a regular map $\overline{S} \to (\Gamma\backslash D)^*$ (see [Bor72]).

3.1.1. *The Satake-Baily-Borel compactification of $\mathcal{A}_g$.* Fix a free $\mathbb{Z}$-module $V_{\mathbb{Z}}$ of rank $2g$, and a nondegenerate skew-symmetric bilinear form Q on $V_{\mathbb{Z}}$. We let D be the classifying space of polarized weight 1 Hodge structures on $V_{\mathbb{Z}}$. To be specific, we have

$$D = \{F \in \mathrm{Gr}(g, V_{\mathbb{C}}) \mid Q(F,F) = 0,\ iQ(F,\overline{F}) > 0\}.$$

This is a Hermitian symmetric domain of type III and we have that

$$D \cong \mathrm{Sp}(2g,\mathbb{R})/\mathrm{U}(g).$$

Taking Q to be the standard symplectic form, the domain D can be identified with the Siegel upper half-space $\mathfrak{H}_g$, the space of symmetric $g \times g$ complex matrices with positive definite imaginary part. The group $\mathrm{Sp}(2g,\mathbb{Z})$ acts on $D \cong \mathfrak{H}_g$ via the fractional linear transformations, and we set

$$A_g := \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathfrak{H}_g.$$

The rational boundary component F_{W_0} corresponds to the choice of a totally isotropic subspace $W_0 \subset V_{\mathbb{Q}}$ ($V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$). Since the group $\mathrm{Sp}(2g,\mathbb{Z})$ acts transitively on the set of isotropic subspaces W_0 of fixed dimension, the boundary component of $\mathcal{A}_g^*$ is indexed by the dimension $\nu \in \{0,\cdots,g\}$ of the isotropic subspaces W_0. Furthermore, the choice of W_0 defines a weight filtration $W_\bullet$ on $V_{\mathbb{Q}}$:

$$W_{-1} := \{0\} \subset W_0 \subset W_1 := (W_0)^{\perp} \subset W_2 := V_{\mathbb{Q}}.$$

The polarization Q induces a nondegenerate symplectic form $\overline{Q}$ on the graded piece $\mathrm{Gr}_1^W := W_1/W_0$. The boundary component F_{W_0} can then be described as the classifying space $D_{g'}$ ($g' = g-\nu$) of $\overline{Q}$-polarized Hodge structure of weight 1 on Gr_1^W (see [Cat84] Page 84), and so $A_{g'} = \mathrm{Sp}(2g',\mathbb{Z})\backslash\mathfrak{H}_{g'} \cong (\mathrm{Sp}(2g,\mathbb{Z}) \cap \mathcal{N}(F_{W_0}))\backslash D_{g'}$ gives a boundary component of $\mathcal{A}_g^*$. The compactification $\mathcal{A}_g^*$ then admits the following stratification

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup A_{g-1} \sqcup \cdots \sqcup A_0.$$

3.1.2. *The Baily-Borel compactifications for modular varieties of orthogonal type.* We let L be an integral lattice (i.e. a free $\mathbb{Z}$-module equipped with a symmetric bilinear form $(-,-)$) of signature $(2,n)$ $(n > 3)$. Consider the Hermitian symmetric domain of type IV

$$D_L := \{x \in \mathbb{P}(L \otimes_{\mathbb{Z}} \mathbb{C}) \mid (x,x) = 0,\ (x,\bar{x}) > 0\}^+,$$

where the superscript $+$ denotes a choice of one of the two connected components. Let $\mathrm{O}^+(L)$ be the subgroup of $\mathrm{O}(L)$ of real spinor norm 1 (in other words, the subgroup fixing the connected components). A *modular variety of orthogonal type* is a quotient

$$\mathcal{F}_L(\Gamma) = \Gamma\backslash D_L$$

of D_L by an arithmetic subgroup Γ of $\mathrm{O}^+(L \otimes \mathbb{Q})$ (that is, $\Gamma \subset \mathrm{O}(L \otimes \mathbb{Q})$ and $\Gamma \cap \mathrm{O}^+(L)$ is of finite index in both Γ and $\mathrm{O}^+(L)$). Some important examples of orthogonal type modular varieties are the period spaces of (lattice) polarized K3 surfaces. For instance, if one takes the lattice L_{2d} of signature $(2,19)$ defined by

$$L_{2d} := \mathrm{U}^{\oplus 2} \oplus \mathrm{E}_8^{\oplus 2} \oplus \langle -2d \rangle,$$

and sets $\Gamma := \mathrm{O}^+(L_{2d}) \cap \widetilde{\mathrm{O}}(L_{2d})$ (where the group $\widetilde{\mathrm{O}}(L_{2d})$ is the kernel of the natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(L^\vee/L)$), then the quotient $\Gamma\backslash D_{L_{2d}}$ is the period space of polarized K3 surfaces of degree $2d$.

The rational maximal parabolic subgroups are the stabilizers of isotropic subspaces of $L_\mathbb{Q} := L \otimes_\mathbb{Z} \mathbb{Q}$. Because of the signature, such spaces have dimensions 2 or 1. The Baily-Borel compactification $\mathcal{F}_L(\Gamma)^*$ decomposes into the disjoint union (see for example [GHS13, Thm. 5.11])

$$\mathcal{F}_L(\Gamma)^* = \mathcal{F}_L(\Gamma) \sqcup \coprod_{\Pi} X_\Pi \sqcup \coprod_{l} Q_l,$$

where l and Π run though representatives of the finitely many Γ-orbits of isotropic lines and isotropic planes in $L_\mathbb{Q}$ respectively. Each X_Π is a modular curve, each Q_l is a point. X_Π and Q_l are usually referred to as 1-dimensional and 0-dimensional boundary components. For the period space $\Gamma\backslash D_{L_{2d}}$ of polarized K3 surfaces of degree $2d$, the 1-dimensional and 0-dimensional boundary components correspond to the degenerate fibers in Kulikov degenerations of type II and type III respectively.

It is also an interesting problem to compare $(\Gamma\backslash D_{L_{2d}})^*$ with certain compactifications of the moduli space using GIT via the period map, see [Laz14, §2.3] for a survey.

3.2. **Toroidal compactifications.** We discuss toroidal compactifications $\overline{\Gamma\backslash D}^{\Sigma}$ (or simply $\overline{\Gamma\backslash D}$) in this section. The general reference is the book [AMRT75]. Roughly speaking, $\overline{\Gamma\backslash D}$ is obtained by adding a divisor at each cusp of $(\Gamma\backslash D)^*$. Locally in the analytic topology near a cusp, the toroidal compactification is a quotient of an open part of a toric variety over the cusp, which depends on a choice of admissible fan in a suitable cone. Let us also note that

(1) The construction of toroidal compactifications depends on certain suitable choices. The compactification $\overline{\Gamma\backslash D}$ can be chosen to be projective, and to have at worst finite quotient singularities.
(2) For every suitable choice, there is a natural morphism $\overline{\Gamma\backslash D} \to (\Gamma\backslash D)^*$, inducing the identity morphism on $\Gamma\backslash D$.

The readers can find the details of the construction of $\overline{\Gamma\backslash D}$ in [AMRT75, §3.5]. We now describe the toroidal compactification of $\mathcal{A}_g$ following [Cat84] (see also Section 1 of [CMGHL14]). We will not discuss their modular meanings here, instead we refer the readers to [Laz14, §2.2.1] and references therein.

Notations as in Section 3.1.1. The construction of $\overline{\mathcal{A}_g}$ is relative over $\mathcal{A}_g^*$. Let W_0 be a totally isotropic subspace of $V_\mathbb{Q}$ of dimension $\nu \le g$, which corresponds to a boundary component F_{W_0} of $\mathcal{A}_g^*$. Consider then the real Lie subalgebra $\mathfrak{n}(W_0)$ of $\mathfrak{sp}(V_\mathbb{R}, Q)$:

$$\mathfrak{n}(W_0) := \{N \in \mathfrak{sp}(V_\mathbb{R}, Q) \mid Im(N) \subset W_0 \otimes_\mathbb{Q} \mathbb{R}\}.$$

For all elements $N \in \mathfrak{n}(W_0)$, we have $Q(N^2(v), w) = -Q(N(v), N(w)) = 0$ where $v, w \in V_\mathbb{R}$, and so $N^2 = 0$. For a rational element N, it defines a weight filtration $W_\bullet(N)$ on $V_\mathbb{Q}$:

$$W_0(N) := Im(N) \subset W_1(N) := \mathrm{Ker}(N) = Im(N)^\perp \subset W_2(N) := V_\mathbb{Q},$$

which is compatible with the weight filtration $W_\bullet$ induced by W_0:

$$W_0(N) \subset W_0 \subset W_1 (= W_0^\perp) \subset W_1(N).$$

In particular, we obtain a natural surjection

$$\mathrm{Gr}_2^W(=V_{\mathbb{Q}}/W_1) \twoheadrightarrow \mathrm{Gr}_2(N)(=V_{\mathbb{Q}}/W_1(N)).$$

Because N is a nilpotent symplectic endomorphism, we have a natural isomorphism

$$\mathrm{Gr}_2(N) \to \mathrm{Gr}_0(N) \to \mathrm{Gr}_2(N)^\vee, \quad v \mapsto N(v) \mapsto Q(N(\cdot), v),$$

which gives a non-degenerate bilinear form Q_N on $\mathrm{Gr}_2(N)$. It is easy to see that Q_N is symmetric. Pulling it back via the natural surjection $\mathrm{Gr}_2^W \to \mathrm{Gr}_2(N)$, we get a symmetric bilinear form $\tilde{Q}_N$ on Gr_2^W. So there is a natural map (defined over $\mathbb{Q}$)

$$\mathfrak{n}(W_0) \xrightarrow{\cong} \mathrm{Hom}(\mathrm{Sym}^2 \mathrm{Gr}_{2,\mathbb{R}}^W, \mathbb{R})$$

which is an isomorphism.

We now define

$$\mathfrak{n}^+(W_0) = \{N \in \mathfrak{n}(W_0) \mid \tilde{Q}_N \text{ is positive definite}\},$$

and denote by $Cl(\mathfrak{n}^+(W_0))$ the closure of $\mathfrak{n}^+(W_0)$ in $\mathfrak{n}(W_0)$. The weight filtration $W_\bullet$ defined by W_0 is the weight filtration $W_\bullet(N)$ of any element $N \in \mathfrak{n}^+(W_0)$, as well as any cone

$$\sigma = \left\{ \sum_{i=1}^r \lambda_i N_i \;\middle|\; \lambda_i \in \mathbb{R}, \lambda_i > 0, N_i \in Cl(\mathfrak{n}^+(W_0)) \right\}$$

which contains an element of $\mathfrak{n}^+(W_0)$ (see Page 78 of [Cat84]). For any such cone σ, we define $B(\sigma) \subset \check{D}$ (note that $\check{D} = \{F \in \mathrm{Gr}(g, V_{\mathbb{C}}) \mid Q(F,F) = 0\}$ is the compact dual of D) to be the set of all those filtrations in $\check{D}$ which, together with $W_\bullet$, define a mixed Hodge structure polarized by every $N \in \mathrm{Int}(\sigma)$. It is not difficult to see that $B(\sigma) = \exp(\sigma_{\mathbb{C}}) \cdot D$, where $\sigma_{\mathbb{C}} = \{\sum_{i=1}^r \lambda_i N_i \mid \lambda_i \in \mathbb{C}, N_i \in Cl(\mathfrak{n}^+(W_0))\}$.

The boundary component associated to the cone σ is then defined as

$$\mathcal{B}(\sigma) := B(\sigma)/\exp(\sigma_{\mathbb{C}}).$$

Since $\exp(\sigma_{\mathbb{C}})$ acts trivially on Gr_1^W, $\mathcal{B}(\sigma)$ factors over the boundary component F_{W_0} of the Baily-Borel compactification by considering the Hodge structure on the graded piece Gr_1^W. Moreover, if τ is a face of σ, then there is a natural morphism $\mathcal{B}(\tau) \to \mathcal{B}(\sigma)$ which respects the projections to F_{W_0}.

In order to attach the boundary components to D so that the extended space (i.e. the disjoint union of D with the boundary components) is compatible with the action of the arithmetic group, we need a suitable choice of a collection Σ of rational polyhedral cones in $Cl(\mathfrak{n}^+(W_0))$, which is called an *admissible rational polyhedral decomposition* of $\mathfrak{n}^+(W_0)$. See for example [CMGHL14, §1.3, 1.4] for further discussions.

One then sets

$$D^\Sigma = \bigcup_{W_0} \bigcup_{\sigma \in \Sigma} \mathcal{B}(\sigma).$$

The action of the arithmetic group $\mathrm{Sp}(2g, \mathbb{Z})$ extends to D^Σ, and so (set-theoretically) we have

$$\overline{\mathcal{A}_g}^\Sigma = \mathrm{Sp}(2g, \mathbb{Z}) \backslash D^\Sigma.$$

References

[ACT02] D. Allcock, J. A. Carlson, and D. Toledo. The complex hyperbolic geometry of the moduli space of cubic surfaces. *J. Algebraic Geom.*, 11(4):659–724, 2002.

[ACT11] D. Allcock, J. A. Carlson, and D. Toledo. The moduli space of cubic threefolds as a ball quotient. *Mem. Amer. Math. Soc.*, 209(985):xii+70, 2011.

[AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. *Smooth compactification of locally symmetric varieties*. Math. Sci. Press, Brookline, Mass., 1975.

[BB66] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[Bor69] A. Borel. *Introduction aux groupes arithmétiques*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341. Hermann, Paris, 1969.

[Bor72] A. Borel. Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. *J. Differential Geometry*, 6:543–560, 1972. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays.

[Bor97] C. Borcea. $K3$ surfaces with involution and mirror pairs of Calabi-Yau manifolds. In *Mirror symmetry, II*, volume 1 of *AMS/IP Stud. Adv. Math.*, pages 717–743. Amer. Math. Soc., Providence, RI, 1997.

[Bou02] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002.

[Cat84] E. H. Cattani. Mixed Hodge structures, compactifications and monodromy weight filtration. In *Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982)*, volume 106 of *Ann. of Math. Stud.*, pages 75–100. Princeton Univ. Press, Princeton, NJ, 1984.

[CMGHL14] S. Casalaina-Martin, S. Grushevsky, K. Hulek, and R. Laza. Extending the Prym map to toroidal compactifications of the moduli space of abelian varieties. arXiv:1403.1938, 2014.

[Con] B. Conrad. Handout on Shimura Data. Electronic notes available at http://math.stanford.edu/∼conrad/shimsem.

[CS11] F. Charles and C. Schnell. Notes on absolute Hodge classes. Notes based on lectures given at the ICTP summer school on Hodge theory and related topics (June 2010, Trieste), available at arXiv:arXiv:1101.3647, 2011.

[Del79] P. Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions*, Proc. Sympos. Pure Math., XXXIII, pages 247–289. AMS, Providence, R.I., 1979.

[Del82] P. Deligne. Hodge cycles on abelian varieties. In *Hodge cycles, motives, and Shimura varieties*, pages 9–100. Springer, 1982.

[DK07] I. V. Dolgachev and S. Kondō. Moduli of $K3$ surfaces and complex ball quotients. In *Arithmetic and geometry around hypergeometric functions*, volume 260 of *Progr. Math.*, pages 43–100. Birkhäuser, Basel, 2007.

[DM86] P. Deligne and G. D. Mostow. Monodromy of hypergeometric functions and nonlattice integral monodromy. *Inst. Hautes Études Sci. Publ. Math.*, 63:5–89, 1986.

[FL13] R. Friedman and R. Laza. Semialgebraic horizontal subvarieties of Calabi-Yau type. *Duke Math. J.*, 162(12):2077–2148, 2013.

[FL14] R. Friedman and R. Laza. On some hermitian variations of hodge structure of calabi-yau type with real multiplication. *The Michigan Mathematical Journal*, 63(1):83–99, 2014.

[GGK12] M. Green, P. A. Griffiths, and M. Kerr. *Mumford-Tate groups and domains: Their geometry and arithmetic*, volume 183 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012.

[GHS13] V. Gritsenko, K. Hulek, and G. K. Sankaran. Moduli of K3 surfaces and irreducible symplectic manifolds. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 459–526. Int. Press, Somerville, MA, 2013.

[Gro94] B. H. Gross. A remark on tube domains. *Math. Res. Lett.*, 1(1):1–9, 1994.

[GvG10] A. Garbagnati and B. van Geemen. Examples of Calabi-Yau threefolds parametrised by Shimura varieties. *Rend. Semin. Mat. Univ. Politec. Torino*, 68(3):271–287, 2010.

[Hel78] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[Kna02] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.

[Kon00] S. Kondō. A complex hyperbolic structure for the moduli space of curves of genus three. *J. Reine Angew. Math.*, 525:219–232, 2000.

[Laz14] R. Laza. Perspectives on the construction and compactification of moduli spaces. arXiv:1403.2105, to appear in CRM Lecture Electronic notes, 2014.

[Lev] B. Levin. Hermitian symmetric domains. Electronic notes available at http://math.stanford.edu/∼conrad/shimsem.

[Loo14] E. Looijenga. Moduli spaces and locally symmetric varieties. Notes based on three lectures given 6th MSJ-SI conference "Development of Moduli Theory" (June 2013, Kyoto), available at arXiv:1404.3854, 2014.

[Mil04] J. S. Milne. Introduction to Shimura Varieties. Electronic notes available at http://www.jmilne.org/math/xnotes/svi.pdf, 2004.

[Mil11] J. S. Milne. Algebraic Groups, Lie Groups, and their Arithmetic Subgroups. Electronic notes available at www.jmilne.org/math/CourseNotes/ALA.pdf, 2011.

[Mil13] J. S. Milne. Shimura varieties and moduli. In *Handbook of Moduli II*, volume 25 of *Advanced Lectures in Mathematics*, pages 467–548. Int. Press, Somerville, MA, 2013.

[Moo99] B. Moonen. Electronic notes on Mumford-Tate groups. Electronic notes available at http://www.math.ru.nl/personal/bmoonen, 1999.

[Roh09] J. C. Rohde. *Cyclic coverings, Calabi-Yau manifolds and complex multiplication*, volume 1975 of *Lecture Electronic notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[Roh10] J. C. Rohde. Maximal automorphisms of Calabi-Yau manifolds versus maximally unipotent monodromy. *Manuscripta Math.*, 131(3-4):459–474, 2010.

[Sat80] I. Satake. *Algebraic structures of symmetric domains*, volume 4 of *Kanô Memorial Lectures*. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N.J., 1980.

[SZ10] M. Sheng and K. Zuo. Polarized variation of Hodge structures of Calabi–Yau type and characteristic subvarieties over bounded symmetric domains. *Math. Ann.*, 348(1):211–236, 2010.

[vG94] B. van Geemen. An introduction to the Hodge conjecture for abelian varieties. In *Algebraic cycles and Hodge theory (Torino, 1993)*, volume 1594 of *Lecture Electronic notes in Math.*, pages 233–252. Springer, Berlin, 1994.

[vG01] B. van Geemen. Half twists of Hodge structures of CM-type. *J. Math. Soc. Japan*, 53(4):813–833, 2001.

[Viv13] F. Viviani. A tour on Hermitian symmetric manifolds. arXiv:1310.3665, 2013.

[Voi93] C. Voisin. Miroirs et involutions sur les surfaces $K3$. *Astérisque*, (218):273–323, 1993. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992).

[Voi03] C. Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003. Translated from the French by Leila Schneps.

[Wol84] J. A. Wolf. *Spaces of constant curvature*. Publish or Perish, Inc., Houston, TX, fifth edition, 1984.

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